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On toroidal compactification of heterotic superstrings

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I comment on the use of background gauge fields and antisymmetric tensor fields in toroidal compactification of heterotic strings, and explicitly construct the background gauge field that interpolates between the $E_8 \times E_8$ and $\mathrm{Spin}(32)/Z_2$ theories defined on $R^9 \times S^1$.

I. INTRODUCTION

In Ref. 1 it was shown that even Lorentzian self-dual lattices with signature $(16+d,d)$ can be used to give a $(16+d)d$-parameter construction of one-loop modular-invariant heterotic string theories with $10-d$ uncompactified dimensions. It was subsequently noted[2] that these theories could be obtained from a standard toroidal compactification of d dimensions of the 10-dimensional heterotic string by varying a metric g_{ij} specifying the radii and angles of the spacetime torus $[\frac{1}{2}d(d+1)$ parameters], and by turning on constant vacuum expectation values for a background antisymmetric tensor field $B_{\mu\nu}$ $[\frac{1}{2}d(d-1)$ parameters] and a background gauge field A^I_μ (the remaining $16d$ parameters). Using the uniqueness of Lorentzian self-dual lattices, it then followed immediately that one could continuously interpolate between compactified versions of the $E_8 \times E_8$ and $\mathrm{Spin}(32)/Z_2$ theories by turning on appropriate background gauge fields and adjusting radii. Since there has been some confusion on this point, and since some of these ideas may prove useful in illuminating relations between various other recent constructions of string theories,[3,4] we shall undertake here the relatively straightforward exercise of making explicit this interpolation.

II. COMPACTIFIED THEORIES WITH BACKGROUND FIELDS

The light-cone field content of the heterotic string consists of the scalar fields x^μ $(\mu=1,\dots,8)$, the right-moving Neveu-Schwarz-Ramond (NSR) fermions λ^μ, and 16 *complex* left-moving gauge fermions ψ^I. The Euclidean-space world-sheet action, including the coupling to background gauge field A^I_μ and background antisymmetric tensor field $B_{\mu\nu}$, is

$$S=\frac{1}{2\pi}\int(\partial x^\mu\bar\partial x^\mu+\lambda^\mu\partial\lambda^\mu+\bar\psi^I\bar\partial\psi^I +i\bar\psi^I A^I_\mu\bar\partial x^\mu\psi^I+B_{\mu\nu}\partial x^\mu\bar\partial x^\nu) \quad (2.1)$$

$[\partial=\partial_z=(i\bar\tau\partial_1-i\partial_2)/\tau_2$, with $z=(\sigma_1+\tau\sigma_2)/2$; $\sigma_1,\sigma_2 \in[0,1]$, $\tau=\tau_1+i\tau_2$ is the one-loop modular parameter, $2\alpha'=1$, and the integration measure is $2i\,dz\wedge d\bar z =\tau_2 d\sigma_1\wedge d\sigma_2]$.

We consider closed-string propagation on a spacetime with topology $R^{10-d}\times(S^1)^d$. The d-dimensional torus $T^d=R^d/2\pi\Lambda$ is generated by modding out d of the spacetime dimensions by a lattice Λ, i.e. identifying the points x^μ and $x^\mu+2\pi n^i(e_i)^\mu$, where the e_i, $i=1,\dots,d$, are a set of basis vectors for the d-dimensional lattice Λ and the n^i are integers. The instanton solutions are given by $x_0=2\pi w\sigma_1+2\pi w'\sigma_2$, with the winding numbers $w,w'\in\Lambda$ $(w=n^ie_i,\ w'=n'^ie_i)$. Defining a metric $g_{ij}=e_i\cdot e_j$ and its inverse $g^{ij}=g_{ij}^{-1}$, we see that the vectors $e^{*i}\equiv g^{ij}e_j$ satisfy $e^{*i}\cdot e_j=\delta^i_j$ (and $e^{*i}\cdot e^{*j}=g^{ij}$), thus generating the dual lattice Λ^*. Momenta p, conjugate to translations, are constrained to lie on the dual lattice $p\in\Lambda^*$, so we have $p=m_ie^{*i}$ for integer m_i. Since we are interested in vacuum solutions of the theory we shall take the background gauge field and antisymmetric tensor fields to be constant. We write their components referred to lattice frames as $A^I_\mu=a^I_i(e^{*i})_\mu$ and $B_{\mu\nu} =b_{ij}(e^{*i})_\mu(e^{*j})_\nu$. In terms of these, the invariant line and surface integrals are given by $\int_i A^I_\mu dx^\mu=2\pi A^I_\mu e^\mu_i =2\pi a^I_i$ and $\int_{ij}B_{\mu\nu}dx^\mu\wedge dx^\nu=4\pi^2 b_{ij}$.

It is easily shown (see Ref. 2 and Appendix A here) that the partition function for the theory in the presence of background A and B fields is proportional to

$$\sum_{\substack{w\in\Lambda,p\in\Lambda^*\\ V\in\Lambda_{\rm int}}} q^{p_L{}^2/2}\bar q^{p_R{}^2/2}\ , \quad (2.2)$$

where $q=e^{2\pi i\tau}$, and $\Lambda_{\rm int}$ is the internal Euclidean self-dual lattice, either the $\mathrm{Spin}(32)/Z_2$ lattice Γ_{16} or the $E_8\times E_8$ lattice $\Gamma_8\oplus\Gamma_8$. The vectors in the exponents

$$p_L=(V+Aw,\tfrac12 p-Bw-\tfrac12 V^KA^K-\tfrac14 A^K(A^Kw)+w)\ ,$$

and (2.3)

$$p_R=(\tfrac12 p-Bw-\tfrac12 V^KA^K-\tfrac14 A^K(A^Kw)-w)$$

are, respectively, $(16+d)$- and d-vectors. Note that if we take the direct sum of p_L and p_R, a $(16+2d)$-vector, as a lattice point in a space with a Lorentzian $(16+d,d)$ signature, then the $(p_L;p_R)$ are elements of an even Lorentzian self-dual lattice. This follows immediately for the case $A=B=0$, because then $(p_L;p_R)=(V,\frac12 p+w;\frac12 p-w)$ lies in a lattice with basis vectors $k^i=(0,\frac12 e^{*i};\frac12 e^{*i})$,

$\bar{k}_i=(0,e_i;-e_i)$, and $l_\alpha=(f_\alpha,0;0)$ [where $(f_\alpha)^I$ are the basis vectors for $\Lambda_{\rm int}$]. The k^i and $\bar{k}_j$ satisfy $k^i\bar{k}_j=\delta^i_j$ and $k^ik^j=\bar{k}_i\bar{k}_j=0$, thus generating a lattice equivalent to d copies of the 2-dimensional even self-dual lattice U with metric $\binom{0\ 1}{1\ 0}$ [and signature (1,1)]. The basis elements also satisfy $k^il_\alpha=\bar{k}_jl_\alpha=0$ and $l_\alpha l_\beta=g^{(\rm int)}_{\alpha\beta}$, so the lattice generated by $(p_L;p_R)$ is equivalent to $U\oplus\cdots\oplus U\oplus\Lambda_{\rm int}$, which is Lorentzian even self-dual as long as $\Lambda_{\rm int}$ is Euclidean even self-dual.

Thus for $A=B=0$, the sum over $w\in\Lambda$, $p\in\Lambda^*$ may be replaced by an equivalent sum over elements $(p_L;p_R)$ of a Lorentzian self-dual lattice. This is the statement that generalizes to the presence of nonzero background A and B fields, since from (2.3) it follows that $(p_L;p_R)$ more generally lie in a lattice with basis vectors

$$k^i=(0,\tfrac{1}{2}e^{*i};\tfrac{1}{2}e^{*i})\ ,$$

$$\bar{k}_i=(a_i^I,-b_{ji}e^{*j}-\tfrac{1}{4}a_j^Ka_i^Ke^{*j}+e_i\ ;\ -b_{ji}e^{*j}-\tfrac{1}{4}a_j^Ka_i^Ke^{*j}-e_i)\ ,$$

and

$$l_\alpha=(f^I_\alpha,-\tfrac{1}{2}f^K_\alpha a_i^Ke^{*i};-\tfrac{1}{2}f^K_\alpha a_i^Ke^{*i})\ .$$

These are easily verified to satisfy the same dot products as for $A=B=0$, and hence continue to generate an even self-dual Lorentzian lattice. Equivalently, (2.3) can be written as a transformation that takes the $(16+d;d)$ column vector $(p_L;p_R)=(V,\frac{1}{2}p+w;\frac{1}{2}p-w)$ to its value for nonzero A,B. The transformation is

$$\begin{bmatrix}\delta^I_J & \tfrac{1}{2}A^I_\nu & -\tfrac{1}{2}A^I_{\nu'}\\ -\tfrac{1}{2}A^J_\mu & \delta_{\mu\nu}-\tfrac{1}{2}B_{\mu\nu}-\tfrac{1}{8}A^K_\mu A^K_\nu & \tfrac{1}{2}B_{\mu\nu'}+\tfrac{1}{8}A^K_\mu A^K_{\nu'}\\ -\tfrac{1}{2}A^J_{\mu'} & -\tfrac{1}{2}B_{\mu'\nu}-\tfrac{1}{8}A^K_{\mu'}A^K_\nu & \delta_{\mu'\nu'}+\tfrac{1}{2}B_{\mu'\nu'}+\tfrac{1}{8}A^K_{\mu'}A^K_{\nu'}\end{bmatrix}=\exp\left[\frac{1}{2}\begin{bmatrix}0 & A^I_\nu & -A^I_{\nu'}\\ -A^J_\mu & -B_{\mu\nu} & B_{\mu\nu'}\\ -A^J_{\mu'} & -B_{\mu'\nu} & B_{\mu'\nu'}\end{bmatrix}\right] \tag{2.4}$$

(where $I,J=1,\ldots,16$ and $\mu,\nu,\mu',\nu'=1,\ldots,d$) and is manifestly an element of $SO(16+d,d)$. [In general $SO(n,m)$ is generated by infinitesimal $(n+m)\times(n+m)$ matrices $\binom{N\ \ P}{P^T\ M}$ with $N=-N^T$, $M=-M^T$, and P an arbitrary $n\times m$ matrix. The matrix in the exponent in (2.4) is a matrix whose cube vanishes.] This shows that the Lorentzian lattices for arbitrary A,B are simply $SO(16+d,d)$ transforms of one another.

Finally, changes in the radii and angles of tori can be parametrized as $e_i\to\exp(-\alpha)e_i$ where $\alpha_{\mu\nu}$ is a symmetric $d\times d$ matrix [$\frac{1}{2}d(d+1)$ parameters]. Acting on $(p_L;p_R)$ this can be expressed as the $SO(16+d,d)$ matrix transformation

$$\begin{bmatrix}\delta^I_J & 0 & 0\\ 0 & \cosh\alpha_{\mu\nu} & \sinh\alpha_{\mu\nu'}\\ 0 & \sinh\alpha_{\mu'\nu} & \cosh\alpha_{\mu'\nu'}\end{bmatrix}=\exp\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \alpha_{\mu\nu'}\\ 0 & \alpha_{\mu'\nu} & 0\end{bmatrix}\ .$$

Composing (2.4) with (2.5) then gives a $(16+d)d$-parameter family of theories with varying radii, angles, A fields, and B fields.

To understand the significance of the remaining parameters in $SO(16+d,d)$, we exhibit the mass operator for the theory

$$\tfrac{1}{4}(\text{mass})^2=(N_L+\tfrac{1}{2}{p_L}^2-1)+(N_R+\tfrac{1}{2}{p_R}^2-c) \tag{2.6a}$$

and the constraint

$$N_L+\tfrac{1}{2}{p_L}^2-1=N_R+\tfrac{1}{2}{p_R}^2-c\ . \tag{2.6b}$$

Here $N_{L,R}$ are the overall occupation numbers for the left-moving bosonic and right-moving bosonic and fermionic oscillators, and c is a normal-ordering constant equal to $\frac{1}{2}$ and 0, respectively, for the antiperiodic (NS) and periodic (R) sectors of the NSR fermions λ^μ. We see that the spectrum of the theory is invariant under $SO(16+d)\times SO(d)$ rotations of $(p_L;p_R)$ (as are the interactions as well, provided we use vertex operators evaluated at rotated momenta). The distinct physical theories are thus parametrized[1] by the $(16+d)d$-dimensional coset space $SO(16+d,d)/(SO(16)\times SO(d))$.

III. EXAMPLES

Before coming to our main point, we illustrate the formalism with two examples mentioned in Ref. 7. For the first, we consider the root lattice Λ_R of some simply laced Lie algebra G of rank d, generated by simple roots $\tilde{e}_i$, normalized to length squared equal to 2. The dual vectors $\tilde{e}^{*i}$ satisfy $\tilde{e}^{*i}e_j=\delta^i_j$ and generate the weight lattice Λ_W. We take the spacetime lattice $\Lambda=\Lambda_R/2$, generated by half root vectors, so that $e_i=\frac{1}{2}\tilde{e}_i$ and winding numbers $w\in\Lambda$ have $w=n^i\tilde{e}_i/2$. Then $e^{*i}=2\tilde{e}^{*i}$ and momenta $p\in\Lambda^*=2\Lambda_W$ satisfy $\frac{1}{2}p=m_i\tilde{e}^{*i}$. Note that $\tilde{g}_{ij}=\tilde{e}_i\cdot\tilde{e}_j$ is the Cartan matrix for G. If we now take

$$2b_{ij}=\begin{cases}\tfrac{1}{2}\tilde{g}_{ij}, & i<j\ ,\\ -\tfrac{1}{2}\tilde{g}_{ij}, & i>j\ ,\\ 0, & i=j\ ,\end{cases} \tag{3.1}$$

then we find the Lorentzian (d,d) lattice vectors $(\frac{1}{2}p-Bw+w;\frac{1}{2}p-Bw-w)$ are generated by $k^i=(\tilde{e}^{*i};\tilde{e}^{*i})$, and

$$\bar{k}_i=(-2b_{ji}\tilde{e}^{*j}+\tfrac{1}{2}\tilde{e}_i;-2b_{ji}\tilde{e}^{*j}-\tfrac{1}{2}\tilde{e}_i)$$
$$=(c_{ij}\tilde{e}^{*j}+\tilde{e}_i;c_{ij}\tilde{e}^{*j})\ ,$$

where $c_{ij}\equiv 2b_{ij}-\frac{1}{2}\tilde{g}_{ij}$ is a matrix of integers. k^i and $\bar{k}_i$ are seen to satisfy $k^i\bar{k}_j=\delta^i_j$, $k^ik^j=\bar{k}_i\bar{k}_j=0$, generating an even self-dual Lorentzian lattice. Inspection of k^i and $\bar{k}_i$ shows that this lattice is composed of vectors $(\tilde{p}_L;\tilde{p}_R)$ with $\tilde{p}_L,\tilde{p}_R \in \Lambda_W$ constrained such that $\tilde{p}_L - \tilde{p}_R \in \Lambda_R$. There are thus additional massless gauge bosons, states with $p_L{}^2=2$, $p_R{}^2=0$, corresponding to the adjoint representation of the group G. We also note that the antisymmetric tensor field $B_{\mu\nu}$ appears in general as a phase in the functional integral and thus constitutes a set of periodic variables (θ angles). Here we see that the theory is invariant under $2b_{ij} \rightarrow 2b_{ij}+m_{ij}$ where m_{ij} is an antisymmetric matrix of integers [since $2b_{ij}$ is a matrix of half-integers, for b_{ij} as in (3.1) this means the theory is also invariant under $b_{ij} \rightarrow -b_{ij}$ (Ref. 5)].

For the second example, we consider the Spin(32)/Z_2 theory with d dimensions compactified on orthogonal S^1's of radius $\frac{1}{2}$. For this we take $(e_i)^\mu = \frac{1}{2}\delta_i^\mu, (e^{*i})_\mu = 2\delta_\mu^i$, giving a spacetime lattice generated by half the weight lattice of SO($2d$), $\Lambda = \frac{1}{2}\Lambda_W(\mathrm{SO}(2d))$. If we take in addition $b_{ij}=0$ and $a_i^I=\delta_i^I$ (so $A_\mu^I = 2\delta_\mu^I$), then we find that our Lorentzian lattice is generated by $l_\alpha = (f_\alpha^I, -f_\alpha^\mu; -f_\alpha^{\mu'})$, $k^i = (0,\delta^{i\mu};\delta^{i\mu'})$, and $\bar{k}_i = (\delta_i^I, 0; -\delta_i^\mu)$. This is a lattice equivalently given by adding to the lattice generated by $(p_L;p_R)$ with $p_L \in \Lambda_R(\mathrm{SO}(32+2d))$, $p_R \in \Lambda_R(\mathrm{SO}(2d))$, the additional basis vectors $(W_s;\tilde{W}_s)$ and $(W_v;\tilde{W}_v)$, where $W_{s,v}, \tilde{W}_{s,v}$ are, respectively, the spinor and vector weights of SO($32+2d$) and SO($2d$). The additional massless gauge bosons, with $p_L{}^2=2$, $p_R{}^2=0$, thus fill out the adjoint of SO($32+2d$).

IV. $E_8 \times E_8$ AND Spin(32)/Z_2

We now discuss the relation between the $E_8 \times E_8$ and Spin(32)/Z_2 theories. For Euclidean signature, the statement is that there exist a finite number of distinct even integer self-dual lattices in dimensions $8k$ (Ref. 6). In dimension 8 there is only the E_8 lattice Γ_8, while in dimension 16 we have both $\Gamma_8 \oplus \Gamma_8$ and the Spin(32)/Z_2 lattice Γ_{16}. If we append to either of these two 16-dimensional Euclidean lattices the even two-dimensional Lorentzian lattice U, we get an even Lorentzian self-dual lattice $\mathrm{II}_{17,1}$ of signature (17,1). But for Lorentzian signature, the statement is that there exist even integer self-dual lattices $\mathrm{II}_{8k+d,d}$ in any dimension $8k+2d$ [with signature $(8k+d,d)$] and they are unique up to SO($8k+d,d$) transformations. This means that there must be some SO(17,1) transformation that relates the basis vectors of $\Gamma_8 \oplus \Gamma_8 \oplus U$ to those of $\Gamma_{16} \oplus U$. Our aim is to exhibit the

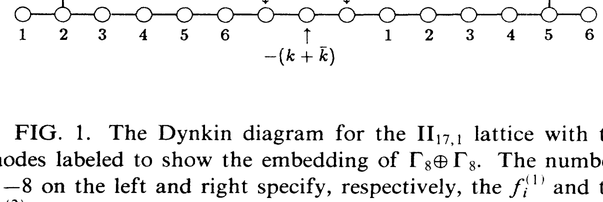

FIG. 1. The Dynkin diagram for the $\mathrm{II}_{17,1}$ lattice with the nodes labeled to show the embedding of $\Gamma_8 \oplus \Gamma_8$. The numbers 1–8 on the left and right specify, respectively, the $f_i^{(1)}$ and the $f_i^{(2)}$ of (4.1a).

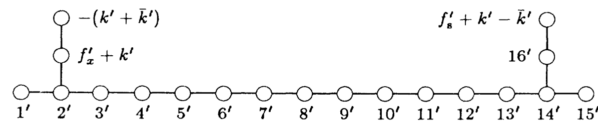

FIG. 2. The Dynkin diagram for the $\mathrm{II}_{17,1}$ lattice with the nodes labeled to show the embedding of Γ_{16}. The numbers $1'-16'$ specify the f'_i of (4.2a).

SO(17,1) transformation that effects this isomorphism, parametrized in terms of a gauge field A (in compactifying only one dimension, there is no B field).

First we need bases for the embedding of $\Gamma_8 \oplus \Gamma_8$ and Γ_{16} in $\mathrm{II}_{17,1}$ [as found, for example, in Ref. 7 (see also Ref. 8)]. Let u_i $(i=1,\ldots,17,0)$ denote an orthonormal basis in $R^{17,1}$ with $u_i \cdot u_j = \eta_{ij} = \mathrm{diag}(1,\ldots,1,-1)$. Then we choose basis vectors for the two Γ_8 lattices embedded in R^{16} as follows:

$$
\begin{aligned}
f_i^{(1)} &= u_i - u_{i+1} \quad (i=1,\ldots,6)\,, \\
f_7^{(1)} &= -u_1 - u_2, \quad f_8^{(1)} \tfrac{1}{2} \sum_{i=1}^{8} u_i\,, \\
f_i^{(2)} &= u_{9+i} - u_{10+i} \quad (i=1,\ldots,6)\,, \\
f_7^{(2)} &= u_{15} + u_{16}, \quad f_8^{(2)} = -\tfrac{1}{2} \sum_{i=9}^{16} u_i\,.
\end{aligned}
\tag{4.1a}
$$

We shall consider the $E_8 \times E_8$ theory compactified on a circle of radius $\frac{1}{2}r$ (circumference πr) so that the spacetime lattice Λ is generated by $e=r/2$ and the dual lattice Λ^* by $e^*=2/r$. Then we define $k=(0^{16},\frac{1}{2}e^*;\frac{1}{2}e^*) = (u_{17}+u_0)/r$ and $\bar{k}=(0,e;-e)=(u_{17}-u_0)r/2$, satisfying $k^2=\bar{k}^2=0$ and $k\bar{k}=1$. To complete the basis for $\mathrm{II}_{17,1}$, we shall also need the vectors $f_x^{(1)}=u_7-u_8$ and $f_x^{(2)}=u_9-u_{10}$, each of which is a linear combination of the basis vectors already appearing in (4.1a), corresponding to the extra root (the lowest root) added to produce an extended E_8 Dynkin diagram. The 16 vectors in (4.1a) together with

$$
f_x^{(1)}+k, \quad f_x^{(2)}+k, \quad \text{and} \quad -(k+\bar{k})\,,
\tag{4.1b}
$$

then provide a Dynkin basis for a $\mathrm{II}_{17,1}$ lattice. In Fig. 1 we give the Dynkin diagram for $\mathrm{II}_{17,1}$ with the roots labeled according to (4.1a) and (4.1b).

To embed the Γ_{16} lattice in $\mathrm{II}_{17,1}$ we first write the basis vectors

$$
\begin{aligned}
f'_i &= u_i - u_{i+1} \quad (i=1,\ldots,15)\,, \\
f'_{16} &= u_{15} + u_{16}\,,
\end{aligned}
\tag{4.2a}
$$

that generate the SO(32) root lattice in R^{16}. Adding as well the spinor weight $f'_s = -\frac{1}{2}\sum_{i=1}^{16} u_i$ would then generate Γ_{16}. The vector corresponding to the extra root in the extended Dynkin diagram for SO(32) in this basis is $f'_x = -u_1 - u_2$. We shall take the Spin(32)/Z_2 theory compactified on a circle of radius r, so that the spacetime lattice Λ is generated by $e'=r$ and the dual lattice Λ^* by

$e'^{*}=1/r$. Then we define $k'=(0^{16},-e;e)=(-u_{17}+u_0)r$ and $\bar{k}'=(0^{16},-\frac{1}{2}e'^{*};-\frac{1}{2}e'^{*})=-(u_{16}+u_0)/2r$ (to agree ultimately with the conventions of Ref. 7), satisfying again $k'^2=\bar{k}'^2=0$ and $k'\bar{k}'=1$. The 16 vectors in (4.2a) together with

$$f_x'+k', \quad -(k'+\bar{k}'), \quad \text{and } f_s'+k'-\bar{k}' \tag{4.2b}$$

then provide a Dynkin basis for another $\mathrm{II}_{17,1}$ lattice, as labeled in Fig. 2. We wish to find the SO(17,1) transformation that relates these two bases.

To begin we shall employ the gauge fields[9]

$$A'^{I}=\frac{2}{r}(0^7,-1,1,0^7), \quad A^{I}=-2r((\tfrac{1}{2})^8,0^8) . \tag{4.3}$$

We denote by

$$W(A)=\exp\left[\frac{1}{2}\begin{bmatrix}0 & A^I & -A^I\\ -A^J & 0 & 0\\ -A^J & 0 & 0\end{bmatrix}\right]=\begin{bmatrix}\delta^i_J & \frac{1}{2}A^I & -\frac{1}{2}A^I\\ -\frac{1}{2}A^J & 1-\frac{1}{8}A^KA^K & \frac{1}{8}A^KA^K\\ -\frac{1}{2}A^J & -\frac{1}{8}A^KA^K & 1+\frac{1}{8}A^KA^K\end{bmatrix}$$

the SO(17,1) transformation associated with turning on the gauge field A^I, and by $R:u_0\to -u_0$ a reflection of the 18th axis. Acting on the $\Gamma_8\oplus\Gamma_8$ basis vectors (4.1a) and (4.1b) first with $W(A')$ and then with $RW(A)R\in SO(17,1)$, we find

$$\begin{aligned}
&f_8^{(1)}=((\tfrac{1}{2})^8,0^9;0)\xrightarrow{W(A')}\left[(\tfrac{1}{2})^8,0^8,\frac{1}{2r};\frac{1}{2r}\right]\xrightarrow{RW(A)R}\left[0^{16},r+\frac{1}{2r};-r+\frac{1}{2r}\right]=-(\bar{k}'+k') ,\\
&f_7^{(1)}=(-1,-1,0^{15};0)\to(-1,-1,0^{15};0)\to(-1,-1,0^{14},-r;r)=f_x'+k' ,\\
&f_x^{(1)}+k=\left[0^6,1,-1,0^8,\frac{1}{r};\frac{1}{r}\right]\to(0^6,1,-1,0^9;0)\to(0^6,1-1,0^9;0)=f_7' ,\\
&-(k+\bar{k})=\left[0^{16},-\frac{1}{r}-\frac{r}{2};-\frac{1}{r}+\frac{r}{2}\right]\to\left[0^7,1,-1,0^7,-\frac{r}{2};\frac{r}{2}\right]\to(0^7,1,-1,0^8;0)=f_8' ,\\
&f_x^{(2)}+k=\left[0^8,1,-1,0^6,\frac{1}{r};\frac{1}{r}\right]\to(0^8,1,-1,0^7;0)\to(0^8,1-1,0^7;0)=f_9' ,\\
&f_8^{(2)}=(0^8,(-\tfrac{1}{2})^8,0;0)\to\left[0^8,(-\tfrac{1}{2})^8,\frac{1}{2r};\frac{1}{2r}\right]\to\left[(-\tfrac{1}{2})^{16},-r+\frac{1}{2r};r+\frac{1}{2r}\right]=f_s'+k'-\bar{k}' .
\end{aligned}\tag{4.4}$$

The remaining basis vectors in (4.1a) are mapped unchanged by the two transformations onto the basis vectors of (4.2a) at the corresponding nodes of the Dynkin diagram. Since the basis vectors generate the entire lattice, we see that the transformation $RW(A)RW(A')$ provides an isomorphism between $\Gamma_8\oplus\Gamma_8\oplus U$ and $\Gamma_{16}\oplus U$.

To write this transformation in terms of a single background gauge field A'', we decompose $RW(A)R\in SO(17,1)$ into a product of an SO(17) rotation S, dilation D, and gauge field transformation W:

$$RW(A)R=\exp\left[\frac{1}{2}\begin{bmatrix}0 & A^I & A^I\\ -A^J & 0 & 0\\ A^J & 0 & 0\end{bmatrix}\right]$$

$$=S(A)D(\alpha)W(\tilde{A}) , \tag{4.5}$$

where

$$S(A)=\exp\left[\xi\begin{bmatrix}0 & A^I & 0\\ -A^J & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right] ,$$

$$\xi=\frac{2}{|A|}\arctan\frac{|A|}{2} \quad (|A|^2=A^IA^I) ,$$

$$D(\alpha)=\exp\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \alpha\\ 0 & \alpha & 0\end{bmatrix}, \quad \alpha=\ln\left[1+\frac{|A|^2}{4}\right] ,$$

and $\tilde{A}=-e^{-\alpha}=-A/(1+|A|^2/4)$ (verifying (4.5) involves noting that

$$\begin{aligned}S&=\exp(M)\\ &=1+M\sin(\xi|A|)/\xi|A|\\ &\quad+M^2[1-\cos(\xi|A|)]/(\xi^2|A|^2) ,\end{aligned}$$

since $M^3=-\xi^2|A|^2M$). When $d=1$, we have $W(A_1)W(A_2)=W(A_1+A_2)$ so for the case of interest it follows that

$$RW(A)RW(A')=S(A)D(\alpha)W\left[-\frac{1}{1+2r^2}A+A'\right]. \tag{4.6}$$

Equation (4.6) shows that we can start from the $E_8\times E_8$ theory with one dimension compactified to a circle of radius $\frac{1}{2}r$, turn on a background gauge field, and decrease the internal size by a factor $e^{-\alpha}=(1+|A|^2/4)^{-1}=(1+2r^2)^{-1}$ to give a theory with the same spectrum and symmetries as the Spin(32)/Z_2 theory compactified on a circle of radius r. It comes out in an obscure SO(17) rotated basis which may then be restored to the canonical form (4.2a) and (4.2b) by the SO(17) transformation $S(A)$. Starting from the other direction with the Spin(32)/Z_2 theory at radius r, on the other hand, after the SO(17) transformation one would have to *increase* the internal size by a factor $e^{\alpha}=1+2r^2=1+r^2/\alpha'$ (restoring the string scale α') and then turn on the opposite gauge field to result in a theory equivalent to the $E_8\times E_8$ theory at radius $\frac{1}{2}r$.

V. COMMENTS

From a mathematical point of view, compactified versions of the $E_8\times E_8$ and Spin(32)/Z_2 theories, insofar as they are continuously related, may thus be regarded as different ground states of the same theory. From a physical point of view, however, although the SO(17) transformation does not alter the spectrum of the theory, it does affect the spacetime interpretation, as is evidenced by the change in radius. The $\mathrm{II}_{17,1}$ lattices in arbitrary intermediate SO(17) transformed bases provide theories which are perfectly sensible as two-dimensional conformal field theories, but which do not necessarily admit interpretations in terms of Riemann surfaces propagating in physical spacetimes. We therefore cannot see, for example, a continuous interpolation of the natural notion of length (of the compactified dimension) between the two situations. From the point of view of the spectrum alone, then, any compactified heterotic string theory with spacetime supersymmetry may be regarded as either a compactified $E_8\times E_8$ or Spin(32)/Z_2 theory with appropriate A,B fields and radii, but a physical observer would choose one or the other as the natural interpretation.

The theories we have considered here are supersymmetric theories with presumably vanishing vacuum energy density, so there is no energetic criterion for selecting any one over another. For the theories without spacetime supersymmetry discussed in Ref. 3, the dependence of the one-loop vacuum energy density can be investigated as a function of the background-field parameters.[10,11] It is further shown in Ref. 11 that compactified versions of the (heterotic-based) theories formulated in Ref. 3 are all connected in the sense described in this note.

Finally, we note that the spectra of both the compactified $E_8\times E_8$ and Spin(32)/Z_2 theories are invariant under the transformation $R: u_0\to -u_0$ used in the above. It is a duality transformation,[12,10] which, for background gauge field $A=0$, interchanges $e\leftrightarrow\frac{1}{2}e^*(e^*=1/e)$, exchanging the roles of winding numbers and momenta. Each of the theories at radius r thus has an identical spectrum at radius α'/r. The self-dual point under this transformation is $r^2=\alpha'$.

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APPENDIX

For completeness we establish Eq. (2.2) used in the text, reproducing the results of Ref. 2. Higher-loop generalizations of the (one-loop) formulas that follow may be found in Ref. 11. We start with the world-sheet action (2.1), rewritten as

$$S=\frac{1}{2\pi}\int d^2\sigma\,\tau_2\left[(\partial_1 x^\mu)^2+\frac{1}{\tau_2{}^2}[(\partial_2-\tau_1\partial_1)x^\mu]^2+\frac{2i}{\tau_2}B_{\mu\nu}\partial_1 x^\mu\partial_2 x^\nu+\bar\psi^I\partial_z\psi^I+i\bar\psi^I A^I_\mu\partial_z x^\mu\psi^I\right]. \tag{A1}$$

We shall sum over the contributions from the different instanton sectors, given by the zero-mode solutions $x_0=2\pi(w\sigma_1+w'\sigma_2)$ $(w,w'\in\Lambda$, the spacetime lattice). In each such sector, the coupling to the constant background gauge field is equivalent to twisted boundary conditions for free fermions

$$\begin{aligned}\psi^I(\sigma_1+1,\sigma_2)&=-(-1)^\alpha e^{2\pi i\theta^I}\psi^I(\sigma_1,\sigma_2),\quad \theta^I=A^I\cdot w,\\ \psi^I(\sigma_1,\sigma_2+1)&=-(-1)^\beta e^{-2\pi i\phi^I}\psi^I(\sigma_1,\sigma_2),\quad \phi^I=-A^I\cdot w',\end{aligned} \tag{A2}$$

where $\alpha,\beta=0,1$ $(=-,+=A,P)$ specifies the spin structure. The path integral is thus proportional to

$$\sum_{w,w'\in\Lambda}e^{-S_0}=\sum_{w,w'\in\Lambda}\exp\left[-2\pi\left[\tau_2 w^2+\frac{1}{\tau_2}(w'-\tau_1 w)^2\right]-4\pi i w'Bw\right]\sum_{\{\alpha,\beta=0,1\}}\prod_I\det{}_{\alpha\beta}(\theta^I,\phi^I), \tag{A3}$$

where the final sum is over the product of fermion determinants partitioned into different spin structures, as will be made explicit shortly.

The determinant for a single complex fermion with boundary conditions twisted by θ,ϕ as in (A2) can be evaluated either by considering a regularized product of eigenvalues or by considering the partition function (as explained, for example, in Ref. 13) for a left-moving fermion with Hamiltonian H and world-sheet fermion number F (satisfying $e^{2\pi i\phi F}\psi e^{-2\pi i\phi F}=e^{-2\pi i\phi}\psi$). For a single complex fermion twisted with respect to the AA spin structure ($\alpha=\beta=0$), for example, we find (up to an overall phase chosen for convenience)

$$\begin{aligned}\det{}_{00}(\theta,\phi)&=\prod_{n,m=-\infty}^{\infty}\frac{2\pi}{\tau_2}[(n+\tfrac{1}{2}+\theta)\tau-(m+\tfrac{1}{2}-\phi)]=\operatorname{tr}e^{2\pi i\phi F}q^{H}\\&=e^{i\pi\theta\phi}q^{\theta^2/2-1/24}\prod_{n=1}^{\infty}(1+q^{n+\theta-1/2}e^{2\pi i\phi})(1+q^{n-\theta-1/2}e^{-2\pi i\phi})\\&=\frac{\sum_{n=-\infty}^{\infty}q^{(n+\theta)^2/2}e^{2\pi i(n+\theta/2)\phi}}{\eta(\tau)}\equiv\frac{\vartheta_{00}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau)}{\eta(\tau)}\ ,\end{aligned}\tag{A4}$$

where $q=e^{2\pi i\tau}$, $\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind η function, and $\theta^2/2-\frac{1}{24}$ is the vacuum normal-ordering constant. [As pointed out in Ref. 13 and as we shall see shortly, (A4) embodies the statement of bosonization at one loop.] The result for other spin structures may be inferred, again up to a phase, by letting $\theta\to\theta+\frac{1}{2},\phi\to\phi+\frac{1}{2}$. We find

$$\det{}_{\alpha\beta}(\theta,\phi)=\frac{\vartheta_{\alpha\beta}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau)}{\eta(\tau)}\ ,\tag{A5}$$

with

$$\vartheta_{\alpha\beta}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau)\equiv e^{i\pi\tau\theta^2+i\pi\theta\phi}\vartheta_{\alpha\beta}(\tau\theta+\phi,\tau)=\sum_{n=-\infty}^{\infty}q^{(n+\alpha/2+\theta)^2/2}\exp[2\pi i(n+\tfrac{1}{2}\alpha+\tfrac{1}{2}\theta)\phi+i\pi(n+\tfrac{1}{2}\alpha)\beta]\ ,\tag{A6}$$

$\alpha,\beta=0,1$, and the $\vartheta_{\alpha\beta}(z,\tau)$ are as defined in Ref. 14. [In the notation of Ref. 15, we have $\vartheta_3(z,\tau)=\vartheta_{00}(z,\tau)$, $\vartheta_4=\vartheta_{01},\vartheta_2=\vartheta_{10}$, and $\vartheta_1=\vartheta_{11}$.] We have chosen the phase in (A6) so that under the modular transformations $\tau\to\tau+1$ and $\tau\to-1/\tau$, these functions conveniently transform as

$$\begin{aligned}\vartheta_{00}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau+1)&=\vartheta_{01}\begin{bmatrix}\theta\\ \theta+\phi\end{bmatrix}(\tau),\quad \vartheta_{00}\begin{bmatrix}\theta\\ \phi\end{bmatrix}\left[-\frac{1}{\tau}\right]=(-i\tau)^{1/2}\vartheta_{00}\begin{bmatrix}\phi\\ -\theta\end{bmatrix}(\tau)\ ,\\ \vartheta_{01}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau+1)&=\vartheta_{00}\begin{bmatrix}\theta\\ \theta+\phi\end{bmatrix}(\tau),\quad \vartheta_{01}\begin{bmatrix}\theta\\ \phi\end{bmatrix}\left[-\frac{1}{\tau}\right]=(-i\tau)^{1/2}\vartheta_{10}\begin{bmatrix}\phi\\ -\theta\end{bmatrix}(\tau)\ ,\\ \vartheta_{10}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau+1)&=e^{i\pi/4}\vartheta_{10}\begin{bmatrix}\theta\\ \theta+\phi\end{bmatrix}(\tau),\quad \vartheta_{10}\begin{bmatrix}\theta\\ \phi\end{bmatrix}\left[-\frac{1}{\tau}\right]=(-i\tau)^{1/2}\vartheta_{01}\begin{bmatrix}\phi\\ -\theta\end{bmatrix}(\tau)\ ,\\ \vartheta_{11}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau+1)&=e^{i\pi/4}\vartheta_{11}\begin{bmatrix}\theta\\ \theta+\phi\end{bmatrix}(\tau),\quad \vartheta_{11}\begin{bmatrix}\theta\\ \phi\end{bmatrix}\left[-\frac{1}{\tau}\right]=-i(-i\tau)^{1/2}\vartheta_{11}\begin{bmatrix}\phi\\ -\theta\end{bmatrix}(\tau)\ .\end{aligned}\tag{A7}$$

They also satisfy $\vartheta_{00}^4[{\theta \atop \phi}](\tau)-\vartheta_{01}^4[{\theta \atop \phi}](\tau)-\vartheta_{10}^4[{\theta \atop \phi}](\tau)+\vartheta_{11}^4[{\theta \atop \phi}](\tau)=0$ for all θ,ϕ [following from relation (R5) on p. 20 of Ref. 14].

To construct a modular-invariant partition function for a string theory, we need to choose an appropriate sum over the spin structures of the fermions. Here we consider the combination

$$\prod_{I=1}^{8k}\vartheta_{00}\begin{bmatrix}\theta^I\\ \phi^I\end{bmatrix}(\tau)+\prod_{I=1}^{8k}\vartheta_{01}\begin{bmatrix}\theta^I\\ \phi^I\end{bmatrix}(\tau)+\prod_{I=1}^{8k}\vartheta_{10}\begin{bmatrix}\theta^I\\ \phi^I\end{bmatrix}(\tau)+\prod_{I=1}^{8k}\vartheta_{11}\begin{bmatrix}\theta^I\\ \phi^I\end{bmatrix}(\tau)\ ,\tag{A8}$$

which is invariant, as required, under the transformations $\tau\to\tau+1$, $(\theta^I,\phi^I)\to(\theta^I,\phi^I-\theta^I)$, and $\tau\to-1/\tau$, $(\theta^I,\phi^I)\to(-\phi^I,\theta^I)$, for arbitrary integer k. The relevant cases will be $k=1,2$, corresponding, respectively, to the bosonic lattices Γ_8 and Γ_{16}. Substituting (A6) into (A8) gives

$$\prod_{I=1}^{8k}\sum_{n^I\in Z} q^{(n^I+\theta^I)^2/2}e^{2\pi i(n^I+\theta^I/2)\phi^I}+\prod_{I=1}^{8k}\sum_{n^I\in Z} q^{(n^I+\theta^I)^2/2}e^{2\pi i(n^I+\theta^I/2)\phi^I}(-1)^{n^I}$$

$$+\prod_{I=1}^{8k}\sum_{n^I\in Z} q^{(n^I+1/2+\theta^I)^2/2}e^{2\pi i(n^I+1/2+\theta^I/2)\phi^I}+\prod_{I=1}^{8k}\sum_{n^I\in Z} q^{(n^I+1/2+\theta^I)^2/2}e^{2\pi i(n^I+1/2+\theta^I/2)\phi^I}(-1)^{n^I+1/2}$$

$$=\sum_{\{n^I\in Z\}} q^{\sum_I(n^I+\theta^I)^2/2}\exp\left[2\pi i\sum_I(n^I+\tfrac{1}{2}\theta^I)\phi^I\right][1+(-1)^{\sum_I n^I}]$$

$$+\sum_{\{n^I\in Z\}} q^{\frac{1}{2}\sum_I(n^I+\frac{1}{2}+\theta^I)^2}\exp\left[2\pi i\sum_I(n^I+\tfrac{1}{2}+\tfrac{1}{2}\theta^I)\phi^I\right][1+(-1)^{\sum_I(n^I+1/2)}]$$

$$=2\sum_{V\in\Gamma_{8k}} q^{(V+A\cdot w)^2/2}e^{-2\pi i(V^I+A^I\cdot w/2)A^I\cdot w'}\ . \tag{A9}$$

The lattice Γ_{8k} in the above is by definition[6] the even self-dual $8k$-dimensional lattice consisting of vectors V whose components V^I are either all integer, $V^I\in Z$, constrained such that $\sum_I V^I$ is even, or all half-integer, $V^I\in Z+\frac{1}{2}$, again such that $\sum_I V^I$ is even. Splitting the complex fermions into two groups of eight and summing independently over the spin structures corresponds to summing over $V\in\Lambda_{\rm int}=\Gamma_8\oplus\Gamma_8$, and summing over the same spin structure for all 16 complex fermions corresponds to summing over $V\in\Lambda_{\rm int}=\Gamma_{16}$.

Inserting the result (A9) for the determinants in (A3), we see that the w' dependence is contained in the function $f(w')=\exp[-(2\pi/\tau_2)(w'-\tau_1 w)^2-2\pi i w'k]$, where

$$k=2Bw+V^IA^I+\tfrac{1}{2}A^I(A^Iw)\ .$$

Fourier transforming gives $\tilde f(p)=\int_x \exp(2\pi i x\cdot p)f(x)\sim\exp[2\pi i\tau_1 w(p-k)-\frac{1}{2}\pi\tau_2(p-k)^2]$. Using the Poisson resummation formula to reexpress the sum of $f(w')$ over $w'\in\Lambda$ by a sum of $\tilde f(p)$ over $p\in\Lambda^*$, and including the additional pieces from (A9) and (A3), gives a partition function proportional to

$$\sum_{\substack{w\in\Lambda,p\in\Lambda^*\\ V\in\Lambda_{\rm int}}}\exp[-2\pi\tau_2w^2+2\pi i\tau_1w(p-k)-\tfrac{1}{2}\pi\tau_2(p-k)^2]q^{(V+Aw)^2/2}$$

$$=\sum_{\substack{w\in\Lambda,p\in\Lambda^*\\ V\in\Lambda_{\rm int}}} q^{[(p-k)/2+w]^2/2}\bar q^{[(p-k)/2-w]^2/2}q^{(V+Aw)^2/2}\ , \tag{A10}$$

thus verifying (2.2) and (2.3).

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