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Topologically massive chromodynamics at finite temperature

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Topologically massive chromodynamics is studied at a finite temperature T. The topological gauge invariance present at $T=0$ is argued to remain valid at $T\neq 0$, but verifying this in the quantum theory is far less direct at $T\neq 0$ than at $T=0$. Debye screening occurs, and has a striking effect on the correlations of static magnetic fields. The behavior of the free energy at high temperature is also computed.

I. INTRODUCTION

In three space-time dimensions, adding a Chern-Simons term[1] to the Lagrangian for a non-Abelian gauge field dramatically changes the theory.[2-6] Most notably, invariance under topologically nontrivial gauge transformations is no longer assured. To do so at the classical level, the coupling of the Chern-Simons terms, which is a mass m, must be equal to an integer q times $g^2/4\pi$, where g is the gauge coupling.[2]

How does this topological gauge invariance manifest itself in the quantum theory? At least in one special case—in the Landau gauge when $q \gg 1$—it is straightforward to calculate the (finite) renormalization of this integer, $q_{\rm ren}$ (Ref. 3). For an SU(N) gauge group without fermions, to one-loop order $q_{\rm ren}=q+N$. Beyond leading order in the loop expansion, $q_{\rm ren}$ does not change, and a new, topological Ward identity emerges to ensure invariance under large gauge transformations. What was not clear before was whether this $q_{\rm ren}$ was a gauge-invariant, and so a physically meaningful, quantity.

In this paper I consider topologically massive chromodynamics at a finite temperature T. My purpose is simply to gain greater insight into the physics of these unusual theories.

This hope is certainly borne out by the first topic I consider, which is the topological gauge symmetry at $T\neq 0$. Common sense tells us that if q and $q_{\rm ren}$ are each integers, then by continuity they must be independent of temperature.

Classically this is obvious, for the same facts of cohomology[1,5] which quantize q as an integer at zero temperature apply without modification when $T\neq 0$.

At first, this does not seem to be true in the quantum theory. If one blindly recalculates at $T\neq 0$ in precisely the same way as at $T=0$, it appears that $q_{\rm ren}$ is not an integer when $T\neq 0$, but just some ugly function of m/T. I shall argue that this is simply because one cannot be blind in calculating $q_{\rm ren}$ at $T\neq 0$.

Understanding how to correctly calculate $q_{\rm ren}$ at $T\neq 0$ suggests a general understanding of $q_{\rm ren}$. I propose that $q_{\rm ren}$ can always be defined by the behavior of the full, effective action under topologically nontrivial gauge transformations. I then conjecture that with this definition, $q_{\rm ren}$ *is* a physical quantity, in that it is independent of the choice of gauge, boundary conditions, and so forth.

For reasons that will become clear, usually it is extremely difficult to calculate $q_{\rm ren}$. If my conjecture is true, this is unnecessary, for one can simply compute in a convenient gauge, such as the Landau gauge, with convenient boundary conditions, such as at zero temperature.

While I shall present some evidence in support of the conjecture, admittedly it is limited. Even so, given the fundamental role that topological gauge invariance plays in these theories, then if they are consistent quantum field theories, it is very hard to see how it could fail.

I am also led to make a second conjecture. The relation $q_{\rm ren}=q+N$ was computed for an SU(N) gauge group in weak coupling, which is large q. Since the bare value of q is itself an integer, there is no guarantee that this relationship could not change discontinuously as q decreased. I suggest that this does not happen—that $q_{\rm ren}=q+N$ as long as the bare value of $q\neq 0$. It might be possible to understand why $q_{\rm ren}-q$ is proportional to N by considering $Z(N)$ monopoles in SU(N) (Ref. 6).

I start in Sec. II by reviewing topological gauge invariance at $T=0$. I pay particular attention to some of the details which were previously swept under the rug,[3] for what is under there is crucial in understanding $q_{\rm ren}$ at $T\neq 0$. Fermions also contribute to $q_{\rm ren}$, and in Sec. III, I show that their contribution is the same at $T\neq 0$ as it is at $T=0$. The methods which I use are not only circuitous, but special to fermions.[7,8] This demonstration is still instructive, for it does show that the way in which the topological gauge invariance works need not, and usually will not, be as transparent as the consequences of invariance under infinitesimal gauge transformations.

I demonstrate that Debye screening occurs at $T\neq 0$ in Sec. IV, and that it has unexpected effects. At $T=0$, the correlations of all fields are screened over large distances by the Chern-Simons mass. For any $T\neq 0$, however, while the long-distance correlations of electric and nonstatic magnetic fields remain screened, those of static magnetic fields are not. Because of the coherent oscillations in the thermal plasma, static B fields do develop long-range correlations.

I discuss some features of the free energy in Sec. V, including how easy it is to compute its behavior at high temperatures.

II. TOPOLOGICAL GAUGE INVARIANCE AT $T=0$

I follow the conventions and notation of Ref. 3. For simplicity, I take the gauge group to be SU(2), and use both the Pauli matrices σ^a and the anti-Hermitian τ^a, $\tau^a=\sigma^a/2i$. I put the fermions in the fundamental representation, with N_f flavors of mass m_f, and use a P-even regulator, so there are no "regulator" fermions.

The Chern-Simons term

$$S_m=-im\int d^3x\,\epsilon^{\mu\nu\lambda}\mathrm{tr}(A_\mu\partial_\nu A_\lambda+\tfrac{2}{3}gA_\mu A_\nu A_\lambda) \tag{2.1}$$

transforms under a gauge transformation Ω as

$$S_m\to S_m+\frac{im}{g}\int d^3x\,\epsilon^{\mu\nu\lambda}\partial_\mu\mathrm{tr}[(\partial_\nu\Omega)\Omega^{-1}A_\lambda]+2\pi i\left[\frac{4\pi m}{g^2}\right]w\,. \tag{2.2}$$

Write Ω in terms of elements of the Lie algebra, Θ^a:

$$\Omega=\exp(i\sigma^a\Theta^a)\,, \tag{2.3}$$

$\Theta^a=\Theta\hat{\Theta}^a$, $(\hat{\Theta}^a)^2=1$. Then w, the winding number, can be expressed as a surface integral:[9]

$$w=\frac{1}{8\pi^2}\int d^3x\,\partial_\mu j^\mu\,,\qquad j^\mu=\left[\Theta-\frac{\sin 2\Theta}{2}\right]\epsilon^{\mu\nu\lambda}\epsilon^{abc}\hat{\Theta}^a\partial_\nu\hat{\Theta}^b\partial_\lambda\hat{\Theta}^c\,. \tag{2.4}$$

If Θ^a is an arbitrary function of $\bar{x}$, w can take on any value, and the theory is not gauge invariant. To ensure invariance under large gauge transformations, it is necessary to assume that space-time can be treated as a compact manifold.

At zero temperature, this means that Ω must approach a unique value Ω_∞ at space-time infinity, and so space-time is isomorphic to the three-sphere S^3. From homotopy theory, the mappings from S^3 into any Lie group are labeled by an integer which is the winding number w.

Since Ω_∞ is constant, as long as all A_μ are pure gauge rotations at infinity, the A_μ-dependent surface term in Eq. (2.2) vanishes. Following Deser, Jackiw, and Templeton,[2] we can then arrange for the partition function to be gauge invariant, at least classically, by choosing $4\pi m/g^2$ equal to an integer q.

It helps to have an example of an Ω with $w\neq 0$. In SU(2), by a global gauge rotation I can take $\Omega_\infty=\pm 1$:

$$\Omega=\exp\left[i\pi n\frac{\bar{x}\cdot\bar{\sigma}}{(\bar{x}^2+\rho^2)^{1/2}}\right]. \tag{2.5}$$

From Eq. (2.4), this Ω has $w=n$, with $\Omega_\infty=(-)^n$.

In Eq. (2.5), ρ is a scale size $0\le\rho\le\infty$. Instantons in four space-time dimensions also have a parameter ρ which characterizes their size, but for instantons, ρ is a physical quantity. Here, ρ is merely another parameter which labels the infinite degeneracy of topologically nontrivial gauge transformations.

It is easy to generalize this to SU(N) by embedding the Ω of Eq. (2.5) into an SU(2) subgroup of SU(N). Up to global color rotations, Ω_∞ is a diagonal matrix with elements $+1$ and -1.

A natural question is whether further restrictions can be placed on the Ω's so as to forbid those with $w\neq 0$. For the Ω of Eq. (2.5), this seems possible, since for this Ω, the contribution to w is entirely from the point at infinity. Perhaps one could impose $w=0$ by requiring Θ, Eq. (2.3), to vanish at infinity.

This hope is misplaced. By a smooth gauge transformation, one can rewrite the Ω of Eq. (2.5) in a form where the point at infinity, represented say by the north pole on the three-sphere, makes no contribution to w. But because w is a topological invariant, this would only be shifting the point which gave $w\neq 0$ to some other point on the three-sphere. In the original space-time, any such point lies at finite $\bar{x}$, and is not affected by the boundary conditions at infinity. Hence in these theories, topological gauge invariance is an unavoidable consequence of the local gauge symmetry.

Things are not so obvious in the quantum theory. One would like to determine q_{ren} from the effective action, computed to some order in the loop expansion, in the presence of a background field A^n_μ with $w=n$. While this A^n_μ will be a pure gauge field, $A^n_\mu=\Omega^{-1}\partial_\mu\Omega/g$, for finite ρ this necessitates calculating the effective action for a background field which is large in magnitude ($A^n_\mu\sim 1/g$) and has a momentum dependence that cannot be neglected.

Suppose, however, that the scale size ρ is very large. By increasing ρ, we can make the magnitude of A^n_μ, and its typical momenta ($\sim\rho^{-1}$), arbitrarily small over any finite region of space-time. As $\rho\to\infty$, all that contributes to the effective action is the renormalization of the Chern-Simons term, computed about the trivial vacuum, $A^n_\mu=0$, at zero momentum.

This means that the effective action must have a well-defined expansion about zero momentum. For covariant gauges, this is only true in the Landau gauge.[3] Using this gauge in weak coupling to one-loop order,[3]

$$q_{\mathrm{ren}}=Z_m\left[\frac{Z}{Z_g}\right]^2 q=q+N+\mathrm{sgn}(m_f)\frac{N_f}{2} \tag{2.6}$$

for an SU(N) gauge group. Z_m is the mass renormalization constant evaluated at zero momentum, etc. For q and q_{ren} to be integral, N_f has to be even,[8] but there is no restriction on the gauge group. (A^n_μ should be put into the Landau gauge, but this can obviously be done.)

Corrections to Eq. (2.6) are the $O(q^{-1})$, so for q_{ren} to remain an integer at large q, Eq. (2.6) must be exact. This implies a topological Ward identity that relates Z_m to the other Z's (Ref. 3). This topological Ward identity is very similar in form to the usual Ward identities of infinitesimal gauge invariance. Nevertheless, it holds only for Z's evaluated at zero momentum in the Landau gauge, since only then in one computing the renormalized action for Ω's with scale size $\rho\to\infty$.

The effects of topological gauge invariance cannot be as simple at nonzero momentum—in any gauge—as they are at zero momentum in the Landau gauge. To probe nonzero momenta, the background A_μ^n must be generated by an Ω with finite scale size. But then A_μ^n is large in magnitude, and so calculating correlation functions about the trivial vacuum will not tell one much about correlation functions in the presence of such an A_μ^n. An explicit example of this will be seen in the next section.

It cannot be easy to show what I conjecture $q_{\rm ren}$ is—namely, a gauge-invariant measure of the (inverse) dimensionless coupling constant. Following Witten,[7] we understand[8] that fermions contribute to $q_{\rm ren}$ because the fermion measure in the functional integral—that is, their effective action $N_f \operatorname{tr}\ln(\not{D}-m_f)$—is not invariant under large gauge transformations. Since $q_{\rm ren}\neq q$ even without fermions, this means that the measure for gauge fields in the functional integral, including ghosts and the like, must also transform nontrivially under large gauge transformations.

So why should the conjecture about $q_{\rm ren}$ hold? For the quantum theory to make sense, $q_{\rm ren}$ must be an integer. It seems unlikely that $q_{\rm ren}$ would change discontinuously as one continuously varied parameters like Ω's scale size, those for gauge fixing, boundary conditions, etc. While difficult, perhaps the proof of the conjecture would also yield some insight into the effects of topological gauge invariance at nonzero momenta.

III. TOPOLOGICAL GAUGE INVARIANCE AT $T\neq 0$

In thermal equilibrium at a finite temperature T, the gauge fields A_μ and the fermions fields ψ obey the boundary conditions

$$\begin{aligned} A_\mu(\mathbf{x},t+\beta)&=+A_\mu(\mathbf{x},t)\ ,\\ \psi(\mathbf{x},t+\beta)&=-\psi(\mathbf{x},t)\ . \end{aligned} \tag{3.1}$$

$\beta=T^{-1}$, and $\bar{x}=(\mathbf{x},t)$: the three-momenta $\bar{p}=(\mathbf{p},p_0)$, where p_0 is an even or odd multiple of πT for bosons or fermions, respectively.

I assume that, like A_μ, the allowed gauge transformations $\Omega=\Omega(\mathbf{x},t)$ are strictly periodic in time. [When there are no fields in the fundamental representation, Ω need only be periodic up to a global $Z(N)$ transformation, but for our purposes, this can be ignored. This $Z(N)$ symmetry does play a role in the presence of monopoles.[6]] Compactifying space-time, we obtain $S^2\times S^1$—but homotopy theory does not tell us how to classify maps from $S^2\times S^1$ into a Lie group.[5] We must resort to cohomology theory,[1,5] which says that the mappings from any compact three-manifold into a Lie group are characterized by an integer, which is of course the winding number w. Since any compact three-manifold will do, q must be an integer for all (physically reasonable) boundary conditions. This includes not just finite temperature, but even if we put the theory in a box which is of finite size in all three directions.

A prescription for constructing a strictly periodic Ω with winding number $w=n$ can be given. Start with a four-dimensional instanton, with instanton number $=n$, at a finite temperature T (Ref. 10). This is an A_μ in singular gauge that is strictly periodic in time. Choose one spatial direction, say x_1, and transform the instanton into the $A_1=0$ gauge. This is done by a gauge transformation $\sim P\exp(\int_{-\infty}^{x_1} A_1 dx_1')$, and so the instanton is still strictly periodic in time in the $A_1=0$ gauge. At $x_1=-\infty$, we can insist that $A_\mu=0$; at $x_1=+\infty$, $A_\mu=\Omega^{-1}\partial_\mu\Omega/g$, $\mu=2,3,4$. Because of the relation between winding number and instanton number, this Ω is what we want—it depends only on the three-space defined by x_2, x_3, and t, is strictly periodic in t, and has winding number $=n$.

In practice, this construction is very cumbersome, and it is easier to guess. Consider

$$\Omega=\exp\left[2\pi i\frac{t}{\beta}\hat{\Theta}\cdot\bar{\sigma}\right]\ . \tag{3.2}$$

$\hat{\Theta}$ depends only on the spatial $\mathbf{x}$, and is chosen to be a two-dimensional instanton[11] with instanton number (for maps of $S^2\rightarrow S^2$) $=n$:

$$n=\frac{1}{8\pi}\int d^2\mathbf{x}\,\epsilon^{ij}\epsilon^{abc}\hat{\Theta}^a\partial_i\hat{\Theta}^b\partial_j\hat{\Theta}^c\ . \tag{3.3}$$

When $\mathbf{x}\rightarrow\infty$, $\hat{\Theta}(\mathbf{x})\rightarrow$a constant, and $\Omega\rightarrow$a constant Ω_∞. Ω_∞ does depend on time, but this is allowed for a manifold isomorphic to $S^2\times S^1$.

As in four dimensions, the two-dimensional instantons $\hat{\Theta}(\mathbf{x})$ come in all scale sizes $\tilde{\rho}$. Using Eq. (2.4) shows that the winding number of Eq. (3.2) is $w=2n$. I have not been able to guess a form for a strictly periodic Ω with odd w—following Eq. (3.2) gives an antiperiodic Ω—but the construction described above shows that such Ω's do exist. The gauge field generated by Eq. (3.2) is

$$\begin{aligned} A_0&=\frac{i}{g}\frac{2\pi}{\beta}\bar{\sigma}\cdot\hat{\Theta}\ ,\\ A_i&=\frac{1}{4g}\left[\exp\left[2\pi i\frac{t}{\beta}\right](1-\bar{\sigma}\cdot\hat{\Theta})\right.\\ &\quad\left.-\exp\left[-2\pi i\frac{t}{\beta}\right](1+\bar{\sigma}\cdot\hat{\Theta})+2\bar{\sigma}\cdot\hat{\Theta}\right]\bar{\sigma}\cdot\partial_i\hat{\Theta}\ . \end{aligned} \tag{3.4}$$

I now make an elementary observation. By making the scale size $\tilde{\rho}$ of $\hat{\Theta}$ arbitrarily small, I can make A_i and the typical spatial momenta of A_0 and A_i arbitrarily small. What I cannot do is make the magnitude of A_0, or the energies of the A_i, small. This is true for any Ω—to have $w\neq 0$, Ω must depend on time, and so have nonzero energy. At $T\neq 0$, no energy $\neq 0$ can be smaller than $2\pi T$.

In considering $q_{\rm ren}$, for the sake of simplicity I concentrate on the effects of fermions. As explained in the previous section, at $T=0$, $q_{\rm ren}$ is determined from Z's evaluated at zero momentum, Eq. (2.6). Fermions contribute equally to Z and Z_g, so only their effect on Z_m matters; $Z_m=1+\Pi_o$, where Π_o is the P-odd part of the gluon self-energy. I do the most naive thing possible, and evaluate $\Pi_o(\mathbf{p},p_0)$ at $T\neq 0$ in exactly the same way as at $T=0$—by taking $p_0=\mathbf{p}=0$:

$$\Pi_o^{\rm ferm}(0,0)=g^2\frac{m_f}{m}N_f\int_-\frac{1}{(\bar{k}^2+m_f^2)^2}\frac{d^3k}{(2\pi)^3}$$
$$=g^2\frac{m_f}{m}N_f\int_-\left[\frac{1}{4E_f^3}-\frac{2}{E_f^3[1+\exp(E_f/T)]}-\frac{E_f}{2T\cosh^2(E_f/2T)}\right]\frac{d^2\mathbf{k}}{(2\pi)^2}\,, \tag{3.5}$$

for fermions in the fundamental representation; $E_f=(\mathbf{k}^2+m_f^2)^{1/2}$. Subscripts $+$ or $-$ on integrals or traces refer to boson or fermion boundary conditions, Eq. (3.1). At low temperatures,

$$\Pi_o^{\rm ferm}(0,0)\underset{T\ll|m_f|}{\sim}\frac{g^2}{8\pi m}{\rm sgn}(m_f)N_f[1-10\exp(-|m_f|/T)+\cdots]\,, \tag{3.6}$$

while at high temperatures,

$$\Pi_o^{\rm ferm}(0,0)\underset{T\gg|m_f|}{\sim}-\frac{3g^2}{16\pi m}{\rm sgn}(m_f)N_f+\cdots\,. \tag{3.7}$$

At $T=0$, $q\Pi_o^{\rm ferm}(0,0)={\rm sgn}(m_f)N_f/2$, Eq. (2.6), but for any $T\neq 0$, $q\Pi_o^{\rm ferm}(0,0)$ is just some involved function of m_f/T.

[As an aside, note that the sign of $\Pi_o^{\rm ferm}$ differs at high temperature from that at low temperature. Hence if one started with a gauge theory with no bare Chern-Simons term, $q=0$, then the sign of the Chern-Simons term induced by the fermions would change as the temperature were raised. Presumably, there is a temperature $T_c\sim|m_f|$, at which the induced term evaporates—$\Pi_o^{\rm ferm}(0,0)=0$ at T_c.]

To correctly calculate the fermion contribution to $q_{\rm ren}$ at $T\neq 0$, I start with the fermion part of the effective action. To one-loop order, this is

$$S_{\rm eff}^{\rm ferm}(A_\mu)=-N_f{\rm tr}_-\ln(\partial\!\!\!/+gA\!\!\!/-m_f)\,. \tag{3.8}$$

Following the definition of $q_{\rm ren}$ proposed in the Introduction,

$$S_{\rm eff}^{\rm ferm}(A_\mu^n)-S_{\rm eff}^{\rm ferm}(0)\equiv(2\pi in)q_{\rm ren}^{\rm ferm}\,, \tag{3.9}$$

where A_μ^n is a pure gauge field of winding number n, as in Eq. (3.4).

$q_{\rm ren}^{\rm ferm}$ could be found by brute force. For a scale size $\tilde{\rho}\gg 1$, $S_{\rm eff}^{\rm ferm}$ can be expanded to quadratic order in the A_i^n, but all orders in the large field A_0^n must be kept. This is why Eqs. (3.5)–(3.8) do not get $q_{\rm ren}^{\rm ferm}$ right: implicitly one is expanding in A_0^n. Although large in magnitude, A_0^n is approximately constant in space when $\tilde{\rho}\gg 1$. Expansion to quadratic order in the spatial derivatives of A_0^n would give $q_{\rm ren}^{\rm ferm}$, but only after much effort and with no insight.

There is an easier way. Consider the variation of $S_{\rm eff}^{\rm ferm}$ with m_f:

$$\frac{\partial}{\partial m_f}[S_{\rm eff}^{\rm ferm}(A_\mu^n)-S_{\rm eff}^{\rm ferm}(0)]$$
$$=N_f{\rm tr}_-\left[\frac{1}{\partial\!\!\!/+gA\!\!\!/^n-m_f}-\frac{1}{\partial\!\!\!/-m_f}\right]. \tag{3.10}$$

Since $gA\!\!\!/^n=\Omega^{-1}\partial\!\!\!/\Omega$,

$$N_f{\rm tr}_-\left[\Omega\frac{1}{\partial\!\!\!/-m_f}\Omega^{-1}-\frac{1}{\partial\!\!\!/-m_f}\right]=0\,, \tag{3.11}$$

by the cyclic property of the trace. Contained in the definition of the trace are the correct boundary conditions for the fermions, Eq. (3.1). As a large gauge transformation, Ω can only be commuted in the trace if it does not alter these boundary conditions; this is why I insisted that Ω be strictly periodic in time.

Equation (3.10) shows that $q_{\rm ren}^{\rm ferm}$ is independent of the magnitude of the fermions' mass, so I can choose to evaluate it at zero mass. Arguments first used by Witten in a related context can then be employed.[7] Remember the construction of an Ω with $w=n$ at $T\neq 0$. Consider the dimension x_1 simply as a parameter that interpolates between $\Omega=0$ at $x_1=-\infty$, and the Ω with $w=n$ at $x_1=+\infty$. Both Ω's are proper gauge transformations, so the eigenvalues of $\partial\!\!\!/+gA\!\!\!/^n$ and $\partial\!\!\!/$ must be identical, but as $x_1:-\infty\to+\infty$, levels can cross from negative to positive energies (or vice versa). The number of such crossings is related to the number of zero modes in the instanton field, which in turn is determined by the instanton number. In this way, one mimics the analysis at $T=0$ to find[8]

$$\det{}_-(\partial\!\!\!/+gA\!\!\!/^n)=(-)^{nN_f}\det{}_-(\partial\!\!\!/)\,. \tag{3.12}$$

Trivially, this result is independent of the scale size $\tilde{\rho}$. From Eqs. (3.9) and (3.11), $q_{\rm ren}^{\rm ferm}$ must be proportional to $N_f/2$. I make the small step of assuming that the constant of proportionality is ${\rm sgn}(m_f)$, and so $q_{\rm ren}^{\rm ferm}={\rm sgn}(m_f)N_f/2$, exactly as at $T=0$, Eq. (2.6). [For this to be consistent, it is necessary to define ${\rm sgn}(m_f)=+1$ or -1 as $m_f\to 0^+$ or 0^-.]

The trick of Eq. (3.10) does not seem to help with gluons. With $S_{\rm eff}$ the total effective action,

$$\frac{\partial}{\partial m}S_{\rm eff}=\frac{1}{m}\langle S_m\rangle\,, \tag{3.13}$$

where $\langle S_m\rangle$ is the vacuum expectation value of the Chern-Simons term, Eq. (2.1). ($\langle S_m\rangle$ is not manifestly invariant under topologically gauge transformations, but becomes so when the sum over positive and negative winding numbers is performed.) Unlike fermions, $\langle S_m\rangle$ depends on both two- and three-point expectation values of A_μ, and so it appears that $S_{\rm eff}$ could depend on the magnitude of m.

IV. DEBYE SCREENING AT $T\neq 0$

The effects of Debye screening follow from the properties of the gluon vacuum polarization tensor, $\Pi_{\mu\nu}(\mathbf{p},p_0)$. As at $T=0$, an infinitesimal Ward identity requires

$$p^\mu \Pi_{\mu\nu}=0 , \tag{4.1}$$

but the consequences of Eq. (4.1) are very different, depending upon whether $p_0=0$ or $p_0\neq 0$ (Ref. 10).

I start with the static limit, $p_0=0$. Equation (4.1) is satisfied by

$$\Pi_{00}=\Pi_{\rm el}(\mathbf{p}^2) ,\quad \Pi_{0i}=m\epsilon_{ij}\mathbf{p}^j\Pi_o(\mathbf{p}^2) ,\quad \Pi_{ij}=(\delta_{ij}\mathbf{p}^2-\mathbf{p}^i\mathbf{p}^j)\Pi_{\rm mag}(\mathbf{p}^2) ; \tag{4.2}$$

$i,j=1,2$ refer to the spatial directions. $\Pi_{\rm el}$ and $\Pi_{\rm mag}$ can be viewed as self-energies for electric and magnetic fields, while Π_o is a P-odd piece like that at $T=0$; there is no P-odd term in Π_{ij}, since it would be $\sim\epsilon_{ij}p_0$. Equation (4.1) does not require Π_{00} to vanish as $\mathbf{p}\to 0$, so we can define an electric mass $m_{\rm el}{}^2=\Pi_{\rm el}(0)$:

$$\Pi_{\rm el}(\mathbf{p}^2)=m_{\rm el}{}^2+\mathbf{p}^2\Pi'_{\rm el}(\mathbf{p}^2) . \tag{4.3}$$

$\Pi'_{\rm el}$ and $\Pi_{\rm mag}$ will contribute to wave-function renormalization for the electric and magnetic fields.

To determine the electric mass, it is easiest to observe that by Eq. (4.2), $m_{\rm el}{}^2=\Pi_{\mu\mu}(0,0)$. I assume the gauge group is SU(N), with N_f flavors of fermions in the fundamental representation. I work in the Landau gauge, since that is the only (covariant) gauge that is infrared finite at $T=0$ (Ref. 3). To one-loop order, $m_{\rm el}{}^2$ is a sum of fermionic and gluon pieces:

$$m_{\rm el}{}^2=-g^2N_f\int_- \frac{\bar k^2+3m_f{}^2}{(\bar k^2+m_f{}^2)^2}\frac{d^3k}{(2\pi)^3} +g^2N\int_+ \frac{\bar k^2+3m^2}{(\bar k^2+m^2)^2}\frac{d^3k}{(2\pi)^3} . \tag{4.4}$$

Performing the sum over energies k_0,

$$m_{\rm el}{}^2=\frac{g^2N_f}{4T}\int\frac{1}{\cosh^2(E_f/2T)}\frac{d^2\mathbf{k}}{(2\pi)^2} +g^2N\int\left[\frac{E^2+m^2}{E^3[\exp(E/T)-1]} +\frac{m^2}{4TE^2\sinh(E/2T)}\right]\frac{d^2\mathbf{k}}{(2\pi)^2} , \tag{4.5}$$

$E_f=(\mathbf{k}^2+m_f{}^2)^{1/2}$, $E=(\mathbf{k}^2+m^2)^{1/2}$. From Eq. (4.5), to leading order the contribution of both fermions and gluons to $m_{\rm el}{}^2$ is a finite and positive quantity for all temperatures. At low temperatures, $T\ll|m_f|$ and $T\ll m$, $m_{\rm el}{}^2$ is exponentially small:

$$m_{\rm el}{}^2\underset{T\to 0}{\sim}\frac{g^2N_f}{2\pi}|m_f|\exp(-|m_f|/T) +\frac{g^2N}{2\pi}m\exp(-m/T)+\cdots . \tag{4.6}$$

The exponential suppression with temperature occurs because all massive particles are essentially nonrelativistic at low T, and so their fluctuations—which give $m_{\rm el}{}^2$—are down by corresponding Boltzmann factors. At high temperature,

$$m_{\rm el}{}^2\underset{T\to\infty}{\sim}\frac{g^2N_f}{2\pi}T\ln 2+\frac{g^2N}{2\pi}T\ln\left[\frac{T}{m}\right]+\cdots . \tag{4.7}$$

Note the factor of $\ln(T/m)$ which appears for gluons but not for fermions. This happens because only bosons can have zero energy at $T\neq 0$: the bosonic part of Eq. (4.4) with $k_0=0$ contains terms which are $\sim\int d^2\mathbf{k}/(k^2+m^2)\sim\ln(T/m)$. The gluon part of Eq. (4.7) agrees with results of d'Hoker.[12]

Up to this point, there are no surprises. Let me now take Eq. (4.2), and compute the renormalized gluon propagator for $p_0=0$, $\mathbf{p}\neq 0$. In the Landau gauge,

$$\Delta_{00}^{\rm ren}=\left[Z_{\rm el}\mathbf{p}^2+m_{\rm el}{}^2+\frac{Z_m{}^2}{Z_{\rm mag}}m^2\right]^{-1} ,\quad \Delta_{0i}^{\rm ren}=-m\epsilon_{ij}\frac{\mathbf{p}^j}{\mathbf{p}^2}\left[\frac{Z_m}{Z_{\rm mag}}\Delta_{00}^{\rm ren}\right] ,\quad \Delta_{ij}^{\rm ren}=\left[\delta_{ij}-\frac{\mathbf{p}^i\mathbf{p}^j}{\mathbf{p}^2}\right]\frac{Z_{\rm el}\mathbf{p}^2+m_{\rm el}{}^2}{Z_{\rm mag}\mathbf{p}^2}\Delta_{00}^{\rm ren} , \tag{4.8}$$

where

$$Z_{\rm el}=1+\Pi'_{\rm el} ,\quad Z_{\rm mag}=1+\Pi_{\rm mag} ,\quad Z_m=1+\Pi_o . \tag{4.9}$$

$Z_{\rm el}$ and $Z_{\rm mag}$ represent wave-function renormalizations of electric and magnetic fields, while Z_m is a mass renormalization constant for the Chern-Simons mass m. Consider Eq. (4.8) about zero spatial momentum. For clarity, I set $Z=1$; I also assume that $m_{\rm el}{}^2\ll m^2$, which is necessary for the loop expansion to be valid. Then as $\mathbf{p}\to 0$,

$$\Delta_{00}^{\rm ren}\sim\frac{1}{m^2} ,\quad \Delta_{0i}^{\rm ren}\sim-\frac{1}{m}\epsilon_{ij}\frac{\mathbf{p}^j}{\mathbf{p}^2} ,\quad \Delta_{ij}^{\rm ren}\sim\left[\delta_{ij}-\frac{\mathbf{p}^i\mathbf{p}^j}{\mathbf{p}^2}\right]\frac{m_{\rm el}{}^2}{m^2}\frac{1}{\mathbf{p}^2} . \tag{4.10}$$

Since I have set $Z=1$, and neglected $m_{\rm el}{}^2$ relative to m^2, one would expect that $\Delta_{\mu\nu}^{\rm ren}$ should be equal to the bare propagator. This is true for $\Delta_{00}^{\rm ren}$ and $\Delta_{0i}^{\rm ren}$, but not for $\Delta_{ij}^{\rm ren}$—instead of a factor of $1/(\mathbf{p}^2+m^2)\sim 1/m^2$, there is a massless pole, with a residue $=m_{\rm el}{}^2/m^2$.

Equation (4.10) shows that the static correlations of A_i behave discontinuously with temperature. At $T=0$, they are exponentially screened over distances $>m^{-1}$, but for *any* $T\neq 0$, they develop long-ranged correlations which are logarithmic in space. The limit $T\to 0$ is still well defined, since the coefficient of the logarithm is $\sim m_{\rm el}{}^2$, and this vanishes exponentially fast as T does, Eq. (4.6).

In a non-Abelian gauge theory neither the A_i nor the field strengths are gauge invariant, so it is not clear what to make of this. The same phenomenon, however, will happen in an Abelian theory with a Chern-Simons mass term. Correlations of A_i are not gauge invariant in an Abelian theory, but those of the magnetic field, $B=\epsilon^{ij}\partial_i A_j$, are and analogous conclusions hold for the correlations of B. Physically, I do not know why Debye screening and the Chern-Simons mass interact in this way. It is certainly counterintuitive to find that the Debye effect—which acts, as always, to help screen electric fields—completely obliterates the screening of static magnetic fields.

This effect only happens in the static limit. In the nonstatic case, $p_0 \neq 0$, Eq. (4.1) implies[10]

$$\Pi_{ij} \sim \overline{m}_{\rm el}^2 \delta_{ij}, \quad \Pi_{0j} \sim -\overline{m}_{\rm el}^2 \mathbf{p}^j / p_0 \; ; \tag{4.11}$$

the other parts to $\Pi_{\mu\nu}$ all vanish as $\mathbf{p} \to 0$, and represent contributions to the renormalization of wave function and mass. $\overline{m}_{\rm el}^2$, which is a function of p_0, is similar to $m_{\rm el}^2$, in that it does not vanish at $\mathbf{p}=0$. In the same spirit as Eq. (4.10), if the renormalized propagator is calculated using just the self-energy of Eq. (4.11), one finds that the renormalized propagator has the same behavior for $\mathbf{p} \to 0$ as the bare one with no singular dependence on $\overline{m}_{\rm el}^2$.

This discussion has overlooked one minor point. The bare propagators for the gluon and ghost have massless poles $\sim 1/k^2$; for the gluon, they are just in the P-odd part. For any temperature or external momentum, whenever virtual gluons or ghosts carry zero energy in a graph, these massless poles will produce terms, $\sim \int d^2\mathbf{k}/\mathbf{k}^2$, that are logarithmically divergent in the infrared. Inserting an infrared cutoff μ into the integrals over $\mathbf{k}$ turns these divergences into terms $\sim \ln(\mu)$. In the gluon polarization tensor $\Pi_{\mu\nu}$, to leading order these logarithms do not appear in $m_{\rm el}^2$ or $\overline{m}_{\rm el}^2$, but they do show up in the factors for wave function and mass renormalization [such as the Z's of Eq. (4.9)]. While μ must be introduced in order to calculate, any dependence on μ should cancel in physical quantites. In some simple cases, I have shown this explicitly.

V. THE FREE ENERGY AT $T \neq 0$

At the tree level for an SU(N) gauge theory, the gluons contribute $(N^2-1)F$ to the free energy, where

$$\beta F = \tfrac{1}{2} {\rm tr}_+ \ln \Delta^{-1}{}_{\mu\nu}(p) - {\rm tr}_+ \ln(\overline{p}^{\,2}) \; . \tag{5.1}$$

$\Delta^{-1}{}_{\mu\nu}$ is the inverse gluon propagator:

$$\Delta^{-1}{}_{\mu\nu}(p) = \delta_{\mu\nu}\overline{p}^{\,2} + \left[\frac{1}{\xi} - 1 \right] p_\mu p_\nu + m \epsilon_{\mu\nu\lambda} p^\lambda \; ; \tag{5.2}$$

ξ is the gauge-fixing parameter for the covariant gauge. This second term in Eq. (5.1), $\sim {\rm tr}_+ \ln(\overline{p}^{\,2})$, is the contribution of ghosts. F is also the correct free energy for an Abelian theory with a Chern-Simons term. While the ghosts decouple in an Abelian theory, Bernard[13] has shown that their contribution must still be included in the total free energy.

The free energy is independent of ξ: $\partial(\beta F)/\partial\xi \sim {\rm tr}_+ p^\mu p^\nu \Delta_{\mu\nu}(p) \sim {\rm tr}_+ 1 \equiv 0$, so I can choose the Feynman gauge, $\xi = 1$. Then

$$\begin{aligned} {\rm tr}_+ \ln \Delta^{-1}{}_{\mu\nu}(p) &= {\rm tr}_+ \ln(\overline{p}^{\,2} \delta_{\mu\nu}) \\ &\quad + {\rm tr}_+ \ln \left[\delta_{\mu\nu} + m \epsilon_{\mu\nu\lambda} \frac{p^\lambda}{\overline{p}^{\,2}} \right] . \end{aligned} \tag{5.3}$$

Performing the trace over the vector indices μ and ν,

$${\rm tr}_+ \ln \Delta^{-1}{}_{\mu\nu}(p) = 3\, {\rm tr}_+ \ln(\overline{p}^{\,2}) + {\rm tr}_+ \ln \left[\frac{\overline{p}^{\,2} + m^2}{\overline{p}^{\,2}} \right] . \tag{5.4}$$

In sum,

$$\beta F = \tfrac{1}{2} {\rm tr}_+ \ln(\overline{p}^{\,2} + m^2) \; , \tag{5.5}$$

which is the free energy for a massive bosonic particle with a single degree of freedom. This was not apparent from Eq. (5.1), but it does agree with the (physical) degrees of freedom found on the mass shell[2,3]—a single massive mode per color index.

Fermions in the fundamental representation contribute $-N_f N \, {\rm tr}_- \ln(\overline{p}^{\,2} + m_f^2)$ to the free energy/T.

For reference, at low temperatures

$${\rm tr}_\pm \ln(\overline{p}^{\,2} + m^2) \underset{T \ll m}{\sim} \mp \frac{mTV}{\pi} \exp(-m/T) + \cdots \tag{5.6}$$

and at high temperature

$${\rm tr}_\pm \ln(\overline{p}^{\,2} + m^2) \underset{T \gg m}{\sim} -\frac{1 \pm 7}{8} T^2 V \epsilon(3) \; . \tag{5.7}$$

$\epsilon(3) = \sum_{n=1}^{\infty} 1/n^3$, and V is the volume of space.

It is much harder to compute perturbative corrections to the free energy. One interesting question is whether it is possible to see any sign of fractional statistics. For distances $\gg m^{-1}$, identical particles interacting with topologically massive gauge fields can be said to exhibit fractional statistics.[3] The parameter which determines how fractional the statistics are is $\delta = g^2/m$. This fractional statistics only occurs over large distances, so at best it could only be important at low temperatures, when the average interparticle separation is large.

Arovas, Schrieffer, Wilczek, and Zee[14] have considered identical particles with fractional statistics of zero size. i.e., point particles. By computing the contribution of two-body correlations in the low-density limit, they showed that the fractional statistics affected the free energy of fermions and bosons very differently. In terms of the fractional parameter δ, the corrections to the free energy about the boson limit are $\sim |\delta|$, while about the fermion limit, they are $\sim \delta^2$.

Two-body correlations will affect the free energy in the present theory at $O(g^2)$. Unfortunately, there does not appear to be any sign of fractional statistics for the free energy computed to $O(g^2)$ at low temperatures. While I have not carried out this computation, matter fields—be they boson or fermion—and gluons will all contribute to the free energy at $O(g^2)$. As long as the matter fields are massive, at low temperatures there is no dramatic difference between how boson, as opposed to fermion, matter fields contribute to the free energy. Indeed, I do not know how any thermodynamic quantity could simply distinguish between the effects of fractional statistics from those of "ordinary" interactions.

It is possible to calculate the behavior of the free energy at high temperature without much trouble. Assume that

$$1 \ll \frac{T}{m} \ll \frac{m}{g^2} \frac{1}{\ln(T/m)} \; ; \tag{5.8}$$

the upper bound on T is necessary so that $m_{\rm el}{}^2$, which is a quantum effect, does not overwhelm the bare m^2. This upper bound can always be satisfied in weak coupling, when $m/g^2 \gg 1$.

To $O(g^2)$, one contribution to the free energy is from an insertion of the gluon polarization tensor $\Pi_{\mu\nu}(\mathbf{p},p_0)$ in a gluon loop.[15] For the term with $p_0=0$ involving Π_{00},

$$F' = -\frac{N^2-1}{2} T \int \Pi_{00}(\mathbf{p},0) \frac{1}{\mathbf{p}^2+m^2} \frac{d^2\mathbf{p}}{(2\pi)^2} = -\frac{N^2-1}{4\pi} T \ln\left[\frac{T}{m}\right] m_{\rm el}{}^2 + \cdots . \quad (5.9)$$

Using Eq. (4.7),

$$F' = -\frac{N(N^2-1)}{8\pi^2} g^2 T^2 \ln^2\left[\frac{T}{m}\right] + \cdots . \quad (5.10)$$

I assert that for high T, F' is the leading correction to the terms $\sim T^3$ in the free energy, Eq. (5.7): relative to F', all other terms will be suppressed at least by $1/\ln(T/M)$ or g^2/T.

Equation (5.9) also gives a term $\sim N_f g^2 T^2 \ln(T/m)$. This is small compared to Eq. (5.10), but in an Abelian theory, this term will dominate at high T. (This assumes there are no scalars. Scalars contribute $\sim g^2T^2\ln^2 T$ in the Abelian and non-Abelian cases.)

Generically, to $O(g^2)$ the purely gluonic contributions to the free energy are of the form[15]

$$F_2 \sim T \sum_{p_0} \int \Delta_{\mu\nu}(\mathbf{p},p_0) \Pi^{\mu\nu}(\mathbf{p},p_0) d^2\mathbf{p} . \quad (5.11)$$

Logarithms $\sim \ln(T/m)$ come from terms $\sim \int d^2\mathbf{k}/(\mathbf{k}^2+m^2)$, so we can concentrate on the term with $p_0=0$ in Eq. (5.11). Similarly, for $\Pi_{\mu\nu}$ the largest contributions in the infrared are from diagrams where the internal energy is zero. For example, by the Ward identity at $p_0=0$, $\Pi_{ij}(\mathbf{p},0)$ is transverse in $\mathbf{p}$, Eq. (4.2). To one-loop order,

$$\Pi_{ij}(\mathbf{p},0) \sim (\delta_{ij}\mathbf{p}^2 - p_i p_j) g^2 T \int \frac{d^2\mathbf{k}}{\mathbf{k}^2(\mathbf{k}+\mathbf{p})^2} \sim g^2 T \ln(\mu) \left[\delta_{ij} - \frac{\mathbf{p}_i\mathbf{p}_j}{\mathbf{p}^2}\right] . \quad (5.12)$$

The estimate of Eq. (5.12) is only qualitative; in the Landau gauge Π_{ij} will not be as singular as this. μ is the infrared cutoff noticed in Sec. IV.

Substituting Eq. (5.12) into Eq. (5.11) gives a term $\sim g^2T^2\ln(T/m)\ln(\mu)$. As was argued before, however, the dependence on μ is an artifact that has to disappear in any physical quantity. Since the free energy is such a quantity, the $\ln(\mu)$'s must cancel, leaving terms $\sim g^2T^2\ln(T/m)$; these are down by $1/\ln(T/m)$ relative to Eq. (5.9). While I have not checked this cancellation explicitly, there is no sleight of hand involved here—μ is just an infrared cutoff, with no relation to T or m, and so a $\ln(\mu)$ cannot become a $\ln(T/m)$.

Corrections to higher order in g^2 are manifestly suppressed by g^2/T, and are negligible. The only higher order that might cause concern is the sum of "necklace" diagrams $F_{\rm necklace}$, since this involves the summation of an infinite number of diagrams.[15] Concentrating on the zero energy term,

$$F_{\rm necklace} \sim T \int \left[\ln\left[1+\frac{1}{\mathbf{p}^2+m^2} m_{\rm el}{}^2\right] - \frac{1}{\mathbf{p}^2+m^2} m_{\rm el}{}^2\right] d^2\mathbf{p} \sim \frac{m^2 m_{\rm el}{}^2}{T} \sim g^2 m^2 \ln(T/m) , \quad (5.13)$$

and $F_{\rm necklace}$ can also be ignored.

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