Consistent factor ordering of constraints may be ambiguous

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It is shown in a simple model that consistency requirements of the Dirac constraint quantization may be satisfied by more than one, essentially different, factor orderings of the constraints and the Hamiltonian.

In the Dirac quantization of a dynamical system with (first-class) constraints, the physical Hilbert space is spanned by states which are annihilated by the constraint operators. Consistency requires that the commutators of such operators again annihilate all physical states and that the Hamilton operator keep them within the physical Hilbert space. It is often difficult to find a factor ordering of the constraints and of the Hamiltonian which satisfies these requirements, especially for systems with open constraint algebras such as the general theory of relativity.[1]

In the face of such difficulties, it is tempting to believe that once a consistent factor ordering is found, it is essentially unique and physically appropriate. The purpose of this paper is to present a simple counterexample to such a soothing conjecture.

Take a nonrelativistic particle moving in a three-dimensional Euclidean space E^3 with Cartesian coordinates $Q^A=(X,Y,Z)$ and let the translation group T(1) act on E^3 as a gauge group by helical motions[2]

$$\begin{aligned} X(\tau)&=X(0)\cos\tau+Y(0)\sin\tau\ ,\\ Y(\tau)&=-X(0)\sin\tau+Y(0)\cos\tau\ ,\\ Z(\tau)&=Z(0)+\tau\ . \end{aligned} \tag{1}$$

The orbits (1) can be considered as points of the physical configuration space $\mathscr{m}$ and labeled by the physical coordinates $q^a=(r,\theta)$,

$$\begin{aligned} r&=R \quad \text{with } R\equiv(X^2+Y^2)^{1/2}\ ,\\ \theta&=(\Theta-Z)\,\mathrm{Mod}2\pi \quad \text{with } \Theta\equiv\arctan X/Y\ , \end{aligned} \tag{2}$$

which are constant along the orbits. The generator

$$\phi^A=\frac{dQ^A(\tau)}{d\tau}=(Y,-X,1) \tag{3}$$

of the group action is a Killing vector of the Euclidean metric $G_{AB}=\delta_{AB}$. The motion of the particle is governed by the Hamiltonian

$$H=\tfrac{1}{2}G^{AB}P_AP_B+V(Q) \tag{4}$$

whose scalar potential V is assumed to be constant along the orbits and hence dependent on Q^A only through the physical coordinates q^a:

$$V(Q)=v(q(Q))\ . \tag{5}$$

The gauge transformations in the phase space T^*E^3 are canonical transformations generated by the dynamical variable

$$\Pi\equiv\phi^A(Q)P_A\ ; \tag{6}$$

their orbits are surface forming on the constraint surface

$$\Pi=0 \tag{7}$$

in T^*E^3. The Hamiltonian (4) is invariant under such gauge transformations:

$$\frac{dH}{d\tau}=\{H,\Pi\}=0\ . \tag{8}$$

The motion in the big phase space T^*E^3 is easily reduced to the physical phase space $T^*\mathscr{m}$. The projector

$$Q^a_A\equiv\partial q^a/\partial Q^A \tag{9}$$

extracts the physical metric

$$g^{ab}=G^{AB}Q^a_AQ^b_B=\mathrm{diag}(1,1+r^{-2}) \tag{10}$$

out of the Euclidean metric $G^{AB}=\delta^{AB}$. The inverse projector

$$Q^A_a=G^{AB}Q^b_Bg_{ba},\quad Q^a_A\phi^A=0,\quad Q^a_AQ^A_b=\delta^a_b \tag{11}$$

enables one to decompose the momentum P_A into the physical momentum p_a and the constraint Π:

$$P_A=Q^a_Ap_a+\phi_A\Pi \quad \text{with } p_a=Q^A_aP_A\ . \tag{12}$$

The Hamiltonian (4) is thereby reduced on the constraint surface (7) to the physical Hamiltonian

$$h=\tfrac{1}{2}g^{ab}p_ap_b+v(q)\ . \tag{13}$$

The metric (10) in the two-dimensional physical configuration space $\mathscr{m}$ is curved, but regular and complete. It becomes flat for $r\ll1$ and again for $r\gg1$, where it corresponds to a cylindrical surface with embedding radius 1. The complete embedding diagram has the shape of a cylindrical vessel curving rather abruptly into a flattened bottom at $r=0$.

Quantum states of the physical system can be described by scalar functions $\psi(q)$ on $\mathscr{m}$, with the inner product

$$\langle\psi_1,\psi_2\rangle=\int_{\mathscr{m}}d^2q\,|\,g\,|^{1/2}\psi_1^*(q)\psi_2(q) \tag{14}$$

whose measure

$$|g|^{1/2}\equiv(\det(g_{ab}))^{1/2}=r(1+r^2)^{-1/2} \tag{15}$$

is given by the determinant $|g|$ of the physical metric (10). The physical coordinates and momenta are represented by the Hermitian operators

$$\hat{q}^a=q^a \quad \text{and} \quad \hat{p}_a=-i|g|^{-1/4}\partial_a|g|^{1/4} \tag{16}$$

and the Hamiltonian (13) by the Hermitian operator

$$\hat{h}=-\tfrac{1}{2}\Delta+v(q) \tag{17}$$

whose kinetic part is the covariant Laplace-Beltrami operator in m:

$$\begin{aligned}\Delta&=|g|^{-1/2}\partial_a|g|^{1/2}g^{ab}\partial_b\\ &=r^{-1}(1+r^2)^{1/2}\partial_r r(1+r^2)^{-1/2}\partial_r\\ &\quad+(1+r^{-2})\partial_\theta{}^2\ .\end{aligned} \tag{18}$$

A consistent quantization of the gauge system whose reduction yielded the physical system (13) requires us to find a factor ordering of the Hamiltonian (4) and of the constraint (6) which would satisfy the quantum version of the condition (8):

$$\frac{1}{i}[\hat{H},\hat{\Pi}]=0\ . \tag{19}$$

This problem has been solved[3] in a manifestly covariant way for an arbitrary gauge system with constraints linear in the momenta and the (first-class) Hamiltonian at most quadratic in the momenta so that yet another physically compelling requirement is fulfilled: Quantization of the gauge system is entirely equivalent to the quantization of the reduced physical system. Applied to our simple model, the procedure can be described as follows.

Using the alternating symbol δ_{ABC} in E^3, form a field of two-flats

$$\phi_{AB}=\delta_{ABC}\phi^C \tag{20}$$

orthogonal to the group orbits and, dividing them by the norm $||\phi||$ of ϕ_{AB},

$$||\phi||^2\equiv\tfrac{1}{2}G^{AC}G^{BD}\phi_{AB}\phi_{CD}\ , \tag{21}$$

construct a "two-dimensional Levi-Civita tensor"

$$\epsilon_{AB}\equiv||\phi||^{-1}\phi_{AB}\ . \tag{22}$$

Turn the constraint (6) into the directional derivative operator along the orbits,

$$\hat{\Pi}=-i\phi^A\partial_A\ , \tag{23}$$

and the Hamiltonian (4) into the second-order differential operator

$$\hat{H}=-\tfrac{1}{2}\epsilon^{AC}\partial_A\epsilon^B{}_C\partial_B+V(Q) \tag{24}$$

acting on the space $\mathscr{F}=C^\infty(E^3,\mathbb{C})$ of complex-valued functions of E^3. Then, the operators (23) and (24) commute, Eq. (19).

The physical states $\Psi(Q)\in\mathscr{F}_0$ are such states in $\mathscr{F}$ which are annihilated by the constraints:

$$\hat{\Pi}\Psi(Q)=0\ . \tag{25}$$

They are constant along the orbits of the group and hence depend on Q^A only through the physical coordinates q^a:

$$\Psi(Q)=\psi(q(Q))\ . \tag{26}$$

The Hamilton operator (24), when acting on a physical state, reduces to the physical Hamilton operator (17):

$$\hat{\Pi}\Psi=0\Longrightarrow\hat{H}\Psi=\hat{h}\psi\ . \tag{27}$$

The space $\mathscr{F}_0$ of physical states can be endowed with an inner product

$$\langle\Psi_1,\Psi_2\rangle=\int_\Sigma\epsilon_{AB}dQ^A\wedge dQ^B\Psi_1^*(Q)\Psi_2(Q)\ . \tag{28}$$

Here, the integration is over an arbitrary two-surface Σ which is transversal to the orbits. The integral (28) does not depend on the choice of the surface and it reduces to the physical inner product (14).

These statements, which follow from the general theory proposed in Ref. 3, can be easily corroborated by direct calculation. In cylindrical coordinates $Q^A=(R,\Theta,Z)$ we have

$$\phi^A=(0,1,1) \quad \text{and} \quad G^{AB}=\text{diag}(1,R^{-2},1)\ . \tag{29}$$

This yields

$$\epsilon_{AB}=R(1+R^2)^{-1/2}\times\left\|\begin{matrix}0&1&-1\\-1&0&0\\1&0&0\end{matrix}\right\| \tag{30}$$

for the Levi-Civita tensor and

$$\begin{aligned}\hat{H}=&-\tfrac{1}{2}R^{-1}(1+R^2)^{1/2}\partial_R R(1+R^2)^{-1/2}\partial_R\\ &-\tfrac{1}{2}R^2(1+R^2)^{-1}(\partial_Z-R^{-2}\partial_\Theta)^2+V\end{aligned} \tag{31}$$

for the Hamilton operator.

Because $V=v(R,\Theta-Z)$, it is obvious that $\hat{H}$ commutes with

$$\hat{\Pi}=-i(\partial_\Theta+\partial_Z)\ . \tag{32}$$

When acting on a physical state

$$\Psi(R,\Theta,Z)=\psi(r=R,\theta=\Theta-Z) \tag{33}$$

the second term in (31) yields $\frac{1}{2}(1+r^2)r^{-2}\partial_\theta{}^2$, which leads to Eq. (27). Finally, by choosing the plane $Z=0$ for Σ and parametrizing it by the polar coordinates R and Θ, we see that $\epsilon_{AB}dQ^A\wedge dQ^B$ gives the correct physical measure (15) and hence (28) reduces to (14).

We have exhibited the factor ordering which is consistent, Eq. (19), and physical, Eq. (28). We want to show now that it is not the only consistent factor ordering. Indeed, a naive way of ordering the Hamiltonian (4) and the constraint (6) would be to implement them as Hermitian operators on a big Hilbert space $\mathscr{H}=L^2(E^3,|G|^{1/2}d^3Q)$ with the measure provided by the determinant $|G|=\det(G_{AB})$ of the flat metric G_{AB}:

$$(\Psi_1,\Psi_2)\equiv\int_{E^3}d^3Q\,|G|^{1/2}\Psi_1^*(Q)\Psi_2(Q)\ . \tag{34}$$

The constraint (6) and the Hamiltonian (4) are then turned into operators which are Hermitian under the inner product (34), namely, into

$$\hat{\Pi}_G = -i(\phi^A \partial_A + \tfrac{1}{2}\,\mathrm{div}_G \boldsymbol{\phi}) \tag{35}$$

and

$$\hat{H}_G = -\tfrac{1}{2}\Delta_G + V , \tag{36}$$

where

$$\Delta_G = |G|^{-1/2}\partial_A |G|^{1/2} G^{AB}\partial_B \tag{37}$$

is the Laplacian in E^3. With this ordering,

$$\frac{1}{i}[\hat{H}_G,\hat{\Pi}_G] = -\tfrac{1}{2}|G|^{-1/2}\partial_A |G|^{1/2}(\pounds_{\phi} G^{AB})\partial_B + \tfrac{1}{4}(\Delta_G\,\mathrm{div}_G\boldsymbol{\phi}) , \tag{38}$$

where $\pounds_{\phi}$ denotes the Lie derivative. For a generic system, the terms on the right-hand side of Eq. (38) make the ordering inconsistent.[3] In our simple model, however, ϕ^A is a Killing vector of G^{AB} and, hence, because

$$\pounds_{\phi} G^{AB} = 0 = \mathrm{div}_G \boldsymbol{\phi} , \tag{39}$$

the consistency condition (19) is actually satisfied. Because of Eq. (39), the orderings (23) and (35) of the constraint Π coincide: $\hat{\Pi}_G = \hat{\Pi}$. On the other hand, the orderings (24) and (36) of the Hamiltonian H differ.

The physical states $\Psi \in \mathscr{H}_0$ are defined as before by the requirement (25). Because $\hat{\Pi}$ has a continuous spectrum, the inner product (34) of two physical states in $\mathscr{H}$ in general diverges, due to the integration along the orbits on which the physical state functions maintain their respective constant values. To see this in detail, complement the physical coordinates $r=R$ and $\theta=\Theta-R$ by the coordinates $z=Z$ labeling the points along the orbits, and write the inner product (34) in the form

$$(\Psi_1,\Psi_2) = \int_{-\infty}^{\infty} dZ \int_0^{\infty} dR \int_0^{2\pi} d\Theta\, R\Psi_1^*\Psi_2 = \int_{-\infty}^{\infty} dz \frac{1}{2\pi} \int_0^{\infty} dr \int_0^{2\pi} d\theta\, r\chi_1^*(r,\theta)\chi_2(r,\theta) , \tag{40}$$

where

$$\Psi(R,\Theta,Z) = (2\pi)^{-1/2}\chi(r=R,\theta=\Theta-Z) . \tag{41}$$

To make the expression (40) finite, we restrict the z integration to a finite stretch $z \in (z_0, z_0+2\pi)$ of the z axis covering exactly one turn of the helical orbits. This amounts to redefining the inner product of two physical states as

$$(\Psi_1,\Psi_2) \equiv (\chi_1,\chi_2) \equiv \int_0^{\infty} dr \int_0^{2\pi} d\theta\, r\chi_1^*\chi_2 . \tag{42}$$

The same expression follows by applying to the physical states the familiar technique of eigendifferentials.[4] The eigenstate of $\hat{\Pi}$ to an arbitrary eigenvalue Π has the form

$$\Psi_{\Pi}(R,\Theta,Z) = (2\pi)^{-1/2}\chi(R,\Theta-Z)e^{i\Pi Z} . \tag{43}$$

The operator $\hat{\Pi}$ has a continuous spectrum and two eigenstates (43) are thus normalized to the δ function:

$$(\Psi_{1\Pi'},\Psi_{2\Pi''}) = (\chi_1,\chi_2)\delta(\Pi'-\Pi'') . \tag{44}$$

Instead of working with the physical states $\Psi_0 = \Psi_{\Pi=0}$, we smear the constraint surface by a small amount ϵ and replace Ψ_0 by an eigendifferential

$$\delta_\epsilon\Psi \equiv \epsilon^{-1/2}\int_{-\epsilon/2}^{\epsilon/2} d\Pi\,\Psi_{\Pi} . \tag{45}$$

The eigendifferentials lie in the original Hilbert space $\mathscr{H}$ and their inner product agrees with the renormalized expression (42):

$$(\delta_\epsilon\Psi_1,\delta_\epsilon\Psi_2) = (\chi_1,\chi_2) . \tag{46}$$

The renormalized inner product (42) does not coincide with the physical inner product (14) and (15) because the big space measure $|G|^{1/2}=r$ differs from the physical measure $|g|^{1/2}=r(1+r^2)^{-1/2}$ by the factor $|\gamma|^{1/2}=(1+r^2)^{-1/2}$ which can be interpreted as the measure of a line element along the orbits:

$$|G|^{1/2} = |\gamma|^{1/2}|g|^{1/2} \quad \text{with } \gamma = G_{AB}\phi^A\phi^B . \tag{47}$$

In other words, a state χ in $\mathscr{H}_0$ corresponds to the state

$$\psi = |\gamma|^{1/4}\chi \tag{48}$$

in $\mathscr{F}_0$ and the Hamilton operator $\hat{H}_G$ in $\mathscr{H}_0$ corresponds (as any other observable) to the Hamilton operator

$$\hat{h}_G = |\gamma|^{1/4}\hat{H}_G|\gamma|^{-1/4} \tag{49}$$

in $\mathscr{F}_0$.

When acting on a physical state (41) in $\mathscr{H}_0$, the Hamilton operator (36) and (37) has the form

$$\hat{H}_G = -\tfrac{1}{2}|\gamma|^{-1/2}|g|^{-1/2}\partial_r|\gamma|^{1/2}|g|^{1/2}\partial_r - \tfrac{1}{2}(1+r^{-2})\partial_\theta{}^2 + v(r,\theta) . \tag{50}$$

Therefore, $\hat{h}_G$ differs from $\hat{h}$ by a potential w,

$$\hat{h}_G = \hat{h} + w , \tag{51}$$

which can be written as

$$w = -\tfrac{1}{2}|G|^{-1/2}|\gamma|^{1/4}(|G|^{1/2}|\gamma|^{-1/4}{}_{,r})_{,r} = -\tfrac{1}{8}(4+r^2)(1+r^2)^{-2} . \tag{52}$$

Notice that the difference between $\hat{h}_G$ and $\hat{h}$ does not amount to a mere inclusion of a curvature term because the potential w is not proportional to the scalar curvature $\mathrm{R}=6(1+r^2)^{-2}$ of the physical metric (10).

We thus see that the naive factor ordering (35)–(37) leads to a quantum theory in the reduced (physical) space which is substantially different from the physical quantum theory (17) and (18). In particular, the reduced Hamilton operator (51) and (52) has typically an entirely different spectrum from the physical Hamilton operator (17) and (18). However, both the naive factor ordering (35)–(37) and the physical factor ordering (23) and (24) satisfy the consistency condition (19). This clearly illustrates the point that consistency alone does not determine the factor ordering uniquely.

The difference (52) between the two orderings is expressed solely through the difference between the measure

$|G|^{1/2}$ in a big unphysical space E^3 and the measure $|g|^{1/2}$ in the small physical space $\mathscr{m}$. The choice of the Laplacian depends on such a choice of the measure. One can say that the naive ordering takes the big space seriously and finds it conceivable that the quantum system can be found outside the constraint surface. This idea permeates the whole procedure. It lies behind the definition (34) of the inner product, behind the requirement that both the constraint operator (35) and the big Hamilton operator (36) and (37) be self-adjoint under this inner product, behind the determination of a complete set of eigenstates (43) of the constraint operator (even those with eigenvalues different from zero) and behind the subsequent formation (45) of the eigendifferentials. On the other hand, the physical factor ordering considers the big space as a mere artifice and treats the physical directions completely differently from the gauge directions so as to ensure that the quantum state never feels anything outside the constraint surface.

Our particular model has an interesting field-theoretical application.[5] In scalar electrodynamics, the action of the gauge group on the real and imaginary parts of the scalar field can be considered as a rotation in the field plane while its action on the electromagnetic potential has a character of a translation. One can thus express the joint action of the gauge transformations of the "first" and "second" kind on the total field space as an infinite number of replicas of the action of T(1) on E^3 by helical motions. Standard quantization of scalar electrodynamics corresponds to the "naive" factor ordering. One may wonder whether the "physical" ordering leads to higher-order corrections which could be in principle detectable. Unfortunately, the difference between the two orderings is obscured by renormalization problems.

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[1]It was first noticed by J. Anderson, in *Eastern Theoretical Conference*, edited by M. E. Rose (Gordon and Breach, New York, 1963), that the standard form of the super-Hamiltonian and supermomentum constraints in canonical gravity cannot be turned into Hermitian operators which would satisfy the consistency requirements. A solution was offered by J. Schwinger, Phys. Rev. **130**, 1253 (1963); **132**, 1317 (1963), and criticized by P. A. M. Dirac, in *Contemporary Physics: Trieste Symposium 1968*, edited by A. Salam (International Atomic Energy Agency, Vienna, 1969), p. 539. The best, and certainly the shortest, exposition of Schwinger's solution may be found in a footnote of the paper by B. S. DeWitt, Phys. Rev. **160**, 1113 (1967), who himself made a different proposal on how to remove the problem by letting any two field operators taken at the same point formally commute. Recently, A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986), has reexpressed the Hamiltonian theory of gravity in a new set of fundamental canonical variables and found a simple factor ordering of the constraints which (disregarding the regularization difficulties) satisfies the consistency requirements.

[2]Basic properties of the helical model are mentioned in the footnote on p. 729 of the article by B. S. DeWitt, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979). The model is studied in detail by K. Kuchař, Phys. Rev. D **34**, 3031 (1986).

[3]K. Kuchař, Phys. Rev. D **34**, 3044 (1986).

[4]See, e.g., A. Messiah, *Quantum Mechanics I* (North-Holland, Amsterdam, 1958).

[5]I. Bialynicki-Birula (private communication).