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Complete treatment of effective actions within the Gaussian approximation and systematic generalizations

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The Gaussian approximation is generalized to include the complete effective action with arbitrary inhomogeneous background fields. This offers new possibilities to treat static and dynamical solitons resigning semiclassical approximation schemes. The analytical continuation to imaginary time allows us to deal with nonclassical instantons. Our procedure is based on time-dependent variational methods of which at least one concept can be extended to a post-Gaussian approximation nonperturbatively. We construct a unique mapping between a quantum field theory of a scalar field in $1+1$ dimensions and a classical field theory of an infinite number of local and multilocal fields. As an application we treat the sine-Gordon system. A preliminary comparison of our numerical results with a semiclassical calculation of the soliton mass is presented and will be extended in a forthcoming publication.

I. INTRODUCTION

The historical evolution in quantum field theory has confronted us with models of increasing complexity. We moved from simple gauge models to unified theories and have now reached a status where the ultimate theory of the world, the superstring theory, seems to be found. These "theories of everything" are so elegant and easy to understand, that we do not discuss them in our publication, but switch to "theories of nothing," which definitely do not describe real nature. Nevertheless, there are some good reasons to study such models. Our understanding of the quantum structure of field theories is very small and most of the arguments are based on semiclassical techniques and perturbative expansions. There are brute-force methods to treat the corresponding lattice field theories via computer simulations, and one tries to relate the results to the continuum version of the theory by renormalization-group arguments. Unfortunately, the lattices are still small, fermions are incorporated only approximately, and, therefore, preliminary results are still doubtful. In continuum field theory the situation is much worse. Besides some exactly or nearly exactly solvable models there are no comparable powerful methods which seem to be reliable in general. Even quite clever approximation methods such as the dilute-gas approximation[1,2] of a gas of instantons does not work independent of a very special model. Although it explains confinement in the Abelian Higgs model[3] it does not in the case of QCD (Refs. 2 and 4). So we conclude that this present status justifies a detailed analysis of models far from reality if we can obtain insight into their quantum structure in the continuum case.

The investigation of simple field-theoretical models has a long tradition. Most of these models are, however, not defined in physical four dimensions. Since the invention of spontaneous compactification 10, 11, or even 25 dimensions do not cause any serious problem and a field theorist may even be called old fashioned if he does physics in four dimensions only. Most of the simple field-theoretical models are defined in two dimensions and, unfortunately, inverse compactification from two to four dimensions has not yet been invented. We assume that there is a not-yet-observed parallel universe with a single space dimension and we would like to describe the field theories of those living in that universe, which may also be considered as a $(1+1)$-dimensional shadow world. But how can we proceed if there are no known exact solutions of these $(1+1)$-dimensional models? A method, which waves the claim of pure quantum-theoretical origin, is the Gaussian approximation.[5-10] At present, nobody can estimate how well this method approximates true wave functionals of a field theory, but it is at least applicable to a wide range of phenomena and not plagued by infinities, which are responsible for the fact that the Gaussian approximation has not been applied succesfully to $(3+1)$-dimensional models. The Gaussian approximation has been studied intensively in the past and there exists already some work done by us[5] to extend this model to a post-Gaussian approximation. The concepts of this generalization will find their successful applications soon.

What is the Gaussian approximation? It is a nonperturbative variational principle based on rather elementary and conceptually simple ideas, well known from quantum mechanics of many-body systems.[5] There is already a large number of publications[5-10] giving many technical details, but it is really astonishing that all authors restrict themselves to one single aspect and ignore quite interesting phenomena.

Let us consider the effective action which is defined as the Legendre transformation of the generating functional of connected Green's functions.[11] This action can be regarded as an almost complete information of the whole

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system which may be obtained from a given quantum field theory. Both perturbative and nonperturbative methods should try to evaluate the effective action from a classical level having applied standard quantization techniques. The Gaussian approximation which belongs to the class of nonperturbative methods is an alternative to the conventional loop expansion[11] which is perturbative in that sense that it expands the effective action in powers of $\hbar$. But, up to now, only a very special part of the effective action, which is commonly known as the effective potential, has been evaluated in the Gaussian approximation. This is acceptable as long as constant background fields are considered exclusively which is sufficient for the vacuum problem. Nevertheless, a lot of interesting phenomena, which require both space- and time-dependent structures, cannot be described by the effective potential alone. Among these structures we find solitons and instantons,[12–15] but also cosmological problems such as the decay of the false vacuuum,[1,16] which is of central importance in the theory of the inflationary universe.[16,17] Dynamical processes of this kind are commonly investigated using classical and semiclassical methods, which are improved by the loop expansion to take quantum effects into account.[17] In this publication we will introduce different procedures, utilizing the Gaussian approximation, to approach problems such as solitons and instantons. We develop the Gaussian approximation in a most general way and restrict ourselves to special cases later on. At first an approximation expression for the effective action will be derived, which includes both static and dynamical field configurations as stationary values of the action. The background field is accompanied by additional classical and multilocal fields, which comprehensively describe all quantum effects. This concept of additional fields can be extended systematically to arbitrary high orders which allows us to go far beyond the standard Gaussian approximation—post-Gaussian approximations are born out of this idea.

This paper, which is the first one of a series, is arranged in six sections. In the next section we describe a time-dependent variational principle as it was originally formulated by Jackiw and Kerman.[18] The case of a free scalar field theory is analytically solvable and the wave functional of that solution is used as a trial wave functional for field theories including interaction. We derive explicit formulas for field theories with nearly arbitrary potentials which can be utilized for the reader's favorite Hamiltonian. Therefore, the second section can also be considered as a cookery book for all the techniques necessary to apply the Gaussian approximation to arbitrary problems. Our tool box contains recipes to calculate the effective action, effective potentials, and $Z[\phi]$ factors for inhomogeneous background fields. We derive general soliton equations and equations for the fluctuation modes around the solitons. Once again, all this can be done without specifying the Hamiltonian which will be done additionally in a later section to make our ideas clear.

In the third section we will give an alternative to the Jackiw-Kerman procedure, which we call the "variational coupled cluster model"[19,20,21] (VCCM). Although this has been known for years it has not yet been applied to concrete problems, mainly because it is more complicated than the ordinary coupled cluster method. On the other hand it allows one to calculate effective actions, whereas the conventional coupled cluster method[5,22–24] does not, it only gives energy eigenvalues of the Hamiltonian. Certainly, the variational version, is accompanied by an increase of numerical and analytical expense. We can rewrite our tool box in terms of a VCCM language but do not proceed in this way. Actually, it turns out that the Jackiw-Kerman formulation is easier than the VCCM, or better said than its equivalent truncation, since the VCCM is exact in principle and more complicated truncation schemes are indeed post-Gaussian approximations. Such generalizations are missing in the Jackiw-Kerman method and that is the advantage offered by the VCCM.

An example of the ideas described in Secs. II and III is presented in Sec. IV. It is quite natural to consider the sine-Gordon system, since this has been studied utilizing the Gaussian approximation and also semiclassical techniques.[25–28] We derive effective actions, the soliton equation, and the equation of the modes around the soliton. The mode expansion is discussed as a standard procedure to solve the mode equation, both for static an time-dependent solitons. The VCCM is more complicated, so we decided to give details only in the case of the effective potential. Section IV finishes with a remark on effective Lagrangians derived by eliminating the background fields instead of the fluctuations around them.

There are some problems to prove the equivalence of the Jackiw-Kerman method and a specially truncated VCCM in the case of a field theory. Problems become easier in the simple case of a quantum-mechanical model, which such a proof is available and will be reported in detail. The isomorphism, mapping variables of each method onto the variables of the other method, can be constructed. We give comments on θ vacua[29] and discuss the difficulties of how to transfer these to field theories. This and other results are presented for the sine-Gordon quantum mechanics in Sec. V. Our conclusions about the Gaussian approximation are summarized in Sec. VI.

II. THE EFFECTIVE ACTION DERIVED FROM A TIME-DEPENDENT VARIATIONAL PRINCIPLE

The effective action $\Gamma[\phi_0]$ is a powerful instrument[11] to give a comprehensive description of a quantum field theory by an equivalent classical field theory. If we denote the classical action of a $(1+1)$-dimensional field theory with $S_{\rm cl}[\phi_0]$, the effective action can be constructed by functional integration and Legendre transformation. All one-particle-irreducible functions $\Gamma_n(x_1,\ldots,x_n)$ can be obtained from $\Gamma[\phi_0]$ by making use of the identity

$$\Gamma[\phi_0]=\sum_n \frac{1}{n!}\int d^2x_1\cdots d^2x_n\phi_0(x_1)\cdots\phi_0(x_n)\Gamma_n(x_1,\ldots,x_n)\ . \tag{2.1}$$

Additionally to this multilocal expansion (2.1) there is a completely local one

$$\Gamma[\phi_0]=\int d^2x\{\tfrac{1}{2}Z[\phi_0]\partial_\mu\phi_0\partial^\mu\phi_0-V_{\text{eff}}[\phi_0]+\tilde{Z}[\phi_0](\partial_\mu\phi_0\partial^\mu\phi_0)^2+\cdots\}\ . \tag{2.2}$$

This representation is useful for slowly varying fields with small gradients. A standard perturbative technique to evaluate $V_{\text{eff}}[\phi_0],Z[\phi_0],\tilde{Z}[\phi_0]$ and further functions determining $\Gamma[\phi_0]$ is given by the loop expansion which has been described by various authors.[11] The Gaussian approximation utilizes a theorem[30] which can be stated as follows.

Among all normalized states $|\psi\rangle$ with $\langle\psi|\phi(x)|\psi\rangle=\phi_0$ with a constant ϕ_0, $V_{\text{eff}}[\phi_0]$ is the minimum of the energy expectation values $\langle\psi|H|\psi\rangle$.

If one restricts the trial states $|\psi\rangle$ to Gaussian wave functions, $V_{\text{eff}}[\phi_0]$ corresponds to the Gaussian approximation. This theorem can be generalized to a representation of the effective action. Such a theorem exists due to the work of Jackiw and Kerman,[18] which states the following.

$|\psi_-(t)\rangle$ and $|\psi_+(t)\rangle$ are time-dependent states with normalized overlap. Both states tend toward the stationary ground state $|\Omega\rangle$ of the Hamiltonian; i.e., $H|\Omega\rangle=0$,

$$\lim_{t\to\pm\infty}|\psi_\mp(t)\rangle=|\Omega\rangle\ .$$

Therefore, let $\langle\psi_-(t)|\phi(x)|\psi_+(t)\rangle=\phi_0(x,t)$ be a fixed exception value. Then, $\Gamma[\phi_0]$ is the stationary value of

$$\Gamma[\phi_0]=\int_{-\infty}^{+\infty}dt\langle\psi_-(t)|(i\partial_t-H)|\psi_+(t)\rangle\ . \tag{2.3}$$

Unfortunately it is impossible to construct the stationary value analytically besides in the trivial case of a free-field theory with Hamiltonian of the type

$$H=\int dx\left[-\frac{1}{2}\frac{\delta^2}{\delta\phi(x)^2}+\tfrac{1}{2}[\nabla\phi(x)]^2+\tfrac{1}{2}m^2\phi^2(x)-E_0\right] \tag{2.4}$$

and

$$E_0=\int\frac{dk}{4\pi}\sqrt{k^2+m^2}\ . \tag{2.5}$$

The states making $\Gamma[\phi_0]$ stationary can be computed explicitly:

$$|\psi_+(t)\rangle=|\psi_-(t)\rangle\ , \tag{2.6}$$

$$|\psi_+(t)\rangle=\frac{1}{N}\exp\left[-\tfrac{1}{2}\int dx\,dy[\phi(x)-\phi_0(x,t)]\Omega(x,y,t)[\phi(y)-\phi_0(y,t)]\right.$$
$$\left.+i\int dx\{[\phi(x)-\phi_0(x,t)]\Pi_0(x,t)+\tfrac{1}{2}\phi_0(x,t)\Pi_0(x,t)\}\right], \tag{2.7}$$

where Ω and Π_0 are defined by

$$\Omega(x,y,t)=\langle x|\sqrt{-\nabla^2+m^2}|y\rangle$$
$$=\int\frac{dk}{2\pi}\sqrt{k^2+m^2}e^{ik(x-y)} \tag{2.8}$$

and

$$\dot{\phi}_0(x,t)=\Pi_0(x,t)\ . \tag{2.9}$$

If we insert these definitions into (2.3) we finish with the expected action functional

$$\Gamma[\phi_0]=\int d^x[\tfrac{1}{2}\partial_\mu\phi_0(x,t)\partial^\mu\phi_0(x,t)-\tfrac{1}{2}m^2\phi_0^2(x,t)]\ . \tag{2.10}$$

Interacting field theories do not allow us to find an explicit solution for the stationary value of $\Gamma[\phi_0]$. But one can still extremize Γ within the restricted set of Gaussian trial states as defined by (2.6) and (2.7). Certainly, Eqs. (2.8) and (2.9) need not be satisfied any longer. This method to approximate $\Gamma[\phi_0]$ is called the Gaussian approximation.

The first necessary remark has to be that $\Omega(x,y,t)$ is no longer a real function. One has to allow that Ω might have an imaginary part:

$$\Omega(x,y,t)=\tfrac{1}{2}G^{-1}(x,y,t)+\frac{i}{2}I(x,y,t)\ . \tag{2.11}$$

It turns out to be an advantage if both operators $\phi(x)$ and $\Pi(x)$ have given expectation values $\phi_0(x,t)$ and $\Pi_0(x,t)$. All further calculations will be done for Hamiltonians of the type

$$H=N_m\int dx\{\tfrac{1}{2}\Pi(x^2)+\tfrac{1}{2}[\nabla\phi(x)]^2+V(\phi(x))\}\ . \tag{2.12}$$

They are normal ordered with respect to some mass m which may be the classical one and is determined by

$$V''(\phi_0)=m^2\ , \tag{2.13}$$

where ϕ_0 is defined by $V'(\phi_0)=0$. It is now straightforward to evaluate (2.3). We give some useful formulas for such a calculation, e.g.,

$$\int dt\langle\psi_-(t)|i\partial_t|\psi_+(t)\rangle$$
$$=\int d^2x[\Pi_0(x,t)\dot{\phi}_0(x,t)-\tfrac{1}{4}\langle x|I(t)\dot{G}(t)|x\rangle]\ , \tag{2.14}$$

where an operator notation has been introduced to work sufficiently with bilocal fields $I(x,y,t)$ and $G(x,y,t)$:

$$I(x,y,t)=\langle x \mid I(t) \mid y \rangle = \langle y \mid I(t) \mid x \rangle \tag{2.15}$$

and

$$\langle x \mid I(t)\dot{G}(t) \mid x \rangle = \int dy \langle x \mid I(t) \mid y \rangle \langle y \mid \dot{G}(t) \mid x \rangle \tag{2.16}$$

and

$$\langle x \mid I^2(t)G(t) \mid x \rangle = \int dy\, dz \langle x \mid I(t) \mid y \rangle \langle y \mid I(t) \mid z \rangle \langle z \mid G(t) \mid x \rangle \ . \tag{2.17}$$

This nomenclature simplifies the following calculations and makes the representation of $\Gamma[\phi_0]$ more comprehensive. Other typical matrix elements appearing in $\Gamma[\phi_0]$ are

$$\int d^2x \langle \psi_-(t) \mid \tfrac{1}{2}N_m[\nabla\phi(x)]^2 \mid \psi_+(t) \rangle = \int d^2x \{ \tfrac{1}{2}[\nabla\phi_0(x,t)]^2 + \tfrac{1}{2}\nabla_x\nabla_y \langle x \mid G(t) \mid y \rangle_{x=y} \} \tag{2.18}$$

and also

$$\int d^2x \langle \psi_-(t) \mid \tfrac{1}{2}N_m \Pi^2(x) \mid \psi_+(t) \rangle = \int d^2x \{ \tfrac{1}{2}\Pi_0{}^2(x,t) + \tfrac{1}{8}\langle x \mid [G^{-1}(t)+I^2(t)G(t)] \mid x \rangle \} \ . \tag{2.19}$$

Finally, the expectation value of $N_m(V(\phi))$ has to be calculated for arbitrary potentials $V(\xi)$. We assume that $V(\xi)$ has a Fourier representation at least in the sense of Schwartz distributions. Then we can use Coleman's theorem[31] and obtain

$$\begin{aligned}\langle \psi_-(t) \mid N_m V(\phi(x)) \mid \psi_+(t) \rangle &= \int d\beta\, \tilde{V}(\beta) \langle \psi_-(t) \mid N_m e^{i\beta\phi(x)} \mid \psi_+(t) \rangle \\ &= \int d\beta\, \tilde{V}(\beta) e^{i\beta\phi_0(x,t)} \exp\{ -\tfrac{1}{2}\beta^2 [\langle x \mid G(t) \mid x \rangle - \langle \phi^2 \rangle_m] \} \\ &= \int \frac{d\alpha}{\sqrt{2\pi}} e^{-\alpha^2} V(\alpha\sqrt{Q(x,t)} + \phi_0(x,t)) \ ,\end{aligned} \tag{2.20}$$

where we defined

$$\langle \phi^2 \rangle_m = \int \frac{dk}{4\pi\sqrt{k^2+m^2}} \tag{2.21}$$

and furthermore

$$Q(x,t) = \langle x \mid G(t) \mid x \rangle - \langle \phi^2 \rangle_m \ . \tag{2.22}$$

Equation (2.20) tells us that all details of the interaction are of minor importance to perform an explicit calculation of $\Gamma[\phi_0]$. Indeed, we show soon that results can be obtained without specifying $V(\xi)$. It is advantageous to define functions of the type

$$v_n(x,y) = \int \frac{d\alpha}{\sqrt{2\pi}} e^{-\alpha^2/2} V^{(n)}(\alpha\sqrt{x} + y) \ , \tag{2.23}$$

where $V^{(n)}$ is the nth derivative of the function $V(\xi)$. These v_n have an easy interpretation which we will give later on. Now we combine all intermediate results to a final effective action:

$$\begin{aligned}\Gamma[\phi_0] = \int d^2x \{ &\Pi_0(x,t)\dot{\phi}_0(x,t) - \tfrac{1}{2}\Pi_0{}^2(x,t) - \tfrac{1}{2}[\nabla\phi_0(x,t)]^2 \\ &- v_0(Q(x,t),\phi_0(x,t)) - \tfrac{1}{4}\langle x \mid I(t)\dot{G}(t) \mid x \rangle - \tfrac{1}{8}\langle x \mid G^{-1}(t) \mid x \rangle \\ &- \tfrac{1}{8}\langle x \mid I^2(t)G(t) \mid x \rangle - \tfrac{1}{2}m^2\langle \phi^2 \rangle_m + E_0 - \tfrac{1}{2}\nabla_x\nabla_y \langle x \mid G(t) \mid y \rangle \mid_{x=y} \} \ .\end{aligned} \tag{2.24}$$

We can either stop at this point or do some cosmetics to find a more elegant notation and representation of $\Gamma[\phi_0]$. The functional $\Gamma[\phi_0]$ can be made stationary with respect to Π_0 and $I(t)$ leading to equations

$$\Pi_0(x,t) = \dot{\phi}_0(x,t) \ , \tag{2.25a}$$

$$I(t) = -\dot{G}(t)G^{-1}(t) \ . \tag{2.25b}$$

It should be noted that $I(t)$ obviously vanishes for the static case. Furthermore, it is favorable to introduce a bilocal field ψ defined by

$$\langle x \mid \psi(t) \mid y \rangle = \psi(x,y,t) = \langle x \mid \sqrt{G(t)} \mid y \rangle \tag{2.26}$$

and another free bilocal field ψ_0 to represent the counterterms E_0 and $\langle \phi^2 \rangle_m$. It can be defined as

$$\psi_0(x,y,t) = \langle x \mid \psi_0(t) \mid y \rangle = \int \frac{dk}{2\pi} \frac{e^{ik(x-y)}}{(2\sqrt{k^2+m^2})^{1/2}} \tag{2.27}$$

and solves the equation

$$(-\partial_\mu\partial^\mu - m^2)\langle x \mid \psi_0(t) \mid y\rangle + \tfrac{1}{4}\langle x \mid \psi_0{}^{-3}(t) \mid y\rangle = 0 \ . \tag{2.28}$$

Eliminating Π_0 and $I(t)$ from $\Gamma[\phi_0]$ by making use of Eqs. (2.25) and expression Γ in terms of ψ and ψ_0 gives an adequate formulation of $\Gamma[\phi_0]$ for an arbitrary potential $V(\xi)$:

$$\Gamma[\phi_0] = \int d^2x [\tfrac{1}{2}\partial_\mu\phi_0(x,t)\partial^\mu\phi_0(x,t) - v_0(Q(x,t),\phi_0(x,t))] + \int dx\, dy\, dt \{ \tfrac{1}{2}\partial_\mu^x\psi(x,y,t)\partial_x^\mu\psi(x,y,t) - \tfrac{1}{8}[\psi^{-1}(x,y,t)]^2 - \tfrac{1}{2}\partial_\mu^x\psi_0(x,y,t)\partial_x^\mu\psi_0(x,y,t) + \tfrac{1}{8}[\psi_0{}^{-1}(x,y,t)]^2 \} \ , \tag{2.29}$$

where $\psi^{-1}(x,y,t)$ is the x representation of the inverse operator of $\psi(t)$. $Q(x,t)$ will be rewritten as

$$Q(x,t) = \langle x \mid \psi^2(t) - \psi_0{}^2(t) \mid x\rangle \ . \tag{2.30}$$

Let us make some remarks concerning the final formula (2.29). It still depends on two fields, the background field $\phi_0(x,t)$ and a bilocal field $\psi(x,y,t)$, which contain all information about the quantum fluctuations around the background field. The fields are coupled via the interaction $v_0(Q,\phi_0)$ whenever terms such as $Q^p(x,t)\phi_0^q(x,t)$ appear. Only for a free-field theory, i.e., $V(\xi)=\frac{1}{2}m^2\xi^2$, can it be shown that the fields do not interact and indeed (2.29) reduces to (2.10), as can certainly be expected. We will now derive some general formulas from (2.29) which will be applied to special models in Sec. IV. $\Gamma[\phi_0]$ is actually not (2.29) but the stationary value of it with respect to the bilocal field $\psi(x,y,t)$. Therefore, ψ has to obey the Euler-Lagrange equation which becomes

$$[-\partial_\mu\partial^\mu - v_2(Q(x,t),\phi_0(x,t))]\langle x \mid \psi(t) \mid y\rangle + \tfrac{1}{4}\langle x \mid \psi^{-3}(t) \mid y\rangle = 0 \ . \tag{2.31}$$

Once again, for a free-field theory $v_2=m^2$ and (2.31) reduces to (2.28). ψ and ψ_0 coincide, $Q(x,t)$ vanishes, and (2.29) agrees with (2.10) as stated before. The more general case of an arbitrary $V(\xi)$ cannot be solved without numerical attempts or expansion methods which we will discuss now. It may be possible to solve (2.31) by a mode expansion

$$\langle x \mid \psi(t) \mid y\rangle = \sum_\alpha \frac{1}{\sqrt{2\epsilon_\alpha}}\eta_\alpha(x,t)\eta_\alpha(y,t) \tag{2.32}$$

which transforms (2.31) into an infinite number of coupled differential equations:

$$[\partial_\mu\partial^\mu + v_2(Q(x,t),\phi_0(x,t))]\eta_\alpha(x,t) = \epsilon_\alpha{}^2\eta_\alpha(x,t) - J_\alpha(x,t) \tag{2.33}$$

and

$$J_\alpha(x,t) = \sum_\beta \left[\frac{\epsilon_\alpha}{\epsilon_\beta}\right]^{1/2} \left[2\dot\eta_\beta(x,t)\int dy\ \eta_\alpha(y,t)\dot\eta_\beta(y,t) + \eta_\beta(x,t)\int dy\ \eta_\alpha(y,t)\ddot\eta_\beta(y,t)\right] . \tag{2.34}$$

Unfortunately, the system is highly coupled and requires a self-consistent method for a unique solution. Both $Q(x,t)$ and $J_\alpha(x,t)$ depend on the modes $\eta_\alpha(x,t)$ themselves. The static case may be more interesting at the moment, where it is known[13–15] that the mode expansion is efficient and converges quite rapidly. Equation (2.33) reduces to

$$[-\nabla^2 + v_2(Q(x),\phi_0(x))]\eta_\alpha(x) = \epsilon_\alpha{}^2\eta_\alpha(x) \tag{2.35}$$

and

$$Q(x) = \sum_\alpha \frac{\eta_\alpha{}^2(x)}{2\epsilon_\alpha} - \int \frac{dk}{4\pi\sqrt{k^2+m^2}} \ . \tag{2.36}$$

We postpone a further discussion of these equations and an additional one, which we will call the soliton equation, to the fourth section. A soliton is a stationary value of $\Gamma[\phi_0]$ and obeys the equation

$$\partial_\mu\partial^\mu\phi_0(x,t) + v_1(Q(x,t),\phi_0(x,t)) = 0 \tag{2.37}$$

with additional requirements concerning nontrivial boundary conditions at infinity. This may be expressed by a topological charge related to the zeros of the function v_1. Instantons are related to the Euclidean versions of our equations which needs no further explanation.

The next point of our discussion is a recipe to construct a local expansion of the effective action as defined by Eq. (2.2). We restrict ourselves here to $V_{\text{eff}}[\phi_0]$ and $Z[\phi_0]$, since they can be extracted in a rather simple way. It is sufficient to consider background fields of the kind

$$\phi_0(x,t) = \phi_0 + \epsilon F(t) \tag{2.38}$$

with constant ϕ_0, infinitesimal ϵ, and an odd function $F(t)$. Equation (2.31) will be solved with these restrictions by a linear response theory. Since the system is still translationally invariant in x space, we search for solutions of (2.31) which are of the type

$$\psi(x,y,t) = \psi(x-y) + \epsilon\lambda(x-y,t) \ . \tag{2.39}$$

The assumption of a linear response theory allows us to stop the ϵ expansion after the linear term. It is possible to calculate the functions $\tilde Z[\phi_0], \ldots$ of (2.2) also, but this requires a nonlinear response theory and all analytical calculations become inelegant. Because of the translational invariance it is advantageous to solve (2.31) in momentum space:

$$\psi(q,t) = \psi(q) + \epsilon\lambda(q,t) \ . \tag{2.40}$$

Inserting (2.39) into (2.31) and utilizing the momentum representation (2.40) leads to two equations for $\psi(q)$ and $\lambda(q,t)$. The $\psi(q)$ equation is solved by

$$\psi(q)=\frac{1}{2(\sqrt{q^2+\mu^2})^{1/2}}\ , \tag{2.41}$$

where the so-called Hartree mass μ obeys to the equation

$$\mu^2=v_2(Q,\phi_0) \tag{2.42}$$

and

$$Q=\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}\ . \tag{2.43}$$

$\lambda(q,t)$ is a solution of a more complicated integro-differential equation

$$\ddot{\lambda}(q,t)+4(q^2+\mu^2)\lambda(q,t)=-\frac{1}{(2\sqrt{q^2+\mu^2})^{1/2}}\left[F(t)v_3(Q,\phi_0)+v_4(Q,\phi_0)\int\frac{dk}{2\pi}\frac{\lambda(k,t)}{(2\sqrt{k^2+\mu^2})^{1/2}}\right]. \tag{2.44}$$

It can be proved that $\ddot{\lambda}(q,t)$ has to be ignored in (2.44) because it does not contribute to the $Z[\phi_0]$ factor of (2.2). It would contribute to another $Z'[\phi_0]$ factor, appearing as a coefficient of $\partial_\mu\partial_\nu\phi_0(x,t)\partial^\mu\partial^\nu\phi_0(x,t)$. Details concerning this complication may be found in Sec. V on sine-Gordon quantum mechanics. Finally, $\lambda(q,t)$ becomes

$$\lambda(q,t)=-\frac{v_3}{4\sqrt{2}}\ \frac{1}{1+v_4/8\pi\mu^2}\ \frac{F(t)}{(q^2+\mu^2)^{5/4}}+O(\ddot{F}(t))\ . \tag{2.45}$$

If we insert $\psi(q,t)=\psi(q)+\epsilon\lambda(q,t)$ into (2.29) and compare with (2.2), $Z[\phi_0]$ and $V_{\text{eff}}[\phi_0]$ are easily obtained:

$$V_{\text{eff}}=\frac{\mu^2-m^2}{8\pi}+v_0(Q,\phi_0)\ , \tag{2.46}$$

$$Z[\phi_0]=1+\frac{{v_3}^2}{48\pi\mu^2}\ \frac{1}{(1+v_4/8\pi\mu^2)^2}\ , \tag{2.47}$$

where, as we already mentioned, μ is a solution of (2.42) and (2.43). Some trivial steps may convince you that $Z[\phi_0]$ can also be written as

$$Z[\phi_0]=1+\frac{1}{48\pi}\left[\frac{d\ln\mu^2[\phi_0]}{d\phi_0}\right]^2 \tag{2.47'}$$

which can be utilized, if the ϕ_0 dependence of μ is known. Finally, we give an interpretation of the c numbers $v_n(Q,\phi_0)$ as effective coupling constants. Let ϕ_0 be a fixed but constant quantity. Instead of taking matrix elements it is also possible to evaluate the effect of a shift from a normal ordering with respect to the original classical mass m to a normal ordering with respect to the Hartree mass μ. The result is simply

$$\begin{aligned}H&=N_m\int dx[\tfrac{1}{2}\Pi^2+\tfrac{1}{2}(\nabla\phi)^2+V(\phi)]\\&=LV_{\text{eff}}[\phi_0]+N_\mu\int dx[\tfrac{1}{2}\Pi^2+\tfrac{1}{2}(\nabla\phi)^2+v_0(Q,\phi_0+\phi)\\&\qquad-v_0(Q,\phi_0)]\ .\end{aligned} \tag{2.48}$$

Obviously, a constant has been added, the normal-ordering mass has been changed, and the potential $V(\phi)$ has been mapped onto a new potential $v_0(Q,\phi_0+\phi)-v_0(Q,\phi_0)$, which may be expanded in powers of $\phi(x)$ to yield

$$\begin{aligned}H=LV_{\text{eff}}+N_\mu\int dx\Bigg[&\tfrac{1}{2}\Pi^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}\mu^2\phi^2\\&+v_1(Q,\phi_0)\phi+\sum_{n\geq 3}\frac{v_n(Q,\phi_0)}{n!}\phi^n\Bigg]\ .\end{aligned} \tag{2.49}$$

It is now very transparent to interpret the v_n as coupling constants of the ϕ^n interaction. The Hamiltonian (2.49) is the starting point for improvements of $V_{\text{eff}}[\phi_0]$ going beyond the Gaussian approximation. This may be done either by ordinary perturbation theory or by more sophisticated techniques such as the coupled cluster method.[5]

III. THE EFFECTIVE ACTION IN THE FRAMEWORK OF THE VARIATIONAL CLUSTER METHOD

The theoretical concepts as developed in the last chapter are basically due to the possibility to evaluate Gaussian path integrals. Trial functionals more complicated than those used before do not allow the construction of $\Gamma[\phi_0]$ without perturbative expansions. It is nevertheless feasible to extend the formalism to a most general time-dependent *Ansatz* which is known as the VCCM. A detailed analysis of this method in connection with applications in many-body theory was given by Arponen[19] in the past. The fundamental idea is to use a special representation of the states $|\psi_-(t)\rangle$ and $|\psi_+(t)\rangle$: namely,

$$|\psi_+(t)\rangle=e^{S(t)}|0\rangle\ , \tag{3.1}$$

where

$$S(t)=\sum_{n=1}^{\infty}S_n(t)=\sum_{n=1}^{\infty}\frac{1}{n!}\int dx_1\cdots dx_nS_n(x_1,\ldots,x_n,t)\phi^{(-)}(x_1)\cdots\phi^{(-)}(x_n)\ . \tag{3.2}$$

The field operator was decomposed into a creation part $\phi^{(-)}(x)$ and an annihilation part $\phi^{(+)}(t)$ with $\phi^{(+)}(t)|0\rangle=0$. Similarly $\langle\psi_-(t)|$ will be expanded as

$$\langle\psi_-(t)|=\langle 0|e^{S''(t)}e^{-S(t)}\ , \tag{3.3}$$

where

$$S''(t)=\sum_{n=1}^{\infty} S_n''(t)=\sum_{n=1}^{\infty}\frac{1}{n!}\int dx_1\cdots dx_n S_n''(x_1,\ldots,x_n,t)\phi^{(+)}(x_1)\cdots\phi^{(+)}(x_n)\ . \tag{3.4}$$

These *Ansätze* are universal and exact as long as an infinite number of so-called correlation amplitudes (S_n,S_n'') is taken into account. Certainly, in practical calculations one has to truncate the infinite set somewhat but systematic improvements of low-order truncations can be obtained by including more and more amplitudes. This is an enormous advantage of the VCCM compared with the Jackiw-Kerman method, which is always restricted to Gaussian trial states only. There is a complete agreement between both procedures if the set of (S_n,S_n'') amplitudes is truncated after (S_2,S_2''), that means $S_n=S_n''=0$ for $n>2$. Let us discuss the special *Ansätze* (3.1)—(3.4) at first. The overlap of both states is obviously normalized. Furthermore, the requirement that the expectation value of $\phi(x)$ is identical with a given function $\phi_0(x,t)$ is a first subsidiary condition for $S_1(x,t)$. If we require that the expectation value of $\Pi(x)$ also agrees with a given function $\Pi_0(x,t)$, this can be translated into another subsidiary condition for $S_1''(x,t)$. Therefore, $S_1(x,t)$ and $S_1''(x,t)$ are both already determined by ϕ_0 and Π_0. The original set $\{S_n,S_n''\}_{n\geq 1}$ can be replaced by an equivalent set $\{\phi_0,\Pi_0,(S_n)_{n>1},(S_n'')_{n>1}\}$ of field variables. Inserting the *Ansätze* (3.1)—(3.4) into (2.3) yields (note the trick to insert $e^{-S''}$ since $S''\,|\,0\rangle=0$)

$$\Gamma[\phi_0,\Pi_0,(S_n)_{n>1},(S_n'')_{n>1}]$$
$$=\int dt\langle 0\,|\,e^{S''(t)}e^{-S(t)}(i\partial_t-H)e^{S(t)}e^{-S''(t)}\,|\,0\rangle\ . \tag{3.5}$$

Equation (3.5) can be considered as a mapping of a quantum field theory with a scalar quantum field $\phi(x)$ onto a classical field theory with an infinite number of classical fields, some of them being local like ϕ_0 and Π_0, the others are multilocal like $(S_n,S_n'')_{n>1}$. This is similar to the Green's-function formulation but there the additional fields are not only multilocal in space but also in the time variables. Our description by Eqs. (3.1)—(3.5) reduces the number of variables by a coincidence of all time variables corresponding to the different locations $x_1\cdots x_n$.

The true effective action $\Gamma[\phi_0]$ is defined by

$$\Gamma[\phi_0]=\min_{\Pi_0,(S_n,S_n'')_{n>1}}\Gamma[\phi_0,\Pi_0,(S_n)_{n>1},(S_n'')_{n>1}] \tag{3.6}$$

and leads to an infinite set of Euler-Lagrange equations for the S_n's and S_n'''s. In concrete calculations utilizing the VCCM it is hard to go beyond low-order truncations of the set $(S_n,S_n'')_{n>1}$. We will restrict ourselves to two cases:

$$(S_n,S_n'')_{n>1}=0 \quad \text{case } A\ ,$$
$$(S_n,S_n'')_{n>2}=0 \quad \text{case } B\ ,$$

and will do most of our calculations in the scheme B. To evaluate (3.5) for Hamiltonians of the type (2.12) similar formulas like those derived in the second section are necessary. In the calculation scheme B we obtain

$$\int dt\langle 0\,|\,e^{S''}e^{-S}i\partial_t e^{S}e^{-S''}\,|\,0\rangle$$
$$=\int d^2x\left[\Pi_0(x,t)\dot{\phi}_0(x,t)+\frac{i}{2}\langle x\,|\,T_2''\dot{T}_2\,|\,x\rangle\right], \tag{3.7}$$

where the operator notation as introduced before has been used again. Furthermore, T_2'' and T_2 are defined by

$$T_2=\sqrt{\Delta}S_2\sqrt{\Delta},\quad T_2''=\sqrt{\Delta}S_2''\sqrt{\Delta}\ , \tag{3.8}$$

and

$$\langle x\,|\,\Delta\,|\,y\rangle=[\phi^{(+)}x,\phi^{(-)}(y)]$$
$$=\int\frac{dk}{4\pi}\frac{e^{ik(x-y)}}{\sqrt{k^2+m^2}}\ . \tag{3.9}$$

Similarly some computations yield

$$\int d^2x\tfrac{1}{2}\langle 0\,|\,e^{S''}e^{-S}N_m\Pi^2(x)e^{S}e^{-S''}\,|\,0\rangle=\int d^2x\left[\tfrac{1}{2}\Pi_0^2(x,t)\right.$$
$$\left.-\tfrac{1}{8}\left\langle x\left|\frac{1}{\sqrt{\Delta}}(T_2+T_2''-T_2T_2''-T_2''T_2+T_2T_2''T_2)\frac{1}{\sqrt{\Delta}}\right|x\right\rangle\right] \tag{3.10}$$

and also

$$\int d^2x\tfrac{1}{2}\langle 0\,|\,e^{S''}e^{-S}N_m[\nabla\phi(x)]^2e^{S}e^{-S''}\,|\,0\rangle=\int d^2x\{\tfrac{1}{2}[\nabla\phi_0(x,t)]^2$$
$$-\tfrac{1}{2}\nabla_x{}^2\langle x\,|\,\sqrt{\Delta}(T_2+T_2''+T_2T_2''+T_2''T_2$$
$$+T_2T_2''T_2)\sqrt{\Delta}\,|\,y\rangle\,|_{x=y}\}\ . \tag{3.11}$$

The matrix element of $V(\phi(x))$ agrees with (2.20), but $Q(x,t)$ is now defined by

$$Q=\sqrt{\Delta}(T_2+T_2''+T_2T_2''+T_2''T_2+T_2T_2''T_2)\sqrt{\Delta}\ . \tag{3.12}$$

This agreement gives us a hint that the truncation scheme B is probably identical with the procedure developed in the second section. Nevertheless, an explicit mapping from (T_2,T_2'') onto ψ has to be constructed to prove this conjecture. We switch to the truncation scheme A for a moment and ignore T_2 and T_2'' in all matrix elements. This maps $\Gamma[\phi_0]$ onto its classical expression

$$\Gamma[\phi_0,\Pi_0]=\int d^2x\{\Pi_0(x,t)\dot{\phi}_0(x,t)-H[\phi_0,\Pi_0]\}\ , \tag{3.13}$$

where $H[\phi_0,\Pi_0]$ is the Hamiltonian after replacing operators $\phi(x)$ and $\Pi(x)$ by their expectation values. Truncation scheme A is the trivial approximation by a classical theory ignoring all quantum effects. Scheme B results in the effective action

$$\Gamma[\phi_0,T_2,T_2'']=\int d^2x[\tfrac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0-v_0(Q,\phi_0)]+\int d^2x\,\mathscr{L}(T_2,T_2'',\dot{T}_2,\dot{T}_2'') \tag{3.14}$$

with

$$\mathscr{L}=\left\langle x\left|\left[\frac{i}{2}T_2''\dot{T}_2+\frac{1}{8}\frac{1}{\sqrt{\Delta}}(T_2+T_2''-T_2T_2''-T_2''T_2+T_2T_2''T_2)\frac{1}{\sqrt{\Delta}}\right]\right|x\right\rangle+\tfrac{1}{2}\nabla_x{}^2\langle x\mid Q\mid y\rangle\mid_{x=y}\ . \tag{3.15}$$

Comparing (2.29) and (3.14) demonstrates that the quantum effects described originally by ψ are now described comprehensively by two fields T_2 and T_2''. It has to be stressed that the first-order Lagrangian (3.15) leads to equations of motion for T_2 and T_2'' which are also first-order differential equations in the time variable. On the other hand the equation of motion for ψ was of second order. To map both procedures onto each other it is necessary to solve one equation of motion for T_2 or T_2'' and insert it into the other equation, which then becomes also a second-order differential equation. Unfortunately, this is harder to execute than to write down.

What else can be said about the VCCM and its usefulness? The formulation is obviously more complicated than the Jackiw-Kerman method. But it is flexible, can be extended systematically to post-Gaussian approximations, and it is challenging to investigate it in a general content. In the next section we return to it in a simpler case to demonstrate its equivalence with the Jackiw-Kerman method at least for constant background fields ϕ_0. Before doing this we present an example how to use all the concepts developed in Secs. II and III for a simple and well-known model.

IV. APPLICATION TO THE SINE-GORDON SYSTEM

This famous model has been studied intensively in the past using many different methods and techniques. It is widely believed that some statements found are probably exact. These are concerned with mass ratios of soliton masses. The model is defined by

$$V(\phi)=\frac{m^4}{g}\left[1-\cos\left[\frac{\sqrt{g}}{m}\phi\right]\right]\ . \tag{4.1}$$

Some simple calculation steps immediately lead to

$$Z[\phi_0]=1+\frac{g}{48\pi m^2}\frac{\tan^2\left[\frac{\sqrt{g}}{m}\phi_0\right]}{\left[1-\frac{g}{8\pi m^2}\right]^2}\ , \tag{4.2}$$

$$V_{\text{eff}}[\phi_0]=\frac{m^2}{8\pi}\left[1-\frac{8\pi m^2}{g}\right]\times\left\{\left[\cos\left[\frac{\sqrt{g}}{m}\phi_0\right]\right]^{1/(1-g/8\pi m^2)}-1\right\}\ . \tag{4.3}$$

Both functions become singular at $g/m^2=8\pi$. This bound was already found by Coleman. He proved that the Hamiltonian becomes unbounded for values of g/m^2 greater than 8π. Furthermore, we have to require $|(\sqrt{g}/m)\phi_0|<\pi/2$. Otherwise $Z[\phi_0]$ develops a pole and $V_{\text{eff}}[\phi_0]$ an imaginary part. This bound can also be found using the loop expansion. The meson mass, which is $\sqrt{V_{\text{eff}}[\phi_0]}$, becomes imaginary at values $\pi/2<|(\sqrt{g}/m^2)\phi_0|<3\pi/2$ indicating instability. For later use we switch to the description of solitons in the sine-Gordon model. The soliton equation (2.7) becomes

$$\partial_\mu\partial^\mu\phi_0(x,t)+\frac{m^3}{\sqrt{g}}\exp\left[-\frac{1}{2}\frac{g}{m^2}Q(x,t)\right]\sin\left[\frac{\sqrt{g}}{m}\phi_0\right]=0\ . \tag{4.4}$$

It should describe time-dependent solitons in the model-like breather modes but also scattering of two or even more solitons. If we ignore the quantum effect $Q(x,t)$ Eq. (4.4) agrees with the classical equation of motion which definitively shows such a type of solutions. Quantum effects themselves are comprehensively formulated by Eq. (2.31), which here leads to

$$\left[\partial_\mu\partial^\mu+m^2\exp\left[-\frac{1}{2}\frac{g}{m^2}Q(x,t)\right]\cos\left[\frac{\sqrt{g}}{m}\phi_0(x,t)\right]\right]\times\langle x\mid\psi(t)\mid y\rangle-\tfrac{1}{4}\langle x\mid\psi^{-3}(t)\mid y\rangle=0\ . \tag{4.5}$$

Since we are interested in static Hartree solitons, which are quantum solitons related to the classical soliton solution

$$\phi_{\text{cl}}(x)=\frac{4m}{\sqrt{g}}\arctan(e^{mx}) \tag{4.6}$$

by an improvement via the fluctuation ψ, we again give the concrete system of equations to be solved:

$$-\nabla^2\phi_0(x)+\frac{m^3}{\sqrt{g}}\exp\left[-\frac{1}{2}\frac{g}{m^2}Q(x)\right]\sin\left[\frac{\sqrt{g}}{m}\phi_0(x)\right]=0\ ,$$

$$Q(x)=\sum_\alpha \frac{\eta_\alpha^2(x)}{2\epsilon_\alpha}-\int\frac{dk}{4\pi\sqrt{k^2+m^2}}\ , \tag{4.7}$$

$$\left[-\nabla_x^2+m^2\exp\left[-\frac{1}{2}\frac{g}{m^2}Q(x)\right]\cos\left[\frac{\sqrt{g}}{m}\phi_0(x)\right]\right]\times\eta_\alpha(x)=\epsilon_\alpha^2\eta_\alpha(x)\ .$$

This system of equations will be analyzed in detail in a forthcoming publication.[15] Here we only sketch its global properties; all numerical details may be found in Ref. 15. After some little algebra we end with the formulas for the soliton mass in units of the classical mass m:

$$\frac{m_{\rm sol}}{m}=\int dy\sum_\alpha \epsilon_\alpha B_{y\alpha}^2\,|\langle\alpha\,|\,y\rangle|^2 + \int d\hat{x}\left\{-\tfrac{1}{2}\hat{\phi}_0(\hat{x})\exp\left[-\frac{1}{2}\frac{g}{m^2}Q(\hat{x})\right]\sin[\hat{\phi}_0(\hat{x})]-\tfrac{1}{2}Q(\hat{x})+\frac{m^2}{g}\left[1-\exp\left[-\frac{1}{2}\frac{g}{m^2}Q(\hat{x})\right]\cos[\hat{\phi}_0(\hat{x})]\right]\right\}, \tag{4.8}$$

where

$$\hat{x}=mx\ ,$$

$$\hat{\phi}_0(\hat{x})=\frac{\sqrt{g}}{m}\phi_0(mx)\ ,$$

$$B_{y\alpha}=\frac{1}{2}\left[\left[\frac{\epsilon_y}{\epsilon_\alpha}\right]^{1/2}-\left[\frac{\epsilon_\alpha}{\epsilon_y}\right]^{1/2}\right]\ ,$$

$$\epsilon_y=\sqrt{y^2+1}\ , \tag{4.9}$$

$$\epsilon_\alpha=\text{modified stability frequencies}\ ,$$

$$Q(x)=\sum_\alpha\frac{1}{2\epsilon_\alpha}|\eta_\alpha(x)|^2-\int\frac{dy}{4\pi\epsilon_y}\ ,$$

$$\langle\alpha\,|\,y\rangle=\int\frac{d\hat{x}}{\sqrt{2\pi}}e^{iy\hat{x}}\eta_\alpha^*(\hat{x})\ .$$

Solving the system of equations (4.7), we observe the following results. The function $\phi_0(x)$ is a modified sine-Gordon soliton which resembles the classical soliton very tightly. The energy of the first excited state is now different from zero and we do not have a zero mode. Additionally, with increasing coupling strength there emerges one more localized excited state out of the continuum part of the spectrum (at $\sqrt{g}/m\approx 1$). If the coupling strength tends toward larger values of g/m^2, i.e., $g/m^2\to 8\pi$, we find more and more localized states of the underlying soliton solution. Our conjecture is that at the "critical point" $g/m^2=8\pi$ we have an infinite number of such states. The quantum sine-Gordon soliton has the same shape as the classical soliton with some minor modifications. We computed the soliton mass as defined by Eq. (6.4) and compared our result with the two-loop formula[33,34] of the semiclassical method (Table I):

$$\frac{m_{\rm sol}}{m}=\frac{8m^2}{g}-\frac{1}{\pi}-\frac{g}{192m^2}\ . \tag{4.10}$$

In the weak-coupling limit both approximations agree excellently. For larger values of the coupling strength we note remarkable deviations which are not surprising as the two-loop expansion describes the mass only for small

TABLE I. Contains the numerical results of the calculation. ω_0 and ω_1 are the energies of the zero mode and the first excited mode. $E_{\rm sol}$ is the contribution of the second integral of Eq. (4.8) whereas FI refers to the first integral. $E_{\rm tot}$ is our final result for the soliton mass and $E_{\rm sc}$ has been obtained using the two-loop formula Eq. (4.10) of the semiclassical calculation.

g	ω_0	ω_1	$E_{\rm sol}$	FI	$E_{\rm tot}$	$E_{\rm sc}$
0.1	0.005	1.049	7.9430×10^{2}	4.3497	7.9865×10^{2}	7.9968×10^{2}
0.2	0.011	1.047	1.9684×10^{2}	2.6320	1.9948×10^{2}	1.9968×10^{2}
0.3	0.018	1.045	8.6588×10^{1}	1.9674	8.8555×10^{1}	8.8570×10^{1}
0.4	0.026	1.042	4.8150×10^{1}	1.5498	4.9700×10^{1}	4.9681×10^{1}
0.5	0.035	1.039	3.0434×10^{1}	1.2703	3.1704×10^{1}	3.1680×10^{1}
1.0	0.078	0.999	7.0539	0.6973	7.7512	7.6765
1.5	0.109	0.978	2.8289	0.5158	3.3447	3.2255
2.0	0.131	0.943	1.3790	0.4322	1.9112	1.6609
2.5	0.144	0.890	7.1107×10^{-1}	0.3896	1.1007	9.2914×10^{-1}
3.0	0.145	0.812	3.3241×10^{-1}	0.3822	7.1458×10^{-1}	5.2370×10^{-1}
3.5	0.136	0.698	6.6314×10^{-2}	0.4127	4.7902×10^{-1}	2.7095×10^{-1}
4.0	0.112	0.541	-1.8933×10^{-1}	0.5093	3.1993×10^{-1}	9.8357×10^{-2}
4.5	0.070	0.322	-6.2866×10^{-1}	0.8247	1.9600×10^{-1}	-2.8717×10^{-2}

values of the coupling strength correctly. The soliton mass is positive in our approach for large values of g/m^2 while the semiclassical approximation is already negative indicating instability of the soliton sector. We calculated the soliton mass for the critical value of g/m^2 and found besides very slow convergence behavior also an "infinite negative" soliton mass in our approach. This singular behavior arises in agreement with the former analysis by Coleman. Thus the Gaussian approximation for the soliton mass of the sine-Gordon model provides us with convincing and very reasonable results.

We now return to the variational coupled cluster method to derive at least $V_{\text{eff}}[\phi_0]$ in that scheme. Since background fields are constant we can make use of a translationally invariant *Ansatz*

$$T_2(x,y,t)=\int \frac{dk}{2\pi} T_2(k) e^{ik(x-y)} \tag{4.11}$$

and similarly for $T_2''(x,y,t)$. Q is simply a constant depending on T_2, T_2'':

$$Q=\int \frac{dk}{4\pi}\{T_2(k)+T_2''(k)[1+T_2(k)]^2\} . \tag{4.12}$$

Varying $\Gamma[\phi_0]$ with respect to T_2'' and T_2 yields the equations

$$\epsilon_k{}^2\left[\frac{1-T_2(k)}{1+T_2(k)}\right]^2=k^2+v_2(Q,\phi_0) \tag{4.13}$$

and

$$\epsilon_k{}^2\frac{1-2T_2''(k)[1-T_2(k)]}{1+2T_2''(k)[1+T_2(k)]}=k^2+v_2(Q,\phi_0) . \tag{4.14}$$

Let use define a mass parameter μ by $v_2(Q,\phi_0)=\mu^2$. Solving Eqs. (4.13) and (4.14) yields explicit representations of T_2 and T_2'':

$$T_2(k)=\frac{\sqrt{k^2+m^2}-\sqrt{k^2+\mu^2}}{\sqrt{k^2+m^2}+\sqrt{k^2+\mu^2}} , \tag{4.15}$$

$$T_2''(k)=\frac{m^2-\mu^2}{4(\sqrt{k^2+m^2}+\sqrt{k^2+\mu^2})} . \tag{4.16}$$

Inserting these into (4.12) results into Eqs. (2.42) and (2.43) as to be expected. Finally, the effective potential expressed in terms of $T_2(k)$ and $T_2''(k)$ becomes

$$\begin{aligned} V_{\text{eff}}[\phi_0]&=v_0(Q,\phi_0)+\int \frac{dk}{2\pi}\left[\frac{\epsilon_k}{4}[T_2+T_2''(1-T_2)^2]\right. \\ &\qquad \left.-\frac{k^2}{4\epsilon_k}[T_2+T_2''(1-T_2)^2]\right] \\ &=v_0(Q,\phi_0)+\frac{\mu^2-m^2}{8\pi} \end{aligned} \tag{4.17}$$

if one inserts the solution (4.15) and (4.16). This completely agrees with the general formula (2.46). Since $\mu^2=v_2(Q,\phi_0)$ can be solved for the sine-Gordon model to yield

$$\mu^2=m^2\left[\cos\left[\frac{\sqrt{g}}{m}\phi\right]\right]^{1/(1-g/8\pi m^2)} \tag{4.18}$$

it is even possible to eliminate μ completely from (4.17) and finish with (4.3) once again. This proves that the effective potential obtained from both calculation schemes do agree. It is not related to the restriction to the sine-Gordon model as (4.13) (4.14) exhibit, since v_2 can be taken from an arbitrary potential of a Hamiltonian.

There is one more remark we would like to make concerning effective Lagrangians. In previous sections we calculated effective actions $\Gamma[\phi_0,\psi]$ describing a background field, e.g., a soliton, and fluctuations ψ around the background-field configuration. $\Gamma[\phi_0]$ was constructed by eliminating ψ due to the Euler-Lagrange equations. That means that mesonic excitations were integrated out and thereby eliminated leaving us with an effective classical field theory of the soliton. One might think to go the other way: to integrate out the soliton to finish with an effective field theory of the mesonic excitations, and to study their spectrum and interactions.

V. REDUCTION TO QUANTUM-MECHANICAL APPLICATIONS

This section, treating formal aspects of the Gaussian approximation, comments on some problems which can be analyzed easily in quantum mechanics, but become somehow difficult in field-theoretical generalizations. We approximate

$$\Gamma[Q]=\int dt\langle\psi_-(t)|[i\partial_t-\tfrac{1}{2}p^2-V(q)]|\psi_+(t)\rangle \tag{5.1}$$

by the value which $\Gamma[Q]$ takes for Gaussian wave functions:

$$\begin{aligned} |\psi_-(t)\rangle&=|\psi_+(t)\rangle \\ &=\left[\frac{\pi}{\rho}\right]^{1/2}\exp\left[-\frac{1}{2}(\rho+i\sigma)(q-Q)^2\right. \\ &\qquad \left.+iP(q-Q)+\frac{i}{2}PQ\right] \end{aligned} \tag{5.2}$$

and $P(t)$, $\rho(t)$, and $\sigma(t)$ are variational parameters, $Q(t)$ is the expectation value of q. After some simple steps, $\Gamma[Q]$ becomes

$$\begin{aligned} \Gamma[Q]=\int dt\left[P\dot{Q}+\frac{1}{4}\frac{\dot{\sigma}}{\rho}-\tfrac{1}{2}P^2-\tfrac{1}{4}\rho-\frac{\sigma^2}{4\rho}\right. \\ \left.-\left[\frac{\rho}{\pi}\right]^{1/2}\int dq\, e^{-\rho q^2}V(q+Q)\right] . \end{aligned} \tag{5.3}$$

It is stationary for $P=\dot{Q}$ and $\sigma=\frac{1}{2}\dot{\rho}\rho^{-1}$. If we define $\psi=1/\sqrt{2\rho}$, $\Gamma[Q]$ takes the conventional form

$$\Gamma[Q]=\int dt\left[\tfrac{1}{2}\dot{Q}^2+\tfrac{1}{2}\dot{\psi}^2-\frac{1}{8\psi^2}-v_0(\psi,Q)\right] , \tag{5.4}$$

where again functions $v_n(\psi,Q)$ have been defined by

$$v_n(\psi,Q)=\int \frac{dq}{\sqrt{2\pi}} e^{-q^2/2} V^{(n)}(q\psi+Q) . \tag{5.5}$$

Now, all formulas of the second section can be derived in the simplified case of quantum mechanics. Since no new insights can be expected, we skip these derivations. But it is possible to do similar calculations in the VCCM. A straightforward evaluation yields

$$\Gamma[Q,P,S_2,S_2'']=\int dt \left[P\dot{Q}-\tfrac{1}{2}P^2+\frac{i}{2}S_2''\dot{S}_2+\tfrac{1}{4}[S_2-1+S_2''(S_2-1)^2]-v_0(\{\tfrac{1}{2}[1+S_2+S_2''(1+S_2)^2]\}^{1/2},Q) \right] . \tag{5.6}$$

To map (5.4) and (5.6) onto each other, we define the variables

$$\psi=\{\tfrac{1}{2}[1+S_2+S_2''(1+S_2)^2]\}^{1/2}, \quad \phi=\frac{1-S_2}{1+S_2} , \tag{5.7}$$

and rewrite $\Gamma[Q,P,S_2,S_2'']$ in terms of P, Q, ψ, and ϕ:

$$\Gamma[Q,P,\psi,\phi]=\int dt[\, P\dot{Q}-\tfrac{1}{2}P^2+\tfrac{1}{2}\dot{\phi}\psi^2-\tfrac{1}{2}\phi + \tfrac{1}{2}\phi^2\psi^2-v_0(\psi,Q)] . \tag{5.8}$$

This functional becomes stationary if

$$\dot{Q}=P, \quad \phi=\psi^{-2}\left[\frac{1}{2}-\frac{i}{2}\frac{d\psi^2}{dt}\right] . \tag{5.9}$$

After inserting of these solutions into (5.8) we arrive at (5.4) which was obtained using the Jackiw-Kerman method; therefore, the equivalence was at least proved for quantum mechanics. The Euler-Lagrange equation of $\psi(t)$ can be solved if we assume $Q(t)=Q_0+\epsilon F(t)$ as before in Eq. (2.38). $\psi(t)$ should be a linear response to $Q(t)$, so we expand in ϵ up to linear order: $\psi(t)=\psi_0+\epsilon\lambda(t)$. The function $\lambda(t)$ is a solution of

$$\ddot{\lambda}(t)+\Omega^2\lambda(t)=\kappa F(t) , \tag{5.10}$$

where Ω and κ are again expressible in terms of the v_n's. We do not need their concrete form here. The general solution of (5.10) may be obtained as a series in F derivatives:

$$\lambda(t)=\frac{\kappa}{\Omega^2}\sum_{n=0}^{\infty}\left[-\frac{1}{\Omega^2}\right]^n\frac{d^{2n}F(t)}{dt^{2n}} \tag{5.11}$$

and we truncated after the $n=0$ term in the second section. The other terms in (5.11) contribute to $\Gamma[Q]$ via

$$\int dt \sum_{n=1}^{\infty}\frac{n\kappa^2}{\Omega^{2n+2}}(Q^{(n+1)})^2 \tag{5.12}$$

and, therefore, are related to higher time derivatives of $Q(t)$. In field theory (5.12) introduces $\partial_{\mu_1}\cdots\partial_{\mu_n}\phi_0(x,t)\partial^{\mu_1}\cdots\partial^{\mu_n}\phi_0(x,t)$ into the effective action which we did not attempt to calculate. The truncation to the $n=0$ term in the second section is justified because of this argument.

In the case of periodic potentials $V(q)=V(q+2\pi/\sqrt{g})$ the wave function (5.2) may be too restricted. There are tunneling effects and the true ground state is a θ vacuum with the wave function

$$\psi_\theta(q)=\frac{1}{N}\sum_{n=-\infty}^{n=+\infty}e^{in\theta}\psi\left[q-\frac{2\pi n}{\sqrt{g}}\right] . \tag{5.13}$$

Sine-Gordon quantum mechanics with the Hamiltonian

$$H=\frac{1}{2}p^2+\frac{1}{g}[1-\cos(\sqrt{g}\,q)] \tag{5.14}$$

is representative of those models which make θ vacua necessary. Tunneling effects are described by a typical tunneling factor

$$\beta=\exp\left[-\frac{\pi^2}{g}\frac{\rho^2+\sigma^2}{\rho}\right] \tag{5.15}$$

being nonanalytical in the coupling constant. The effective action $\Gamma[Q]$ can be split up into a term $\Gamma_{n=0}[Q]$ agreeing with (5.4) and additional contributions $\Gamma_{n\neq 0}[Q]$ which are complicated expressions in Jacobi's ϑ functions. If one likes to transfer such θ-vacuum wave functions to field theory, a tunneling factor

$$\beta=\exp\left[-\frac{2\pi^2}{g}\int dx\, dy \langle x \mid \Omega^* G \Omega \mid y \rangle\right] \tag{5.16}$$

is encountered. It vanishes in the infinite-volume limit; therefore, $\Gamma[\phi_0]$ is always $\Gamma_{n=0}[\phi_0]$. Tunneling effects due to instantons are more difficult to incorporate in the Gaussian approximation of field-theoretical models. A successful incorporation of tunneling effects in the Gaussian approximation has been described recently by Cooper, Pi, and Stancioff[32] but these authors restrict themselves on quantum-mechanical applications.

VI. DISCUSSION AND OUTLOOK

In this paper we presented a complete discussion how to apply the Gaussian approximation to arbitrary space- and time-dependent phenomena in (1 + 1)-dimensional quantum field theory of a single scalar field. It was possible to obtain general expressions for the effective potential, $Z[\phi_0]$ factors of the effective action, quantum soliton equations, and comprehensively formulated stability equations. All this was formulated in the Jackiw-Kerman variational principle which seems to be suited best in case of a restriction on a definite approximation scheme is taken into account explicitly. It is not possible to go beyond that concept relying completely on Gaussian trial wave functionals.

As a new and extendible alternative the variational coupled cluster method has been proposed. Here, post-Gaussian approximations are easy to formulate and emerge quite naturally from the general description which

is exact in principle. We are convinced that all these ideas based on the variational coupled cluster method will have their successful applications soon since they allow to have access to the complete effective action in a nonperturbative way. It is now possible to extend the standard coupled cluster method, which is restricted to the calculation of eigenvalues of a Hamiltonian, to a variational method which gives much more insight in the structure of a field theory.

As a first application of our method we propose to estimate the soliton mass of the $(\phi^4)_2$ quantum field theory in case of a post-Gaussian approximation. A previous estimation[14] of the soliton mass utilizing the standard coupled cluster method already indicates that the mass is significantly changed by higher-order correlations at least very close to the critical point.

There is one more test of the Gaussian approximation still to be performed. Mass ratios of soliton masses in the sine-Gordon system may be known exactly due to the WKB approximation. It is necessary to calculate masses of different static solitons of that model and compare their ratios with these perhaps exact results. This would underpin the statement of the high reliability of the Gaussian approximation. Work in this direction has been started recently and will be reported soon.[15]

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