Approaching the critical region of two-dimensional ϕ^4 quantum field theory with post-Gaussian approximations

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We investigate the vacuum state of $(1+1)$-dimensional ϕ^4 quantum field theory utilizing a modification of the powerful coupled cluster method by the additional maximum-overlap condition. This permits us to construct the ground state of that field theory for nearly all values of the coupling strength. Only a small region has to be excluded where our method still fails. This is most probably due to critical behavior showing up in a change of the order parameter of the model. Our procedure predicts a behavior of the $(\phi^4)_2$ model in complete agreement with some rigorous mathematical statements which is not possible in the case of a Gaussian approximation only. Perhaps somewhat unexpectedly, the symmetry-breaking Hamiltonian does not have any critical point.

I. INTRODUCTION

In a series of publications[1-6] the vacuum sector of the $(\phi^4)_2$ model was studied intensively in the past. Mathematicians could prove rigorously the existence of a unique phase transition of second order,[7-9] but they did not give an estimation where it should occur. On the other hand, physicists utilized the nonperturbative Gaussian approximation to make concrete predictions.[1-3] Unfortunately, all these predictions, as they are concerned with critical behavior, are totally wrong. Instead of a second-order transition only a first-order transition is obtained, which is in disagreement with the famous Simon-Griffith theorem.[10] Therefore, the Gaussian approximation is not capable of describing correctly the most interesting features of the $(\phi^4)_2$ model. On the other hand, that approximation becomes exact far away from the critical region.

In our first attempt[11] to overcome these shortcomings we decided to use a powerful method of many-body theory, which is known among experts as the coupled cluster method.[12,13] Although we had a lot of experience with its application to typical problems in many-body theory, it was never applied to quantum field theory. Because of this lack of experience the first numerical calculations were not as successful as expected. The complicated equations were solved iteratively and convergence could not always be observed. Therefore, our first predictions were not as reliable as necessary to make some precise statements about the critical point. In the meantime we have improved our concepts[14] and can now report on these superior techniques, which have a wider range of validity and seem to be precise enough to better localize the critical point, which itself is still not accessible without further modifications of the procedure.

This paper is arranged in five sections. In the next section we discuss properties of the $(\phi^4)_2$ model in detail. We emphasize the duality of models to relate states and energy eigenvalues of three different Hamiltonians to each other. Scaling identities follow naturally from these considerations. As a consequence, only a very small range of parameters has to be analyzed to extrapolate the behavior of the $(\phi^4)_2$ model to all other parameter values. There exists a small interval of the coupling parameter where duality is absent. Most probably, the critical point is contained in this interval which is strongly supported by our numerical data. The uniqueness of the critical point is automatically guaranteed. In the third section we make the reader familiar with a formalism corresponding to post-Gaussian approximations. The ordinary Gaussian approximation can be recovered as an extremely crude reduction of our most general procedure. We utilize an approximation which seems to be sufficient to describe all relevant properties of the $(\phi^4)_2$ model correctly. Only close to the critical point do expected problems appear. Numerical results concerning the energy density and the vacuum expectation value of the field operator are discussed in detail in the fourth section. An estimation of the critical point is derived. The exact duality relations and scaling identities can be recovered from the numerical data. In the fifth section we draw our conclusions and stress the necessity to make the coupled cluster method available in the critical region.

II. THE MODEL, DUALITY, AND SCALING IDENTITIES

The model has already been described several times. Therefore we restrict ourselves to those relevant aspects which are of utmost importance. At first, we define two Hamiltonians which will be called symmetric ($\tilde{H}$) and asymmetric (H):

$$H(m,\lambda,L)=N_{\sqrt{2}m}\int_{-L/2}^{L/2}dx\left[\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\nabla\phi)^2+\frac{\lambda}{4}\left[\phi^2-\frac{m^2}{\lambda}\right]^2\right], \tag{2.1}$$

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$$\tilde{H}(\mu,\lambda,L)=N_\mu \int_{-L/2}^{L/2} dx \left[\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 \right] . \tag{2.2}$$

These are pure names which have to be interpreted with great care. Utilizing the formulas of Ref. 3 it is easily verified that both Hamiltonians agree up to a constant. In this case we call the two models described by these Hamiltonians dual to each other. Here we obtain

$$H(m,\lambda,L)=\tilde{H}(\mu,\lambda,L)+L\left[\frac{\mu^2-2m^2}{8\pi}+\frac{3\lambda}{4}\Delta^2 - \frac{m^2\Delta}{2}+\frac{m^4}{4\lambda} \right] , \tag{2.3}$$

if

$$\mu^2=-m^2+3\lambda\Delta , \qquad \Delta=\frac{1}{4\pi}\ln\frac{2m^2}{\mu^2} . \tag{2.4}$$

Equations (2.4) have solutions only if λ/μ^2 exceeds a critical value $(\lambda/\mu^2)_{\text{crit}}=g_3^*=9.045\,978$. Therefore, given a coupling strength $g_3=\lambda/\mu^2$ larger than this lower bound g_3^*, a special symmetric Hamiltonian $\tilde{H}(\mu^2,\lambda,L)$ is dual to an asymmetric Hamiltonian H. But even more, it is dual to two asymmetric Hamiltonians, which will be denoted as $H(m_1,\lambda,L)$ and $H(m_2,\lambda,L)$. In the next step we scale all dimensionful quantities with powers of m_1, m_2, or μ to obtain dimensionless quantities. As before, $g_1=\lambda/{m_1}^2$ and $g_2=\lambda/{m_2}^2$ are adequate abbreviations we make use of several times. Pure dimensional considerations lead to ($i=1,2$)

$$H(m_i,\lambda,L)=m_i\mathscr{H}(g_i,\hat{L}_i) , \qquad \tilde{H}(\mu,\lambda,L)=\mu\tilde{\mathscr{H}}(g_3,\hat{L}_3) , \tag{2.5}$$

where $\mathscr{H}$ and $\tilde{\mathscr{H}}$ are dimensionless and need no further definition. $\hat{L}_i=m_iL$ and $\hat{L}_3=\mu L$ were introduced furthermore. Now we rewrite the duality relations in terms of these scaled operators and parameters. Equation (2.3) transforms into

$$\mathscr{H}(g_i,\hat{L}_i)=\left(\frac{g_i}{g_3}\right)^{1/2}\tilde{\mathscr{H}}(g_3,\hat{L}_3) +g_i\hat{L}_i\left[\frac{1}{8\pi}\left(\frac{1}{g_3}-\frac{2}{g_i}\right)+\tfrac{3}{4}{\Delta_i}^2 -\frac{1}{2}\frac{\Delta_i}{g_i}+\frac{1}{4{g_i}^2}\right] , \tag{2.6}$$

and Δ_i is a solution of the equation

$$e^{-4\pi\Delta_i}=-\tfrac{1}{2}+\tfrac{3}{2}g_i\Delta_i . \tag{2.7}$$

Furthermore, g_3 and g_i depend on each other via

$$\frac{1}{g_3}+\frac{1}{g_i}-3\Delta_i=0 . \tag{2.8}$$

Since $\mathscr{H}(g_1,\hat{L}_1)$ and $\mathscr{H}(g_2,\hat{L}_2)$ are both dual to the same Hamiltonian $\tilde{\mathscr{H}}(g_3,\hat{L}_3)$, they should be dual to themselves. This is expressed as an important self-duality condition for the Hamiltonian $\mathscr{H}(g,\hat{L})$:

$$\mathscr{H}(g_1,\hat{L}_1)=\left(\frac{g_1}{g_2}\right)^{1/2}\mathscr{H}(g_2,\hat{L}_2)+g_1\hat{L}_1w(g_1,g_2) , \tag{2.9}$$

where

$$w(g_1,g_2)=\frac{1}{4\pi}\left[\frac{1}{g_2}-\frac{1}{g_1}\right]+\tfrac{3}{4}({\Delta_1}^2-{\Delta_2}^2) -\frac{\Delta_1}{2g_1}+\frac{\Delta_2}{2g_2}+\frac{1}{4{g_1}^2}-\frac{1}{4{g_2}^2} , \tag{2.10}$$

and g_1 is related to g_2 by the equation

$$\frac{1}{g_2}-\frac{1}{g_1}+\frac{3}{4\pi}\ln\frac{g_2}{g_1}=0 . \tag{2.11}$$

Let us turn to a qualitative discussion of these exact identities. $\tilde{\mathscr{H}}(g_3,\hat{L}_3)$ is dual to $\mathscr{H}(g_1,\hat{L}_1)$ if $g_3\geq g_3^*$ and $0<g_1\leq 4\pi/3$. It is dual to $\mathscr{H}(g_2,\hat{L}_2)$ if $g_3\geq g_3^*$ and $4\pi/3\leq g_2<\infty$. Finally, $\mathscr{H}$ is dual to itself if $0<g_1\leq 4\pi/3$ and $4\pi/3\leq g_2<\infty$. It is therefore sufficient to investigate $\mathscr{H}(g_1,\hat{L}_1)$ in the range $0<g_1\leq 4\pi/3$ to get all information about $\mathscr{H}$ in the range $4\pi/3\leq g_2$ and about $\tilde{\mathscr{H}}$ in the range $g_3\geq g_3^*$ also. On the other hand, $\tilde{\mathscr{H}}(g_3,\hat{L}_3)$ for $0<g_3<g_3^*$ is not dual to any other Hamiltonian. So we conclude that a complete investigation of both the symmetric and the asymmetric model has to be performed for $0<g_1\leq 4\pi/3$ and $0<g_3<g_3^*$. These ranges cover all possible different Hamiltonians. This will be done in the fourth section. We want to become more concrete now and relate states and energy eigenvalues of dual models to each other. As a first result the agreement of states has to be mentioned

$$|\psi_n(g_1,\hat{L}_1)\rangle=|\psi_n(g_2,\hat{L}_2)\rangle =|\tilde{\psi}_n(g_3,\hat{L}_3)\rangle , \tag{2.12}$$

where $|\psi_n\rangle$ are eigenstates of $\mathscr{H}$ with eigenvalues $E_n(g_i,\hat{L}_i)$ whereas $|\tilde{\psi}_n\rangle$ is an eigenstate of $\tilde{\mathscr{H}}$ with eigenvalue $\tilde{E}_n(g_3,\hat{L}_3)$. In addition to (2.12) one has to assume that the expectation values of the field operator $\phi(x)$ also agree. Since the states may be degenerate one has to relate corresponding states [via (2.12)]. Because of the duality relations, these energy eigenvalues are dependent on each other: namely,

$$E_n(g_1,\hat{L}_1)=\left(\frac{g_1}{g_2}\right)^{1/2}E_n(g_2,\hat{L}_2)+g_1\hat{L}_1w(g_1,g_2) , \tag{2.13}$$

$$E_n(g_1,\hat{L}_1)=\left[\frac{g_1}{g_3}\right]^{1/2}\tilde{E}_n(g_3,\hat{L}_3)+g_1\hat{L}_1\left[\frac{1}{8\pi}\left[\frac{1}{g_3}-\frac{2}{g_1}\right]+\tfrac{3}{4}\Delta_1{}^2-\frac{1}{2}\frac{\Delta_1}{g_1}+\frac{1}{4g_1{}^2}\right]. \tag{2.14}$$

These equations have consequences for the energy density of the vacuum state of the dual models. Since E_0 and $\tilde{E}_0$ should be proportional to the volume, we obtain

$$\mathscr{E}_0(g_1)=\frac{g_1}{g_2}\mathscr{E}_0(g_2)+g_1 w(g_1 g_2) \tag{2.15}$$

and

$$\mathscr{E}_0(g_1)=\frac{g_1}{g_3}\tilde{\mathscr{E}}_0(g_3)+g_1\left[\frac{1}{8\pi}\left[\frac{1}{g_3}-\frac{2}{g_1}\right]+\tfrac{3}{4}\Delta_1{}^2-\frac{\Delta_1}{2g_1}+\frac{1}{4g_1{}^2}\right], \tag{2.16}$$

where $\mathscr{E}_0$ and $\tilde{\mathscr{E}}_0$ are the vacuum energy densities of $\mathscr{H}$ and $\tilde{\mathscr{H}}$. Both are dimensionless quantities. Similar considerations applied to the mass gaps $M(g_i)$ and $\tilde{M}(g_3)$ yield

$$M(g_1)=\left[\frac{g_1}{g_2}\right]^{1/2}M(g_2)=\left[\frac{g_1}{g_3}\right]^{1/2}\tilde{M}(g_3)\,. \tag{2.17}$$

Despite the fact that these are exact identities they do not tell us how large $\mathscr{E}_0(g_1)$ and $\tilde{\mathscr{E}}_0(g_3)$ really are. This has to be clarified in an explicit calculation scheme as to be introduced in the next section. As mentioned before it is sufficient to do these calculations in the ranges $0<g_1\leq 4\pi/3$ and $0<g_3<g_3^*$.

III. COUPLED CLUSTER THEORY OF THE VACUUM STATE

Since we do not intend to give a boring introduction into all the ideas and techniques of the coupled cluster method once again, we only mention the existence of a paper, telling the interested reader all necessary details.[14] Furthermore, a numerical investigation of the $(\phi^4)_2$ model has been performed some time ago making use of the standard coupled cluster method.[11] It may be advantageous to compare our procedure with this original one, which is both analytically and numerically much more involved.

We will concentrate on the asymmetric scaled Hamiltonian $\mathscr{H}(g_1,\hat{L}_1)$ to derive the maximum overlap coupled cluster equations. A similar analytical calculation will yield the corresponding equation for the symmetric model with scaled Hamiltonian $\tilde{\mathscr{H}}(g_3,\hat{L}_3)$. Since no new insights can thereby be obtained, we only sketch the minor differences in the latter system of equations. Furthermore, not the complete spectrum E_n and $\tilde{E}_n$ will be evaluated but only the vacuum state and the vacuum energy densities. This restriction is sufficient to answer the important question of whether or not a phase transition occurs in one of the two models described by the Hamiltonians $\mathscr{H}$ or $\tilde{\mathscr{H}}$. The mass gap can in principle be extracted from a coupled cluster calculation of the first excited state or from the correlation function of two field operators. In the present paper we shall not deal with this problem, however. As discussed in the second section we intend to solve the Schrödinger equation

$$\mathscr{H}(g_1,\hat{L}_1)\,|\,\Omega(g_1,\hat{L}_1)\rangle=E_0(g_1,\hat{L}_1)\,|\,\Omega(g_1,\hat{L}_1)\rangle\,. \tag{3.1}$$

The relevant energy density $\mathscr{E}_0(g_1)$ is related to the eigenvalue $E_0(g_1,\hat{L}_1)$ via

$$\mathscr{E}_0(g_1)=\lim_{\hat{L}_1\to\infty}\hat{L}_1{}^{-1}E_0(g_1,\hat{L}_1)\,. \tag{3.2}$$

In the coupled cluster method $|\,\Omega(g_1,\hat{L}_1)\rangle$ is assumed to be a generalized coherent state

$$|\,\Omega(g_1,\hat{L}_1)\rangle=e^{s(g_1,\hat{L}_1)}\,|\,0\rangle\,, \tag{3.3}$$

where $|\,0\rangle$ is a Fock vacuum state still to be defined. The e^s form of the vacuum state reminds us of the typical properties of coherent states. Before we make (3.3) more concrete we turn to the Hamiltonian $\mathscr{H}(g_1,\hat{L}_1)$ once again. It is specified by normal ordering with respect to a set of creation and annihilation operators. The field operator is expanded in this set which defines a Fock space in the customary way. But this Fock space may not be the most efficient one. A unitary transformation to a new basis in Fock space can improve the convergence of perturbative or nonperturbative calculation schemes. Optimization of the basis is exactly the concept utilized by the Gaussian approximation, which is a variational principle in this sense. But once again, this is equivalent to a unitary Bogoliubov transformation of the creation and annihilation operators as was pointed out in Ref. 14.

Therefore, we proceed in the following way.

(1) The original Hamiltonian is defined in terms of operators $a_k,a_k^\dagger$ and normal ordered with respect to these. We perform a Bogoliubov transformation to a new basis of operators $b_k,b_k^\dagger$ and, additionally, shift the field operator, since $\langle\Omega\,|\,\phi(x)\,|\,\Omega\rangle\neq 0$ has to be expected in one phase of the $(\phi^4)_2$ model.

(2) The Hamiltonian is rewritten in terms of the $b_k,b_k^\dagger$ operators. Both the energies of the Bogoliubov quasiparticles and the shift parameter of the field operator will serve as free parameters in later calculations. We emphasize that the energy spectrum of the quasiparticles need not be relativistic any longer. This is not a serious drawback of the method because the quasiparticles have no physical meaning.

(3) The coupled cluster method is applied to the Hamiltonian resulting from the manipulations of step (2). It turns out that the description is not unique at all. This allows us to set some correlation amplitudes to zero, resulting in a unique and relatively simple system of equations.

Proceeding in this way is only a matter of lengthy calculations. As mentioned in step (1), $\phi(x)$ is expanded as

$$\phi(x)=t+\int\frac{dk}{\sqrt{4\pi\mathscr{E}_k}}(b_k e^{ikx}+b_k^\dagger e^{-ikx})\,. \tag{3.4}$$

Inserting this definition and a similar one for the operators $\pi(x)$ into $\mathscr{H}(g_1,\hat{L}_1)$ and changing the normal ordering from a operators to b operators results in

$$\mathscr{H}(g_1,\hat{L}_1)=\hat{L}_1\left[\int\frac{dk}{8\pi\mathscr{E}_k}(\mathscr{E}_k-\omega_k)^2+\frac{g}{4}\left[t^2-\frac{1}{g}\right]^2+\tfrac{3}{4}g\Delta^2+\tfrac{3}{2}g\Delta\left[t^2-\frac{1}{g}\right]\right]+\kappa(b_0+b_0^\dagger)$$
$$+\int dk[E_k b_k^\dagger b_k+\Omega_k(b_k b_{-k}+b_k^\dagger b_{-k}^\dagger)]+N_b\int_{-\hat{L}_1/2}^{\hat{L}_1/2}dx\left[\frac{g}{4}\phi^4(x)+gt\phi^3(x)\right]. \tag{3.5}$$

There is one necessary remark to be made concerning a slightly inconsistent treatment which can be traced back to a usage of a finite volume but a continuous momentum. A correct procedure should use discrete momenta. Since the infinite-volume limit is somehow trivial to perform we decided to work with continuous momenta already from the beginning.

The quantities κ, E_k, Ω_k, w_k, and Δ are given by

$$\kappa=gt\left[\frac{\pi}{\mathscr{E}_{k=0}}\right]^{1/2}\left[t^2-\frac{1}{g}+3\Delta\right],$$
$$E_k=\frac{1}{2\mathscr{E}_k}\left[\omega_k{}^2+\mathscr{E}_k{}^2+3g\left[t^2-\frac{1}{g}+\Delta\right]\right],$$
$$\Omega_k=\frac{1}{4\mathscr{E}_k}\left[\omega_k{}^2-\mathscr{E}_k{}^2+3g\left[t^2-\frac{1}{g}+\Delta\right]\right], \tag{3.6}$$
$$\Delta=\int\frac{dk}{2\pi}\left[\frac{1}{2\mathscr{E}_k}-\frac{1}{2\omega_k}\right],\quad \omega_k=(k^2+2)^{1/2}.$$

It is obvious from expression (3.5) that neither $\mathscr{E}_k$ nor E_k can be reliably identified with a physical particle energy. $\mathscr{H}$ as defined by (3.5) seems to be a drawback since there are terms proportional to $b_k b_{-k}$ and $b_k^\dagger b_{-k}^\dagger$ as well as terms proportional to b_0 and $b_0^\dagger$. The Gaussian approximation fixes t and $\mathscr{E}_k$ by minimizing the energy. This leads to the requirement that these operators are absent from $\mathscr{H}$. Here we do not insist that both κ and Ω_k vanish but try to choose a method more intimately connected with the coupled cluster method. We return to the ansatz (3.3) and expand the operator S in terms of creation operators with coefficients which are commonly known as correlation amplitudes:

$$S(g_1,\hat{L}_1)=\sum_{n=1}^{\infty}S_n(g_1,\hat{L}_1)$$
$$=\sum_{n=1}^{\infty}\int dk_1\cdots dk_n\left[\left[\prod_{i=1}^{n}2\mathscr{E}_{k_i}\right]^{1/2}\delta\left[\sum_{i=1}^{n}k_i\right]S_n(k_1\cdots k_{n-1})b_{k_1}^\dagger\cdots b_{k_n}^\dagger\right]. \tag{3.7}$$

As long as an infinite number of correlation amplitudes is taken into account, the ansatz (3.3) permits an exact treatment of a physical vacuum state, although some compromise has to be found later on. In practical calculations it will not be possible to evaluate all correlation amplitudes exactly. We insert the ansatz (3.3) into the Schrödinger equation and project onto states $\langle 0\,|\,b_{k_1}\cdots b_{k_n}e^{-s}$, where $b_k\,|\,0\rangle=0$:

$$\langle 0\,|\,b_{k_1}\cdots b_{k_n}e^{-s}\mathscr{H}e^{s}\,|\,0\rangle=0,\quad n=1,2,\ldots, \tag{3.8}$$

$$\langle 0\,|\,\mathscr{H}e^{s}\,|\,0\rangle=E_0(g_1,\hat{L}_1). \tag{3.9}$$

The nth equation of the complete and infinite system of Eqs. (3.8) will be abbreviated by $[n]=0$ to shorten the further discussion. Equation (3.9) defines the exact vacuum energy in terms of $t,\mathscr{E}_k$ and correlation functions S_n. A simple calculation of (3.9) yields in the case of the vacuum energy density (for reasons which will become obvious later we choose $S_1=S_2=0$) the exact expansion

$$\mathscr{E}_0(g_1)=\int\frac{dk}{8\pi\mathscr{E}_k}(\omega_k-\mathscr{E}_k)^2+\frac{g}{4}\left[t^2-\frac{1}{g}\right]^2$$
$$+\tfrac{3}{4}g\Delta^2+\tfrac{3}{2}g\Delta\left[t^2-\frac{1}{g}\right]+\frac{g}{16\pi^2}S_{4,3}$$
$$+\frac{gt}{(2\pi)^{3/2}}S_{3,2}, \tag{3.10}$$

where we set up $S_{n,k}$ amplitudes via

$$S_{n,k}(p_1\cdots p_{n-1-k})=\int dl_1\cdots dl_k$$
$$\times S_n(p_1\cdots p_{n-1-k},l_1\cdots l_k).$$

This introduces partially averaged correlation amplitudes. Since $\mathscr{E}_0(g_1)$ does not depend on any momentum it is quite convincing that only $S_{4,3}$ and $S_{3,2}$ enter the expression of the vacuum energy density. If we had not scaled the momenta, these numbers would be dimensionless;

therefore, this leads to

$$S_{n,n-1}(g_1)=S_{n,n-1}(g_2)\ . \tag{3.11}$$

The numerical calculations should obey this fundamental relation serving as a test of numerical accuracy.

At this point we have to return to Eq. (3.10) where $S_1=S_2=0$ was used which is exactly the maximum overlap condition.[15] Why can we put these two functions equal to zero? There is a one to one correspondence between amplitudes S_n and equations $[n]=0$. Each amplitude is completely specified by its equation. But t and $\mathscr{E}_k$ are also free parameters which cannot be fixed in this way. The shift t and the amplitude S_1, both simple numbers, can somehow be interchanged. We can set t to zero and use $S_1\neq 0$ or we set S_1 to zero and assume $t\neq 0$. In the maximum overlap condition S_1 is set to zero and $[1]=0$ is the equation fixing the parameter t. Similarly $S_2=0$ fixes the energies $\mathscr{E}_k$ via $[2]=0$. Only amplitudes $S_3,S_4,\ldots$ appear in the expansion of $S(g_1,\hat{L}_1)$. From now on we replace the original definition of S by a modified one where the sum over n starts at $n=3$. $[1]=0$ will be called the t equation and $[2]=0$ will be called the $\mathscr{E}_k$ equation. $[3]=0,[4]=0,\ldots$ are $S_3,S_4,\ldots$ equations. This approach is based on "the maximum overlap" (MO) condition, well known in many-body theory.[13] Since for boson systems the most general Gaussian wave function is of the form[14] $\phi=\exp(S_1+S_2)\,|\,0\rangle$ the variation $\delta\,|\,\langle\phi\,|\,\Omega\rangle\,|=0$ leads to $S_1=S_2=0$.

Let us return to the Gaussian approximation for a moment. If we ignore $S(g_1,\hat{L}_1)$, $[1]=[2]=0$ are the Hartree conditions of the Gaussian approximation. The other equations $[n]=0$ with $n\geq 3$ are not satisfied. We will use an approximation which includes S_3 and S_4. Therefore, we can satisfy $[1]=[2]=[3]=[4]=0$, but $[n]\neq 0$ for $n\geq 5$. This is obviously a post-Gaussian approximation which goes beyond the standard Gaussian one as defined before in terms of "$[n]=0$" equations.

As a result of lengthy but straightforward evaluations we arrive at the following.

t equation:

$$0=\frac{1}{2\mathscr{E}_{k=0}}\left[\frac{gS_{3,2}}{2\pi}+\sqrt{2\mathscr{E}_{k=0}}\kappa\right]+\frac{gtS_{4,2}(0)}{\sqrt{2\pi}}+\int dk\ \Omega_k^0 S_3(0,k)\ . \tag{3.12}$$

$\mathscr{E}_k$ equation:

$$0=\frac{g}{2\pi}\left[\frac{S_{4,2}(q)}{\mathscr{E}_q}+S_{3,2}S_3(0,q)+\tfrac{3}{2}S_{3,1}^2(q)\right]+\frac{3gt}{\sqrt{2\pi}}\frac{S_{3,1}(q)}{\mathscr{E}_q}+\int dk\ \Omega_k^0 S_4(q,-q,k)+\frac{\Omega_q}{\mathscr{E}_q}+\sqrt{2\mathscr{E}_{k=0}}\kappa S_3(0,q)\ . \tag{3.13}$$

S_3 equation:

$$0=\left[\sqrt{2\mathscr{E}_{k=0}}\kappa+\frac{gS_{3,2}}{2\pi}\right]S_4(0,q_1,q_2)+\sum_{\alpha=1}^{3}[E(q_\alpha)+\sigma(q_\alpha)]S_3(q_1,q_2)+\frac{3g}{4\pi}\sum_{\substack{1\\ \mu<\nu}}^{3}\left[\left[\frac{1}{2\mathscr{E}_{q_\mu}\mathscr{E}_{q_\nu}}+S_{4,1}(q_\mu,q_\nu)\right]S_{3,1}(\{q_1,q_2,q_3\}\setminus\{q_\mu,q_\nu\})\right]+\frac{3gt}{4\sqrt{2\pi}\mathscr{E}_{q_1}\mathscr{E}_{q_2}\mathscr{E}_{q_3}}+\sum_{\mu=1}^{3}\left[\frac{3gt}{\sqrt{2\pi}}\frac{1}{2\mathscr{E}_{q_\mu}}S_{4,1}(\{q_1,q_2,q_3\}\setminus\{q_\mu\})\right]\ , \tag{3.14}$$

where $q_1+q_2+q_3=0$.

S_4 equation:

$$0=\sum_{\alpha=1}^{4}[E(q_\alpha)+\sigma(q_\alpha)]S_4(q_1,q_2,q_3)+\frac{3g}{16\pi\mathscr{E}_{q_1}\mathscr{E}_{q_2}\mathscr{E}_{q_3}\mathscr{E}_{q_4}}+\sum_{\substack{1\\ \mu<\nu}}^{4}\left\{S_3(\{q\}_4\setminus\{q_\mu,q_\nu\})\left[S_3(q_\mu,q_\nu)\Omega^0_{q_\mu+q_\nu}+\frac{3g}{4\pi}\left[\frac{S_{3,1}(q_\nu)}{\mathscr{E}_{q_\mu}}+\frac{S_{3,1}(q_\mu)}{\mathscr{E}_{q_\nu}}\right]+\frac{3gt}{\sqrt{2\pi}}\left[\frac{1}{2\mathscr{E}_{q_\mu}\mathscr{E}_{q_\nu}}+S_{4,1}(q_\mu,q_\nu)\right]\right]\right\}+\frac{3g}{8\pi}\sum_{\substack{1\\ \mu<\nu}}^{4}\left[\left[\frac{1}{\mathscr{E}_{q_\mu}\mathscr{E}_{q_\nu}}+S_{4,1}(q_\mu,q_\nu)\right]S_{4,1}(\{q\}_4\setminus\{q_\mu,q_\nu\})\right]\ , \tag{3.15}$$

where $q_1+q_2+q_3+q_4=0$ and, e.g., $\{q\}_4\setminus\{q_1,q_2\}=\{q_3,q_4\}$.

Furthermore

$$\sigma(q)=\frac{g}{2\pi}S_{4,2}(q)+\frac{3gt}{\sqrt{2\pi}}S_{3,1}(q) \tag{3.16a}$$

and

$$\Omega_q^0=2\mathscr{E}_q\Omega_q \ . \tag{3.16b}$$

Equations (3.12)—(3.16) constitute a basis to analyze the physical vacuum and its properties nonperturbatively. The complexity of the ansatz for a realistic description of the vacuum state has been significantly increased in comparison with Gaussian trial states. It is not obvious at all that this improvement is already sufficient to reproduce the known and provable features of the $(\phi^4)_2$ model with acceptable accuracy. This problem has to be postponed to the next section. Hopefully, the inclusion of more and more amplitudes will give a convergent behavior of the resulting series of physical quantities. Unfortunately, already the inclusion of S_5 will increase the numerical expense enormously and convergence is very hard to prove.

A phase transition can be identified by inspection of two physical quantities, the field expectation value $\langle\Omega\,|\,\phi(x)\,|\,\Omega\rangle$ and the mass gap

$$M(g_1)=\lim_{|x|\to\infty}\frac{-\ln[\langle\Omega\,|\,\phi(x)\phi(0)\,|\,\Omega\rangle_{\rm con}]}{|x|} \ . \tag{3.17}$$

If we insert the calculated vacuum $|\Omega\rangle$ in the coupled cluster representation, the resulting matrix element cannot be evaluated exactly. We expand in the correlation amplitudes up to quadratic order, since S_3 and S_4 should be very small which is guaranteed by the numerical solution for a wide range of coupling parameters.

This results in

$$\langle\Omega\,|\,\phi(x)\,|\,\Omega\rangle=t+\frac{4}{3}\left[\frac{2}{\pi}\right]^{1/2}\int dk_1dk_2[S_3(k_1,k_2)S_4(k_1,k_2,0)\mathscr{E}_{k_1}\mathscr{E}_{k_2}\mathscr{E}_{k_1+k_2}] \ . \tag{3.18}$$

It should be noted that $\langle\phi\rangle(g_1)=\langle\phi\rangle(g_2)=\langle\phi\rangle(g_3)$ in the range of duality because $\langle\phi\rangle$ is a dimensionless quantity of the theory. Furthermore

$$\langle\Omega\,|\,\phi(x)\phi(0)\,|\,\Omega\rangle_{\rm con}=\int\frac{dp}{2\pi}e^{ipx}\left[\frac{1}{2\mathscr{E}_p}+4\left[\int dk[\mathscr{E}_k\mathscr{E}_{k+p}S_3{}^2(k,p)]\right.\right.$$

$$\left.\left.+\tfrac{2}{3}\int dk_1dk_2[S_4{}^2(k_1,k_2,p)\mathscr{E}_{k_1}\mathscr{E}_{k_2}\mathscr{E}_{k_1+k_2+p}]\right]\right] \ . \tag{3.19}$$

Formula (3.18) will be utilized to extract the critical point from the numerical data. Equation (3.19) was not helpful in obtaining the physical meson mass since a Fourier transformation is required first and the asymptotic behavior could not be reliable derived from it numerically because of fast fluctuations of the integrand at very large x. We conjecture that there is a special value $g_{3,\rm crit}$ in the range $0<g_{3,\rm crit}<g_3^*$ having the property that $\langle\phi\rangle=0$ is true in the range $0<g_3<g_{3,\rm crit}$ and $\langle\phi\rangle\neq0$ will be the case in the range $g_3>g_{3,\rm crit}$. This means, the critical behavior can be found somewhere in the symmetric version of the $(\phi^4)_2$ model which will be briefly discussed now. All operators, states, and functions $(\mathscr{H},|\,\Omega\rangle,E_0,\ldots)$ are replaced by those of the symmetric model $(\tilde{\mathscr{H}},|\,\tilde{\Omega}\rangle,\tilde{E}_0,\ldots)$.

$\mathscr{H}$ and $\tilde{\mathscr{H}}$ differ in κ, E_k, ω_k, Δ, and Ω_k as well as in the constant proportional to $\hat{L}_3$. All the objects of the symmetric model are indexed by a tilde above them. $\tilde{\mathscr{H}}$ has the same representation as $\mathscr{H}$ of (3.5), but

$$\tilde{\kappa}=g_3\tilde{t}\left[\frac{\pi}{\tilde{\mathscr{E}}_{k=0}}\right]^{1/2}\left[3\tilde{\Delta}+\tilde{t}^2+\frac{1}{g_3}\right] ,$$

$$\tilde{E}_k=\frac{1}{2\tilde{\mathscr{E}}_k}[\tilde{\omega}_k{}^2+\tilde{\mathscr{E}}_k{}^2+3g_3(\tilde{t}^2+\tilde{\Delta})] ,$$

$$\tilde{\Omega}_k=\frac{1}{4\tilde{\mathscr{E}}_k}[\tilde{\omega}_k{}^2-\tilde{\mathscr{E}}_k{}^2+3g_3(\tilde{t}^2+\tilde{\Delta})] , \tag{3.20}$$

$$\tilde{\omega}_k=(k^2+1)^{1/2} ,$$

$$\tilde{\Delta}=\int\frac{dk}{2\pi}\left[\frac{1}{2\tilde{\mathscr{E}}_k}-\frac{1}{2\tilde{\omega}_k}\right] .$$

Equations (3.12)—(3.16) are still correct with these new definitions if $S_{n,k}$'s are also replaced by $\tilde{S}_{n,k}$'s. The vacuum energy density $\tilde{\mathscr{E}}_0(g_3)$ is slightly different: namely,

$$\tilde{\mathscr{E}}_0(g_3)=\int \frac{dk}{8\pi\tilde{\mathscr{E}}_k}(\tilde{\omega}_k-\tilde{\mathscr{E}}_k)^2+\frac{g_3}{4}\tilde{t}^4+\tfrac{1}{2}\tilde{t}^2+\tfrac{3}{4}g_3\tilde{\Delta}^2$$
$$+\tfrac{3}{2}g_3\tilde{\Delta}\tilde{t}^2+\frac{g_3}{16\pi^2}\tilde{S}_{4,3}+\frac{g_3\tilde{t}}{(2\pi)^{3/2}}\tilde{S}_{3,2}\ . \qquad (3.21)$$

In the phase of a unique vacuum state both $\tilde{t}$ and $\tilde{S}_3$ should vanish. A nonvanishing value of $\tilde{t}$, and therefore of $\tilde{S}_3$, indicates a phase transition from a $\langle\phi\rangle=0$ to a $\langle\phi\rangle\neq 0$ phase.

Before we turn to the numerical part of this paper, we would like to make a last remark about asymptotically exact properties inherent in the system of equations (3.10) and (3.12)—(3.16). The cases $g_1\to 0$ [and $g_2\to\infty$ via the connection (2.11)] should allow a perturbative treatment by expanding in powers of g_1.

This leads us back to the Gaussian approximation plus higher-order corrections. Equations (2.15) and (2.16) are satisfied in the Gaussian approximation, but $\mathscr{E}_0(g_1)=0$ for the $\langle\phi\rangle\neq 0$ sector in the whole range $0<g_1\leq 4\pi/3$. If we do perturbation theory in lowest order, $\mathscr{E}_0(g_1)$ will be replaced by the contribution coming from $S_{3,2}$:

$$\mathscr{E}_0(g_1)=\frac{\sqrt{g_1}}{(2\pi)^{3/2}}S_{3,2}(g_1)=-0.0225g_1\ , \qquad (3.22)$$

where $S_{3,2}(g_1)$ was extracted from Eq. (3.14) in lowest order also. (3.22) is reproduced by our more involved method in the limit $g_1\to 0$ and serves as a numerical test case. On the other hand, the limit $g_2\to\infty$ is also now available. Making use of the identity (2.15) leads to

$$\mathscr{E}_0(g_2)=-0.0225g_2-g_2w(g_1(g_2),g_2)\ , \qquad (3.23)$$

where $g_1(g_2)$ is a solution of (2.11). Once more, (3.23) can be tested and turns out to be correct very precisely. We now switch to the numerical investigations supporting all our statements made before.

IV. NUMERICAL RESULTS

A superficial comparison of the system of equations (3.12) to (3.15) with the original system as derived in the standard coupled cluster method[11] must lead to the conclusion that the numerical expense has been reduced significantly. But even more advantages can be attributed to the use of the maximum overlap condition. There is a nearly unique iteration scheme since all S_n equations can be written as

$$S_n(p_1,p_2,\ldots,p_{n-1})=F_n[S_3,\ldots,S_{n-1},S_{n+1},\ldots;$$
$$S_{n_1},\ldots,S_{n,n-1}]\ , \qquad (4.1)$$

where S_n itself does not occur on the right-hand side any longer. Only averages $S_{n,k}$ are allowed as arguments of F_n. Starting with (4.1), an adequate iteration scheme is given by

$$S_n^{(i+1)}(p_1,p_2,\ldots,p_{n-1})=F_n[S_3^{(i)},\ldots,S_{n-1}^{(i)},S_{n+1}^{(i)},\ldots;$$
$$S_{n,1}^{(i)},\ldots,S_{n,n-1}^{(i)}]\ . \qquad (4.2)$$

The maximum overlap condition allows exactly one possibility to write the S_3 and S_4 equations in such a way. This is in contrast with the standard coupled cluster equations which allow many different formulations. Computer experiments applied to the anharmonic oscillator model have convinced us recently[16] that convergence or divergence is strongly dependent on the correct choice of an iteration scheme. This is in obvious agreement with observations made in nonlinear dynamics of iterative maps. Finding the best functions F_n of Eq. (4.2) is highly nontrivial and the partial failure of the standard coupled cluster method in our first attempt may be traced back to this fact. We have overcome this problem due to the maximum overlap condition and its unique description.

The t equation is a cubic equation which can be solved by standard algorithms. It may possess up to three distinct solutions. In the Gaussian approximation there is always a solution at $t=0$ indicating three distinguishable vacuum sectors of the asymmetric $(\phi^4)_2$ model. This sector even becomes the one with the lowest energy in some range of the coupling strength. This causes a phase transition of first order in the asymmetric model.[1] Already in our first attempt we could never construct this spurious sector and concluded that it is a disastrously wrong prediction of the Gaussian approximation. In the maximum overlap case things do not change, and there is no hint that a vacuum sector at $\langle\phi\rangle=0$ exists in the asymmetric model. On the other hand, the symmetric model has a unique vacuum at $\langle\phi\rangle=\tilde{t}=0$ at least for small values of the coupling strength. This is in agreement with the Gaussian approximation. But this sector becomes instable for larger values of the coupling strength and dynamical symmetry breaking takes place. The unique vacuum at $\langle\phi\rangle=0$ splits up into two degenerate vacua at $\langle\phi\rangle\neq 0$. We will try to estimate at which value of g_3 this phase transition appears. At least qualitatively we can understand why $g_{3,\mathrm{crit}}<g_3^*$ results from our more sophisticated method. The $\tilde{t}$ equation has no solution at $\tilde{t}\neq 0$ in the Gaussian approximation if $g_3<g_3^*$. Because of higher-order correlations solutions of the modified $\tilde{t}$ equation occur already at values g_3 smaller than this Gaussian g_3^*. Therefore, a phase transition can be observed in the post-Gaussian approximation at some $g_{3,\mathrm{crit}}<g_3^*$.

The $\mathscr{E}_k$ equation is solved iteratively also. We can rewrite this equation as a quadratic equation for $\mathscr{E}_k$:

$$0=A[t,\mathscr{E}_k,S_3,S_4]\mathscr{E}_k^2+B[t,S_3,S_4]\mathscr{E}_k+C[t,\mathscr{E}_k]\ , \qquad (4.3)$$

where A and C may depend on $\mathscr{E}_k$ themselves at least in a functional form via Δ or other integrals containing $\mathscr{E}_k$. (4.3) will be solved as a series $\mathscr{E}_k^{(i)}$ defined by

$$0=A[t,\mathscr{E}_k^{(i)},S_3,S_4](\mathscr{E}_k^{(i+1)})^2$$
$$+B[t,S_3,S_4]\mathscr{E}_k^{(i+1)}+C[t,\mathscr{E}_k^{(i)}] \qquad (4.4)$$

and usually convergence is established after a few iterations. Sometimes problems appeared because (4.4) has no real solutions if $B^2-4AC<0$. This can be expressed as a bound on the higher correlations S_3 and S_4. If they exceed some critical value, complex pseudoparticle energies will result.

TABLE I. Numerical results for three Hamiltonians dual to each other. First three columns list the coupling constants for the three Hamiltonians. Columns four to eight give the energy densities.

g_1	g_2	g_3	$\mathscr{E}_0(g_1)$	$\mathscr{E}_0(g_2)$	$\tilde{\mathscr{E}}_0(g_3)$	$\mathscr{E}_0(g_2)_{\text{pred}}$	$\tilde{\mathscr{E}}_0(g_3)_{\text{pred}}$
0.5	2170.044 5	1091.299 4	−0.011 52	−1191	−599	−1151	−579
1.0	61.608 493	36.931 457	−0.023 72	−6.904	−4.135	−6.904	−4.134
$\pi/2$	17.886 871	14.956 747	−0.038 28	−0.808 3	−0.6625	−0.808 2	−0.6624
2.0	11.157 619	11.638 368	−0.048 59	−0.356 7	−0.3516	−0.356 7	−0.3516
$4\pi/5$	7.751 199	10.080 763	−0.058 24	−0.196 8	−0.2285	−0.196 8	−0.2285
3.0	6.098 053	9.443 272	−0.065 73	−0.137 0	−0.1799	−0.137 0	−0.1799
3.5	5.070 816	9.152 828	−0.073 55	−0.107 0	−0.1577	−0.107 0	−0.1577
4.0	4.389 650	9.052 659	−0.082 47	−0.090 51	−0.1498	−0.090 51	−0.1498
$4\pi/3$	$4\pi/3$	9.045 978	−0.086 22	−0.086 22	−0.1493	−0.086 22	−0.1493

We finish the description of numerical technicalities at this point and turn to the numerical results concerning the vacuum energy density and field expectation value. We can either restrict ourselves to the range $0<g_1\leq 4\pi/3$ and relate the model at values of the coupling strength $g_2\geq 4\pi/3$ to this range by the duality transformation, or we forget duality and analyze the model at all values $g>0$. Then, duality is a numerical result of our calculations and can serve as a test of numerical accuracy. If we use g as a coupling strength, the complete interval $0<g<\infty$ is meant, whereas g_1 and g_2 always cover only a limited range as it was often defined before.

Table I tests duality of three different Hamiltonians with coupling strengths g_1 and g_2 of the asymmetric model and coupling strength g_3 of the symmetric one. We give both the numerical data $\mathscr{E}_0(g_1)$, $\mathscr{E}_0(g_2)$, and $\tilde{\mathscr{E}}_0(g_3)$ and the prediction of duality for $\mathscr{E}_0(g_2)$ and $\tilde{\mathscr{E}}_0(g_3)$. The agreement is excellent in all cases supporting our belief that our numerics is completely reliable. A disagreement in the last digit can always be traced back to the integration method and complete agreement can easily be obtained by adjusting it. Scaling identities, telling us that t, $S_{4,3}$, and $S_{3,2}$ should agree for the three dual models, are shown in Table II. We need not emphasize the high precision at which the identities can be reproduced by our procedure. Therefore, numerical problems should never occur due to an inadequate numerics but more likely due to some fundamental shortcoming of the method or a chosen approximation scheme. Divergent iteration schemes are probably a consequence of missing higher-order correlations $S_5, S_6, \ldots$ being extremely desirable close to the critical point.

Figure 1 shows the vacuum energy density of both the symmetric (dashed curve) and the asymmetric model (solid curve). At first we have to mention that the numerical data are in complete agreement with Eqs. (3.22) and (3.23) if g_1 is small and g_2 very large. These two formulas describe the vacuum energy density nearly perfectly. At intermediate coupling strengths further corrections are necessary but still small. Therefore, many higher-order contributions sum up to a small quantity making infinite summations, as they are described by the coupled cluster method, absolutely obligatory. Things are worse in the symmetric model where no solutions at values $3.8<g_3<8.6$ were reliably obtained because the iteration scheme did not converge. This may be understood from the large correlations which we will describe later on. Nevertheless, the incomplete curve $\tilde{\mathscr{E}}_0(g_3)$ allows a simple interpolation in the range $3.8<g_3<8.6$ if we assume a totally smooth behavior. This is indicated by the dotted curve.

Figure 2 exhibits the field expectation value as a function of the coupling strength g or g_3. It is remarkable that t or $\tilde{t}$ describes these expectation values nearly perfectly and the corrections of order S_3S_4 are almost less than 1%. This supports our conjecture that an expansion of a matrix element as a power series in correlation amplitudes is reliable even if the series is truncated already after the quadratic terms. $\langle\phi\rangle(g)$ is represented by the solid curve of Fig. 2. There is no hint of a phase transition in the asymmetric model. We searched for the spurious vacuum sector at $\langle\phi\rangle=0$ as it is predicted by the simpler

TABLE II. Test of scaling identities for three Hamiltonians dual to each other. The labels i of t, $S_{3,2}$, and $S_{4,3}$ refer to the three Hamiltonians.

g_1	t_1	t_2	$\tilde{t}_3$	$(S_{3,2})_1$	$(S_{3,2})_2$	$(\tilde{S}_{3,2})_3$	$(S_{4,3})_1$	$(S_{4,3})_2$	$(\tilde{S}_{4,3})_3$
0.5	1.410	1.421	1.421	−0.2694	−0.2669	−0.2669	−0.005 44	−0.005 38	−0.005 38
1.0	0.9868	0.9868	0.9868	−0.4197	−0.4197	−0.4197	−0.007 32	−0.007 32	−0.007 32
$\pi/2$	0.7679	0.7679	0.7679	−0.6007	−0.6008	−0.6008	−0.016 2	−0.016 2	−0.016 2
2.0	0.6662	0.6662	0.6662	−0.7305	−0.7305	−0.7305	−0.065 8	−0.065 7	−0.065 7
$4\pi/5$	0.5883	0.5883	0.5883	−0.8135	−0.8135	−0.8135	−0.207 5	−0.207 5	−0.207 5
3.0	0.5415	0.5415	0.5415	−0.8327	−0.8327	−0.8327	−0.362 3	−0.362 3	−0.362 3
3.5	0.5130	0.5130	0.5130	−0.8309	−0.8309	−0.8309	−0.479 4	−0.479 4	−0.479 4
4.0	0.5013	0.5013	0.5013	−0.8275	−0.8275	−0.8275	−0.531 4	−0.531 4	−0.531 4
$4\pi/3$	0.5004	0.5004	0.5004	−0.8271	−0.8271	−0.8271	−0.535 0	−0.535 0	−0.535 0

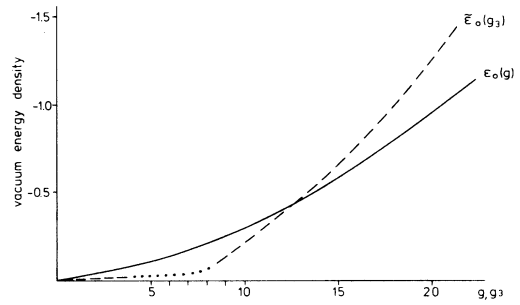

FIG. 1. Vacuum energy densities for the symmetric and asymmetric models.

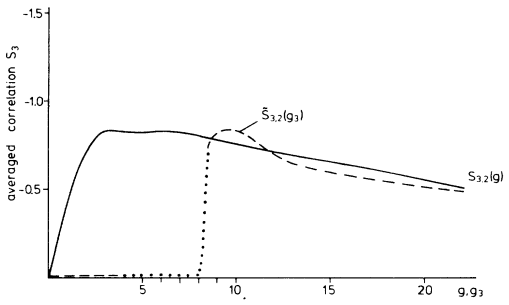

FIG. 3. Correlation amplitudes $S_{3,2}$ for the symmetric and asymmetric models.

Gaussian approximation, but we never got a solution of the corresponding $\mathscr{E}_k$ and S_4 equations. Certainly, this does not prove the absence of this sector. But it cannot exist at small values of S_4, and, therefore, the wave functional is far away from any Gaussian shape. The symmetric model exhibits completely different properties. At small values of g_3 we have $\langle\phi\rangle=0$ as expected from semiclassical calculations. Increasing g_3 has enormous consequences because at some still unknown coupling strength $g_{3,\mathrm{crit}}$ the symmetric model is dynamically broken by radiative corrections. Our calculations demonstrate that this occurs already at $g_3=8.6$ and, therefore, $3.8<g_{3,\mathrm{crit}}<8.6$. This is a safe prediction but the curve indicates that $g_{3,\mathrm{crit}}$ is probably about 8.0. The curve drops down suddenly if g_3 is decreased beyond $g_3^*=9.045\,978$. An extrapolation as shown by the dotted curve yields this estimation of $g_{3,\mathrm{crit}}$ although we had to use the assumption that $\langle\phi\rangle(g_3)$ behaves similarly as the magnetization as a function of the temperature in spin models. $g_{3,\mathrm{crit}}\approx 8.0$ should be considered as a conjecture which needs much more work to become a serious result.

Figures 3 and 4 show the relevance of higher-order correlations S_3 and S_4 in the vacuum wave functional.

The correlation amplitudes S_3 and S_4 themselves are both difficult to interpret and hard to draw because of their dependence on many variables. Special cuts such as $S_3(k,-k)$ or $S_4(k,-k,0)$ or even $S_{4,1}(k,-k)$ show no dramatic behavior of these functions. All amplitudes have a maximum at $K=0$ and decrease to zero rather fast via a power law. $S_{3,2}$ and $S_{4,3}$ represent the total correlation and constitute a measure for the accuracy of the approximation scheme used. If $S_{3,2}$ and $S_{4,3}$ are small the wave functional is nearly Gaussian. Large amplitudes do not necessarily mean that the coupled cluster method is no longer applicable, but $S_{4,3}$ should be smaller than $S_{3,2}$, otherwise there is no reason to believe that $S_{5,4}, S_{6,5}, \ldots$ decrease to zero and can be ignored. This condition is satisfied in the asymmetric model for all values of the coupling strength, although it becomes a little bit doubtful at about $g=4\pi/3$, where correlations reach their maximum in the asymmetric model. In the symmetric model correlations are small for $g_3>g_3^*$. But they become very large in the complete range $0<g_3<g_3^*$ indicating that the wave functional is far away from any Gaussian shape. At the critical point $\tilde{S}_{4,3}(g_3)$ seems to approach infinity. This may be described by a power law as it is typical for critical behavior. We conjecture that all even amplitudes $\tilde{S}_{2n}$ will show this property and introduce critical exponents a_n and b_n:

$$\tilde{S}_{2n,2n-1}(g_3) \underset{g_3\to\tilde{g}_{3,\mathrm{crit}}}{\sim} \begin{cases} A_-\,|\,g_3-g_{3,\mathrm{crit}}\,|^{-a_n}, & g_3<g_{3,\mathrm{crit}}, \\ A_+\,|\,g_3-g_{3,\mathrm{crit}}\,|^{-b_n}, & g_3>g_{3,\mathrm{crit}} \end{cases} \tag{4.5}$$

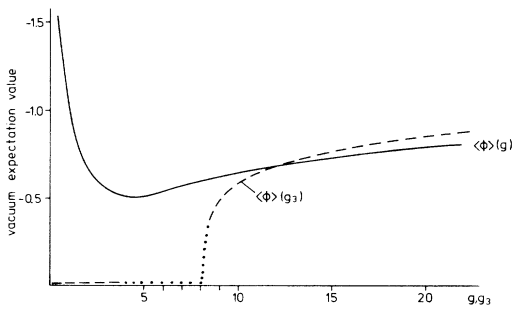

FIG. 2. Field expectation value for the symmetric and asymmetric models.

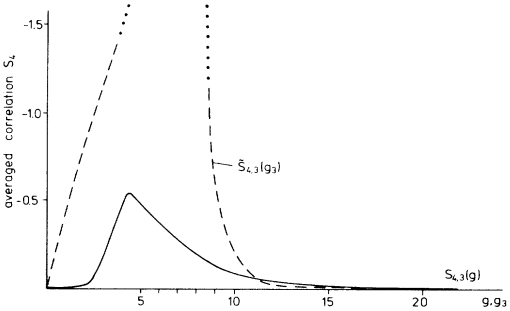

FIG. 4. Same as Fig. 3 for $S_{4,3}$.

Certainly, we cannot evaluate a_n and b_n from our data. But the existence of such power laws must have consequences for our method which is basically analytical. Therefore, nonanalytical behavior will never result from such simple truncation schemes as they were used in this publication. The critical region needs more sophisticated techniques which have not yet been formulated.

V. DISCUSSION AND OUTLOOK

In this paper we have presented a method to treat the continuum $(\phi^4)_2$ quantum field theory with post-Gaussian approximations. In contrast with the ordinary Gaussian approximation most features of that model can now be reproduced, only the critical point is still out of reach without further modifications of the general procedure. But we have succeeded in localizing the critical point better than ever before. It is most probably contained in the symmetric $(\phi^4)_2$ model at a coupling strength $3.8 < g_{3,\text{crit}} < 8.6$. Our data indicate a value of $g_{3,\text{crit}} \approx 8.0$ although some assumptions have to be made to come to this conclusion. At values $g_3 < g_{3,\text{crit}}$ the symmetric model possesses a unique vacuum state and becomes dynamically broken by radiative corrections at coupling strengths $g_3 > g_{3,\text{crit}}$. The asymmetric model has always two phases for all values of the coupling strength, and thus no critical point.

The maximum overlap version of the coupled cluster method is a useful and precise tool for the investigation of quantum field theories at least if we exclude critical phenomena. This method is based on three principles: namely, the Bogoliubov transformation, the maximum overlap condition, and the coupled cluster method. But this is not the end of the story and further well-known concepts of many-body theory can be included to constitute even more complex nonperturbative calculation schemes in quantum field theory. It is quite natural to think of pairing-phenomena as they are described by a BCS theory. This is one of our future projects.

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