Nonperturbative method in field theory: The gauge technique

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We present a derivation of the gauge-technique ansatz, which emphasizes the approximations made in regard to the truncation of the Dyson-Schwinger equations. The nonperturbative nature of the method is elucidated. Finally we extend the technique to nongauge settings. This is possible because for fermionic theories which are not necessarily gauge theories Ward identities exist in the infrared limit. To illustrate these features a simple model is solved in the infrared region.

INTRODUCTION

Field theories, while steadfastly held to be the way of describing the real world, are notoriously difficult to fathom outside of perturbation theory. The content of a field theory lies in the Dyson-Schwinger equations, which, unfortunately, are an infinite set of coupled integral equations making exact solutions impossible, except in the most special of cases. To obtain information then, a resort is usually made to perturbation theory—taking the coupling to be small; this leads to some success for electroweak theory and for quantum chromodynamics in the asymptotic region. An alternative to perturbation theory is to obtain nonperturbative results by truncating the Dyson-Schwinger equations in a meaningful way. It is the purpose of this paper to discuss such a method, to give a motivation for the truncation and a discussion of the various approximations made. The method is known as the gauge technique,[1-3] though, as will become clear in the following discussion, it is applicable beyond the original setting of gauge theories where it was first introduced.

The original idea of the gauge technique (in the form given by Delbourgo and West[2,3]) was simply to give a nonperturbative solution to the Green-Takahashi identity for the vertex in terms of the propagators for the charged particles. The gauge identity does not specify the transverse (to the photon momentum) part of the vertex, so this was left undetermined. Subsequent work on introducing transverse vertex corrections may be found in Refs. 4 and 5. Having said this, it is important to note that in the infrared limit the original Ward identity (note that throughout we will be making a distinction between the Green-Takahashi and Ward identities) holds and the complete vertex may be expressed in terms of the particle propagator. Hence in this limit the gauge technique must at least yield reliable results, as indeed it does.

Other areas where the technique has been applied successfully include two-dimensional models[6] where the vertex function may be specified completely and in three-dimensional quantum electrodynamics where the leading-logarithmic behavior of the electron propagator is determined nonperturbatively.[7]

What has not been explained is in what sense the gauge technique is a method for truncating the Dyson-Schwinger equations (and hence, how one goes about improving on it). In the next section we will explain both the perturbative and nonperturbative content of the approximations made and outline how to go beyond the first gauge approximation. This is in the context of gauge theories. The following section is devoted to a discussion of nongauge theories and of why the technique may be implementable there as well. As an example we solve a simple model of a spinor field interacting with a scalar field through a Yukawa coupling, and obtain the following infrared behavior:

$$S(p)\sim(\not p-m)^{-1+g^2/4\pi^2} \quad \text{as } \not p\to m$$

for the fermion propagator. Finally we end up with some conclusions and an outlook.

GAUGE TECHNIQUE

The original method may be found in Refs. 2 and 3; here we present a variant based on truncating the Dyson-Schwinger equations so as to highlight the approximations being made. (We deal with electrodynamics for ease of presentation, but a similar analysis may be carried out for other gauge theories.) The Dyson-Schwinger equation for the three-point photon amputated Green's function G_μ may be written in two different ways:[4]

$$G_\mu(p',p)(\not p-m_0)=S(p')\gamma_\mu-ie^2\int d^4k\,G_{\mu\nu}(p',p-k;p'-p,k)\gamma_\rho D^{\nu\rho}(k)\ , \tag{1}$$

$$(\not p'-m_0)G_\mu(p',p)=\gamma_\mu S(p)-ie^2\int d^4k\,\gamma_\rho G_{\mu\nu}(p'+k,p;p'-p,k)D^{\nu\rho}(k)\ . \tag{2}$$

The first gauge approximation is to neglect the integrals in both (1) and (2), so that in using the vertex we are correct perturbatively in all the momentum region to order e^0. One may combine Eqs. (1) and (2) to give

$$G_\mu(p',p)=[\not p'F_\mu(p',p)+F_\mu(p',p)\not p]/(p'^2-p^2)\ , \tag{3}$$

where $F_\mu(p',p)=\gamma_\mu S(p)-S(p')\gamma_\mu$. This choice for G_μ is dictated by the fact that we wish the Green-Takahashi identity to be satisfied. [There are many correct choices to ~ 1, e.g., $S(p')\gamma_\mu/(\not p - m)$ will do. However, if one wants to go beyond perturbation theory $S(p')\gamma_\mu/(\not p - m)$ does not suffice as it violates charge-conjugation invariance. Indeed one is forced into the choice made in (3).] Perturbatively correct to ~ 1 at all momenta it is in fact correct to all orders in the coupling for $p' \to p$ in that in this limit is satisfies the Ward identity

$$G_\mu(p,p) = -\frac{\partial S(p)}{\partial p^\mu} ,$$

(which is what the Green-Takahashi identity reduces to). Now the spectral representation for the fermion propagator

$$S(p) = \int d\omega \frac{\rho(\omega)}{\not p - \omega + i\epsilon(\omega)0^+} \tag{4}$$

can be employed to express (3) as

$$G_\mu(p',p) = \int d\omega\, \rho(\omega) \frac{1}{\not p' - \omega} \gamma_\mu \frac{1}{\not p - \omega} . \tag{5}$$

This is nothing more than the ansatz of Delbourgo and West,[2] and is a possible solution of the Green-Takahashi identity for the vertex. The importance of our derivation is that it exhibits clearly in what way one actually approximates the Dyson-Schwinger equations while self-consistently satisfying the Ward and Green-Takahashi identities.

The next step is to solve for the spectral function by going to the Dyson-Schwinger equation for the electron propagator:

$$Z_\psi{}^{-1} = S(p)(\not p - m_0) - ie^2 \int d^4k\, G_\mu(p,p-k)\gamma_\nu D^{\mu\nu}(k) . \tag{6}$$

For a given $D^{\mu\nu}(k)$ one obtains a linear equation in $\rho(\omega)$ which sometimes makes it possible to obtain an analytic solution for the electron propagator. For example, when the photon propagator is taken to be bare, an analytic solution is available and one obtains the correct infrared limit for $S(p)$ (Refs. 3, 4, and 8). It is interesting to note that as this equation is identical to the one for the quark propagator in QCD, so that there as well it represents a linear equation (in the axial gauges, so to avoid complications due to ghosts), one may now choose one's favorite behavior for the gluon propagator and *check* if a solution exists.

The procedure outlined above works well for the charged part of the theory, but fails for the self-energy of non-Abelian vector bosons.[9] The reason for this failure is that in the infrared limit the part of the vertex determined by the Ward identity, in fact, does not contribute to the self-energy. This spells doom not only for the gauge technique but also for related methods when applied to this sector.[10]

Special care must also be taken for theories which have higher point interactions at the bare level or interactions not specific to the gauge invariance. This situation is exemplified in scalar electrodynamics in four dimensions.[11] There it is found that *no* $\lambda\phi^4$ counterterm is required to render finite the renormalized particle propagator. At the same time, the four-point vertex $A_\mu{}^2\phi^2$ is neglected. An analysis was carried out to determine whether this had an effect (at next order in perturbation theory) on the infrared behavior; it turns out that in this limit the four-point vertex is safely ignored.[4,5] Clearly, each model must be treated on its own merits.

It is now time to face up to the question of how to systematically improve the approximation. The idea is to truncate the next Dyson-Schwinger equation, that is, the one for the four-point vertex $G_{\mu\nu}$ (with both photon legs amputated) so that the vertex thus obtained (now in terms of G_μ) once more satisfies the Ward-Takahashi identity. To see how this will feed into G_μ consider once more Eqs. (1) and (2); they may be combined to give

$$G_\mu(p',p) = \int d\omega\, \rho(\omega) \frac{1}{\not p' - \omega} \gamma_\mu \frac{1}{\not p - \omega} + e^2 \frac{\not p' H_\mu(p',p) + H_\mu(p',p)\not p}{p'^2 - p^2} , \tag{7}$$

where

$$H_\mu(p',p) = -i \int d^4k\, D^{\nu\rho}(k) \times [\gamma_\rho G_{\mu\nu}(p'+k,p;p'-p,k) - G_{\mu\nu}(p',p-k;p'-p,k)\gamma_\rho] . \tag{8}$$

The Dyson-Schwinger equations for $G_{\mu\nu}$ are

$$(\not p' - m_0)G_{\mu\nu}(p',p;l,p'-p-l) + \gamma_\mu G_\nu(p'-l,p;p'-p-l) + \gamma_\nu G_\mu(p+l,p;l) + O(e^2) = 0 , \tag{9}$$

$$G_{\mu\nu}(p',p;l,p'-p-l)(\not p - m_0) + G_\mu(p',p'-l;l)\gamma_\nu + G_\nu(p',p+l;p'-p-l)\gamma_\mu + O(e^2) = 0 , \tag{10}$$

where $O(e^2)$ represents higher-order Green's-function contributions. As with G_μ, we may now solve for $G_{\mu\nu}$ to give

$$G_{\mu\nu}(p',p;l,p'-p-l) = [\not p' M_{\mu\nu}(p',p,l) + M_{\mu\nu}(p',p,l)\not p]/(p'^2 - p^2) + O(e^2) . \tag{11}$$

with

$$M_{\mu\nu}(p',p,l) = [G_\mu(p',p'-l;l)\gamma_\nu + G_\nu(p',p+l;p'-p-l)\gamma_\mu] - [\gamma_\mu G_\nu(p'-l,p;p'-p-l) + \gamma_\nu G_\mu(p+l,p;l)] . \tag{12}$$

The form of (11) is once more dictated by the Ward-Takahashi identity:

$$k^\mu G_{\mu\nu}(p',p;k,l) = G_\nu(p',p+k) - G_\nu(p'-k,p) .$$

In terms of the spectral density we find

$$G_{\mu\nu}(p',p;k,l) = \int d\omega\,\rho(\omega)\frac{1}{\not p'-\omega}\left[\gamma_\mu\frac{1}{\not p'-\not k-\omega}\gamma_\nu + \gamma_\nu\frac{1}{\not p+\not k-\omega}\gamma_\nu\right]\frac{1}{\not p-\omega} + O(e^2)\ . \tag{13}$$

This may be inserted into (8), so that G_μ is once more given as an integral linear in the spectral function. So that one finds, for G_μ,

$$G_\mu(p',p) = \int d\omega\,\rho(\omega) g_\mu(p',p\mid\omega)\ , \tag{14}$$

where

$$g_\mu(p',p\mid\omega) = \sum_{n=1}^{\infty}(e^2)^{n-1} g_\mu^{(n)}(p',p\mid\omega)\ . \tag{15}$$

$g_\mu^{(1)}$ comes from the original ansatz. $g_\mu^{(2)}$ now is determined via Eqs. (13) and (8); these have been previously determined by Delbourgo and Zhang.[4] Now it is clear that as the process of truncating the Dyson-Schwinger equations to higher and higher point functions (i.e., more photon leg insertions) one gets spectral weights over the *bare* Green's functions as seen for G_μ in Eq. (5) and for $G_{\mu\nu}$ in Eq. (13). Hence we see the general validity of Eq. (14). Further, one should note that the $g_\mu^{(n)}$ satisfy

$$\left.\begin{aligned}&(p'-p)^\mu g_\mu^{(n)}(p',p\mid\omega)=0\ ,\\ &\lim_{p'\to p} g_\mu^{(n)}(p',p\mid\omega)=0\ ,\end{aligned}\right\}\quad \forall n>1$$

as a consequence of the Green-Takahashi and Ward identities, respectively, and so the infrared behavior of the electron propagator is not altered beyond the first approximation. That result is exact.

For the photon propagator, one could in principle use the form for G_μ obtained by the gauge technique and feed this into its Dyson-Schwinger equation.

$$D^{-1}{}_{\mu\nu}(k) = Z_\psi D^{(0)-1}{}_{\mu\nu}(k) + ie^2 Z_\psi\int d^4p\,\mathrm{tr}[\gamma_\mu G_\nu(p+k,p)]$$

then couple this to the Dyson-Schwinger equation for the electron to form a self-consistent set of integral equations for $\rho(\omega)$. However, the resulting equation is now no longer linear in $\rho(\omega)$, so that it becomes extremely difficult to solve. Instead one treats the photon propagator perturbatively. This is not a bad approximation in quantum electrodynamics in four dimensions where the behavior of the photon propagator remains $(k^2)^{-1}$ to all orders of perturbation theory. On the other hand, in two and three dimensions some dressing of the photon propagator must be taken into account. As an alternative one may feed in for G_μ the explicit form for G_μ that was obtained in the previous truncation as corrections to $D^{-1}{}_{\mu\nu}$ come in as e^2G_μ and this yields correct perturbative results while at the same time giving rise to nonperturbative information. It is quite clear that in practice any of these programs will be very difficult to follow through, the point is that in principle it may be done.

NONGAUGE THEORIES

The gauge technique has already been applied in nongauge settings in two dimensions.[6] These models such as the Thirring model are cast into a "gauge" form. In this section we are interested in theories of fermion couplings that may not have a gauge equivalent form, for example, a $\bar\psi\phi\psi$ interaction term.

It is acceptable in order to get a handle on the Dyson-Schwinger equations to once more truncate at some order for this theory. The only problem with such a procedure is that it is perturbative in nature, as one now seems no longer able to choose the truncation so as to automatically obtain some nonperturbative information. By this we mean that (apparently) contrary to the preceding section one cannot say that in some momentum range, the truncation is indeed exact, or at the very least a good approximation. The reason for this is the lack of a guiding Ward identity. We have put "apparently" into the sentence above because indeed, in the infrared limit for fermion theories, there always is a Ward identity that one may reasonably expect to hold. Though this seems almost to be contradictory, it is at the basis of some very old ideas on the infrared problem. One appeals to the Bloch-Nordseich conjecture.[12]

The idea is the following. Fermions when emitting real and virtual quanta of the boson field suffer a recoil. Such a recoil will be very small if the energy (or mass) of the fermion is much larger than the boson energy. In this limit one may replace the Dirac matrices γ_μ by constant vectors (timelike) V_μ which represent an averaged fermion four-momentum. This is precisely the infrared limit. In this limit it is possible to derive a Ward identity. We now turn to an example field theory to display how this comes about in practice.

The model is given by the Lagrangian

$$\mathscr{L}(x) = \bar\psi(i\not\partial - m_0 - g_0\phi)\psi - \tfrac{1}{2}\phi(\partial^2+\mu_0{}^2)\phi\ . \tag{16}$$

In the infrared limit we may replace γ_μ with V_μ; then (16) is invariant under

$$\psi\to e^{-i\Lambda g_0}\psi\ ,\quad \phi\to V^\mu\partial_\mu\Lambda \tag{17}$$

except for the ϕ^2 term which acts like a conventional gauge-fixing term in this regard. Provided one takes $\gamma_\mu\equiv V_\mu$, then following the standard path the Green-Takahashi identity

$$V\cdot\partial\frac{\delta\Gamma}{\delta\phi} = V\cdot\partial(\partial^2+\mu^2)\phi + g\bar\psi\frac{\delta\Gamma}{\delta\bar\psi} - g\frac{\delta\Gamma}{\delta\psi}\psi \tag{18}$$

is derived, where Γ is the one-particle-irreducible vertex functional and the fields are treated classically. On reinstating γ_μ for V_μ we obtain for the inverse propagator and the vertex the corresponding identities in the infrared limit in momentum space:

$$\Delta^{-1}(k) = k^2-\mu^2 \tag{19}$$

and

$$\Gamma(p',p) = \frac{\partial}{\partial\not p}S^{-1}(p)\ . \tag{20}$$

Strictly speaking, identities derived from (18) are *only* true in the infrared limit and we take this as understood. Also as the boson is allowed to have a mass we must only consider the case where this is much smaller than the fermion mass, otherwise the "no-recoil" assumption is invalidated. Indeed, for simplicity we concentrate on the special choice of the renormalized mass for the boson vanishing (i.e., $\mu_0^2=\delta\mu^2$).

But now the full apparatus developed in the preceding section becomes applicable. That is, one truncates the Dyson-Schwinger equation for the vertex by neglecting the integral convolution and keeping only the propagator terms, cf. Eqs. (2) and (3); correspondingly one finds

$$G(p',p)=(S\Gamma S)(p',p)=\frac{\not p' F(p',p)+F(p',p)\not p}{p'^2-p^2}\ , \qquad (21)$$

with

$$F(p',p)=S(p)-S(p')\ ,$$

where this choice satisfies (20). (In this example, charge-conjugation invariance does not pose any constraints on the choice made for G. This is to be contrasted with the gauge theory case.) Once more (21) is easily rewritten as

$$G(p',p)=\int d\omega\,\rho(\omega)\frac{1}{\not p'-\omega}\frac{1}{\not p-\omega}\ , \qquad (22)$$

a spectral weighting of the Born diagram. Higher-order correction will follow directly from the analysis given for gauge theories. As before higher vertex functions (i.e., with more ϕ legs) will be spectrally weighted sums of the corresponding Born diagrams.

We now turn to the solution for the fermion propagator in the infrared limit. Begin with the Dyson-Schwinger equation for the propagator

$$Z_\psi{}^{-1}=S(p)(\not p-m_0)-ig^2\int d^4k\,G(p,p-k)\Delta(k)\ . \qquad (23)$$

Employ (22) and (19) in (23) to arrive at

$$\int d\omega\frac{\rho(\omega)}{\not p-\omega}[\omega-m_0+\Sigma(p,\omega)]=0\ , \qquad (24)$$

where $Z_\psi^{-1}=\int d\omega\,\rho(\omega)$ has been used and

$$\Sigma(p,\omega)=-ig^2\int d^4k\frac{1}{\not p-\not k-\omega}\frac{1}{k^2}\ .$$

On taking the imaginary part of (24) we arrive at the conventional gauge-technique equation:[3,8]

$$\epsilon(\omega)\rho(\omega)(\omega-m)=\frac{1}{\pi}\int d\omega'\frac{\rho(\omega')}{\omega-\omega'}\,\mathrm{Im}\Sigma(\omega,\omega')\ . \qquad (25)$$

The imaginary part of the self-energy is straightforwardly calculated:

$$\mathrm{Im}\Sigma(\omega,\omega')=\pi\left[\frac{g}{4\pi}\right]^2\frac{1}{2\omega^3}(\omega+\omega')^2 \times(\omega^2-\omega'^2)\theta(\omega^2-\omega'^2)\ . \qquad (26)$$

Equation (25) reduces to

$$\epsilon(\omega)\rho(\omega)(\omega-m)=\left[\frac{g}{4\pi}\right]^2\left[\int_{-|\omega|}^{-m}+\int_m^{|\omega|}\right] \times d\omega'\rho(\omega')\frac{(\omega+\omega')^3}{2\omega^3} \qquad (27)$$

which has the solution, as $\omega\to m$,

$$\rho(\omega)\simeq C(\omega-m)^{-1+g^2/4\pi^2}\ ,\quad \omega\to m\ . \qquad (28)$$

Recalling previous remarks we may say that (28) is exact in the infrared, this produces the asymptotic fermion propagator

$$S(p)\sim(\not p-m)^{-1+g^2/4\pi^2}\ . \qquad (29)$$

The behavior of the propagator mimics that found for the electron propagator in QED. There the exponent is gauge dependent and will for an appropriate choice of gauge coincide with the power in (29).

CONCLUSION

This paper has displayed how the gauge technique which has had considerable success, may be understood as a truncation of the Dyson-Schwinger equations. Further, its nonperturbative character has been elucidated. A well-prescribed method for improving the approximation has also been presented. The case of fermionic theories without gauge couplings has been discussed and we have shown how in such theories it is also possible to employ the gauge technique. Future work may be envisaged as the application of this method to more theories, gauge and nongauge. Also we have not mentioned in the text other momentum limits, for example, the high-momentum limit of the fermion propagator. The only information we have on this, at least for the case of electrodynamics, is that the gauge technique in both the infrared and ultraviolet limits satisfies the Zumino identities.[13,14] This is not enough in itself to imply that the high-energy behavior is correct but it does give some confidence.

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