Structure of a composite system in motion in relativistic quantum mechanics

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Unlike in nonrelativistic quantum mechanics, the structure of a relativistic system of bound particles is intrinsically coupled with the overall translational motion of the system. This is illustrated by means of a solvable two-body model in one space dimension. The model is described in terms of the two-body Dirac equation with an interaction in the form of the δ function. Although the equation is not manifestly covariant, relativistic covariance of the model is confirmed by constructing the Lorentz-boost operator. When boosted the system exhibits exact Lorentz contraction. It is pointed out that the "form factor" of the bound state, which simulates the form factor of the deuteron determined by electron scattering, is not as simply related to the density distribution in the system as is often taken for granted.

I. INTRODUCTION

Consider a system of two interacting particles. Assume that the interaction depends on the particle positions only through the relative coordinates. Then the total momentum is a constant of the motion, and the system is translationally invariant. In nonrelativistic quantum mechanics, the center-of-mass (c.m.) and relative coordinates can be separated, and the two-body problem is exactly reduced to a one-body problem. The structure, e.g., the size, of the bound state (if any) does not depend on the c.m. motion. In relativistic quantum mechanics, however, translational invariance does not imply separability of the c.m. motion and internal structure; rather the latter is expected to change when the system is boosted. The main purpose of this paper is to examine a type of solvable model which illustrates the structure of a relativistic two-body system and its relation to the overall translational motion.

The model that we consider is defined in terms of the two-body Dirac equation in one space dimension with a direct instantaneous interaction. This belongs to what Dirac called "the instant form" of the formulation of the relativistic two-body problem.[1] The equation is not manifestly covariant because the two particles are at equal times. However, if the interaction is in the from of $\delta(x)$, where $x=x_1-x_2$ is the relative coordinate, the system is in fact covariant.[2] We confirm this by constructing the Lorentz-boost operator. For such an interaction the two-body Dirac equation can be solved analytically for bound and scattering states. For the bound state the relativistic relation between energy and momentum is satisfied and the wave function exhibits exact Lorentz construction. We also examine the form factor, which simulates the form factor of the deuteron determined by electron scattering. The model reveals a rather intriguing relation between the form factor and the density distribution of the bound system.

II. MODEL

The one-body Dirac Hamiltonian in one space dimension reads

$$H_i=\alpha_i p_i+\beta_i m_i \ , \tag{2.1}$$

where $i=1$ and 2 refer to the two particles. We denote the particle position by x_i; $[x_i,\, p_j]=i\delta_{ij}$. The units are such that $c=\hbar=1$. For the Dirac matrices we use $\alpha=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. The masses are assumed to be equal, i.e., $m_1=m_2=m$, although unequal masses can easily be handled.

Our basic equation for the two-body system is

$$H\psi=E\psi \ , \tag{2.2}$$

where

$$H=H_1+H_2+V \ , \tag{2.3}$$

H_1 and H_2 being defined by Eq. (2.1).[2] The interaction V will be specified below. It is understood that the two particles are at equal times. The wave function ψ has four components, $\psi_{\mu\nu}$, which we specify in terms of the eigenvalues of β_1 and β_2, i.e.,

$$\psi_{\pm\nu}=\tfrac{1}{2}(1\pm\beta_1)\psi \ , \quad \psi_{\mu\pm}=\tfrac{1}{2}(1\pm\beta_2)\psi \ . \tag{2.4}$$

We introduce the c.m. and relative coordinates and momenta in the usual manner, i.e., $X=\frac{1}{2}(x_1+x_2)$, $x=x_1-x_2$, $P=p_1+p_2$, and $p=\frac{1}{2}(p_1-p_2)$. Then $H_0\equiv H_1+H_2$ can be rewritten as

$$H_0=\tfrac{1}{2}(\alpha_1+\alpha_2)P+(\alpha_1-\alpha_2)p+(\beta_1+\beta_2)m \ . \tag{2.5}$$

For the interation V we consider the following three types:

$$V=\begin{cases}\frac{1}{2}(1-\alpha_1\alpha_2)f_V(x) \ ,\\ \beta_1\beta_2 f_S(x) \ ,\\ \alpha_1\alpha_2\beta_1\beta_2 f_P(x) \ .\end{cases} \tag{2.6}$$

Here the combinations of the Dirac matrices are those of the covariant ones which correspond to the interaction types of vector, scalar, and pseudoscalar, repsectively.[2] Since V of Eq. (2.6) does not depend on X, $[P,H]=0$; i.e., the system is translationally invariant. We assume that $f(x)$'s are even functions of x so that parity is a good quantum number.

III. BOOST OPERATOR

Since the two particles are at equal times, Eq. (2.2) obviously refers to a specific Lorentz frame. Even so the system conforms to relativistic covariance if a boost operator K exists such that

$$[H,K]=-iP\ , \tag{3.1}$$

$$[P,K]=-iH\ . \tag{3.2}$$

This K is the generator of the Lorentz transformation. The energy H and momentum P transform according to

$$\begin{bmatrix} H' \\ P' \end{bmatrix} = e^{iuK} \begin{bmatrix} H \\ P \end{bmatrix} e^{-iuK} = \begin{bmatrix} \cosh u & -\sinh u \\ -\sinh u & \cosh u \end{bmatrix} \begin{bmatrix} H \\ P \end{bmatrix}, \tag{3.3}$$

where u is related to v, the velocity of the transformed coordinate system with respect to the original one, by $\cosh u=(1-v^2)^{-1/2}$ and $\sinh u=v(1-v^2)^{-1/2}$.

When there is no interaction, it can easily be checked that K is given by

$$K_0=\tfrac{1}{2}(\{x_1,H_1\}+\{x_2,H_2\})\ , \tag{3.4}$$

where $\{a,b\}=ab+ba$. In the presence of the interaction V, let us start with the ansatz

$$K=K_0+XV+L\ , \tag{3.5}$$

where $[P,L]=0$. This K satisfies Eq. (3.2). By setting Eq. (3.5) into Eq. (3.1) we find that L has to satisfy the equation

$$[H,L]=\begin{cases} \tfrac{1}{2}m\alpha_1\alpha_2(\beta_1-\beta_2)xf_V(x)\ , \\ -\tfrac{1}{2}\beta_1\beta_2[(\alpha_1+\alpha_2)\{p,xf_S(x)\} \\ \qquad +(\alpha_1-\alpha_2)Pxf_S(x)]\ , \\ -\tfrac{1}{2}\beta_1\beta_2[(\alpha_1+\alpha_2)\{p,xf_P(x)\} \\ \qquad -(\alpha_1-\alpha_2)Pxf_P(x)] \\ \qquad +m\alpha_1\alpha_2(\beta_1-\beta_2)xf_P(x)\ . \end{cases} \tag{3.6}$$

For all types of the interaction, the right-hand side (RHS) of Eq. (3.6) vanishes if $f(x)\propto\delta(x)$; then $L=0$ and the boost operator K is determined.[3] We have not been able to find any other form of $f(x)$ such that K can be determined. We therefore assume that $f(x)$ is in the form of $\delta(x)$. Then, as shown in the next section, the two-body Dirac equation (2.2) can be solved for all three types of interaction.[4]

IV. SOLUTIONS

Existence of the boost operator K implies relativistic covariance of the system. However, since the basic equation (2.2) is not manifestly covariant it would be interesting to see how the solutions conform to covariance.

We assume that all $f(x)$'s of Eq. (2.6) are in the form of $\delta(x)$. In solving Eq. (2.2), however, we start with a square-well potential:

$$f(x)=\begin{cases} -D & \text{for } |x|<a\ , \\ 0 & \text{for } |x|>a\ . \end{cases} \tag{4.1}$$

After solving the equation we let the width of the potential reduce to zero, keeping the "area" $2aD=g$ constant. Then $f(x)$ becomes

$$\lim_{a\to 0} f(x)=-g\delta(x)\ ,\quad g=2aD\ . \tag{4.2}$$

If one uses the δ-function potential from the outset one encounters various inconsistencies. This has recently been illustrated for a one-body Dirac equation in one dimension;[5] hence we refrain from delving into this problem here.

In solving Eq. (2.2) we find it convenient to use the following combinations of $\psi_{\mu\nu}$:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} e^{iPX} = \frac{1}{\sqrt{2}} \begin{bmatrix} \psi_{++}+\psi_{--} \\ \psi_{++}-\psi_{--} \\ \psi_{+-}+\psi_{-+} \\ \psi_{+-}-\psi_{-+} \end{bmatrix}, \tag{4.3}$$

where the dependence on the c.m. coordinate X has been separated. Equation (2.2) leads to the following four equations:

$$P\phi_3+2m\phi_2=(E-C_1)\phi_1\ , \tag{4.4}$$

$$-2p\phi_4+2m\phi_1=(E-C_2)\phi_2\ , \tag{4.5}$$

$$P\phi_1=(E-C_3)\phi_3\ , \tag{4.6}$$

$$-2p\phi_2=(E-C_4)\phi_4\ , \tag{4.7}$$

where the C's are related to the depth of the potential by

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = - \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} D_V \\ D_S \\ D_P \end{bmatrix}. \tag{4.8}$$

It is understood that the C's are set to zero for $|x|>a$. Equations (4.4)–(4.7) can be reduced to an equation for a single component of ϕ. We find it most convenient to choose ϕ_2, and obtain

$$4p^2\phi_2=(E-C_4)\left[E-C_2-\frac{4m^2(E-C_3)}{(E-C_1)(E-C_3)-P^2}\right]\phi_2\ . \tag{4.9}$$

In Eq. (4.9) we disregarded a δ-function term at $|x|=a$ which stems from the noncommutativity between p and the potential. This can be taken care of by appropriately matching the ϕ's at $|x|=a$ (Ref. 6).

Rather than keeping all types of interactions let us examine the case of type V alone in some detail. In this case $C_2=C_4=-D_V$ and $C_1=C_3=0$. We assume that $D_V>0$ (attractive) so that we have a bound state which simulates the deuteron.

Let us consider the even-parity bound state. It is expected that ϕ_2 is an even function of x, which can be taken as

$$\phi_2=\begin{cases}\cos\mathcal{K}x & \text{for } |x|<a ,\\ (\cos\mathcal{K}a)e^{-\kappa(|x|-a)} & \text{for } |x|>a .\end{cases} \tag{4.10}$$

From Eq. (4.9) κ and $\mathcal{K}$ are determined as

$$\kappa^2=\left[\frac{E}{2}\right]^2\left[\frac{4m^2}{E^2-P^2}-1\right] , \tag{4.11}$$

and

$$\mathcal{K}^2=\frac{E+D_V}{4}\left[E+D_V-\frac{4m^2E}{E^2-P^2}\right] . \tag{4.12}$$

By substituting the above ϕ_2 into Eq. (4.7) we obtain

$$\phi_4=\begin{cases}-\dfrac{2i}{E+D_V}\mathcal{K}\sin\mathcal{K}x & \text{for } |x|<a ,\\ -\dfrac{2i}{E}(\cos\mathcal{K}a)\kappa\dfrac{x}{|x|}e^{-\kappa(|x|-a)} & \text{for } |x|>a .\end{cases} \tag{4.13}$$

The other components ϕ_1 and ϕ_3 are related algebraically to ϕ_2 and ϕ_4. The requirement of continuity for ϕ_4 at $|x|=a$ leads to

$$\tan\mathcal{K}a=\frac{E+D_V}{E}\frac{\kappa}{\mathcal{K}} . \tag{4.14}$$

This essentially completes the solution for the square-well potential.

Let us now take the narrow-width limit of Eq. (4.2). Then $\mathcal{K}\to D_V/2$ and $\mathcal{K}a\to g/4$, and Eq. (4.14) becomes

$$\tan\frac{g}{4}=\frac{2\kappa}{E} . \tag{4.15}$$

Combining Eqs. (4.11) and (4.15) we find

$$E^2={E_0}^2+P^2 , \tag{4.16}$$

where

$$E_0=2m\cos(g/4) . \tag{4.17}$$

The relativistic energy-momentum relation is satisfied as expected. Since m is the only parameter with dimension in the model, E_0 is proportional to m.

In the way we obtained the boost operator K in Sec. III, it was crucial that $f(x)$ is in the form of $\delta(x)$. That suggests that the two-body system is not covariant if the well width is finite. This can be seen as follows. In taking the limit of $a\to 0$, if we retain terms next to the leading, we obtain

$$E^2=\frac{4m^2\cos^2(g/4)}{1+aE\tan(g/4)}+P^2 . \tag{4.18}$$

Hence, the relativistic energy-momentum relation is not satisfied.

Returning to the δ-function limit, we obtain the wave function

$$\phi=N\begin{bmatrix}2mE/{E_0}^2\\ 1\\ 2mP/{E_0}^2\\ -2i\eta x/|x|\end{bmatrix}e^{-\kappa|x|} , \tag{4.19}$$

where N is the normalization factor which we determine later, and

$$\eta=\frac{\kappa}{E}=\frac{(4m^2-{E_0}^2)^{1/2}}{2E_0}=2\tan\frac{g}{4} . \tag{4.20}$$

The relative magnitude of the components of ϕ varies depending on P. Suppose that $\eta\ll 1$. Then ϕ_1 and ϕ_2 are the main components when $P\ll m$. If $P\gg m$, then ϕ_1 and ϕ_3 become more important. This has an important bearing on the form factor of the bound state which we will examine in the next section. The x dependence of $\phi^\dagger\phi$ is given by $e^{-2\kappa|x|}$ where $\kappa=\eta E$. Therefore, when the system is boosted, its size simply scales according to $1/E$. Since $P/E=v$,

$$\frac{E_0}{E}=\left[1-\left[\frac{P}{E}\right]^2\right]^{1/2} \tag{4.21}$$

is exactly the Lorentz-contraction factor. Although the boost operator K has the interaction term VX, VX has no effect on ϕ.

Next let us briefly examine the relation between ϕ and the boost opertor K. Consider ϕ_P such that $H\phi_P=E\phi_P$ and $\hat{P}\phi_P=P\phi_P$. Here the caret on P is to emphasize that $\hat{P}$ is an operator. Now introduce $\phi_{P'}$ defined by

$$\phi_{P'}=e^{iuK}\phi_P . \tag{4.22}$$

Equation (3.3) then implies that $\phi_{P'}$ should satisfy $H\phi_{P'}=E'\phi_{P'}$ and $\hat{P}\phi_{P'}=P'\phi_{P'}$, where (E',P') are related to (E,P) by

$$\begin{bmatrix}E'\\ P'\end{bmatrix}=\begin{bmatrix}\cosh u & \sinh u\\ \sinh u & \cosh u\end{bmatrix}\begin{bmatrix}E\\ P\end{bmatrix} . \tag{4.23}$$

We have confirmed this for an infinitesimal transformation, i.e., for $e^{iuK}=(1+iuK)$, by operating with K that we determined in Sec. III on ϕ. In doing so we realized that ϕ has to be normalized in an invariant way, i.e.,

$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dX\,\phi_{P'}^\dagger\phi_P e^{-i(P'-P)X}=2\pi(E/E_0)\delta(P'-P) . \tag{4.24}$$

Of course, $E\delta(P'-P)$ is a relativistic invariant. The transformation (4.22) retains the normalization of ϕ in the sense of Eq. (4.24). The normalization factor N of Eq. (4.19) determined in this way is

$$N^2=\eta{E_0}^3/(8m^2) . \tag{4.25}$$

In addition to the even-parity bound state so far discussed, there is an odd-parity bound state. For this state ϕ_2 is an odd function of x, and hence $\cos\mathcal{K}x$ of Eq. (4.10) is replaced with $\sin\mathcal{K}x$. Consequently, tan in Eqs. (4.14) and (4.15) are both replaced with $-$cot. Equation (4.16) remains valid, but E_0 becomes

$$E_0=-2m\sin(g/4)\ . \tag{4.26}$$

This solution, with $E_0<0$, should be reinterpreted in the light of the vacuum hole theory.[7] We do not attempt it in this paper, but we do not think that this solution can simply be dismissed either. Let us add that the solutions for scattering states can also be obtained. The scattering phase shifts for even- and odd-parity waves can be defined in a way exactly analogous to the nonrelativistic case.[8]

We have discussed the vector-type interaction at length. Other interactions can be handled in the same manner. For types S and P, all trigonometric functions are replaced with corresponding hyperbolic functions. For example Eq. (4.15) becomes $\tanh(g/4)=2\kappa/E$. The wave function ϕ of Eq. (4.19) applies to all three types of interactions. In other words, from the form of the wave function alone one cannot tell which of the three interactions has been used.

V. FORM FACTOR

Let us first motivate ourselves to look into the form factor. The structure of a composite system such as the deuteron can be probed by means of electron scattering. From the measured cross section one deduces a form factor $F(Q^2)$ where Q is the four-momentum transferred by a virtual photon and is related to the three-momentum transfer $\mathbf{q}$ by $Q^2=q_0{}^2-\mathbf{q}^2$. One then relates $F(Q^2)$ to, say, the charge density distribution in the target. If the target is assumed to be fixed and its charge density with respect to its center is $\rho(r)$, then $F(Q^2)$ is given by

$$\begin{aligned}F(Q^2)&=\int d\mathbf{r}\,\rho(r)e^{i\mathbf{q}\cdot\mathbf{r}}\\&=1-\tfrac{1}{6}\langle r^2\rangle\mathbf{q}^2+\cdots\ .\end{aligned}\tag{5.1}$$

Here it is understood that $q_0=0$, $\mathbf{q}^2=-Q^2$, $\int d\mathbf{r}\,\rho(r)=1$, and $\langle r^2\rangle=\int d\mathbf{r}\,\rho(r)r^2$. If $F(Q^2)$ is known up to large values of Q^2, one can determine $\rho(r)$ by Fourier transforming $F(Q^2)$.

On the theoretical side the usual practice is to solve the Schrödinger equation for the deuteron with some appropriate nucleon-nucleon potential, and calculate $\rho(r)$ and $F(Q^2)$ by means of Eq. (5.1). This is of course an approximation. In the actual scattering process the deuteron is not at rest. In particular, when the momentum transfer q is very large the deuteron suffers a big recoil. Can one still use the static distribution $\rho(r)$ to calculate $F(Q^2)$? This question cannot be answered within nonrelativistic quantum mechanics in which the structure of a bound system does not depend on the c.m. momentum.

Let us consider a one-dimensional simulation of this problem by means of our relativistic model. We call our two-body bound system the "deuteron" in this section. Imagine that the deuteron with momentum P meets a virtual "photon" with momentum q. The photon is absorbed by the proton (particle 1) in the deuteron, transferring momentum q. Let the deuteron momentum in the final state be P'.

We have to introduce the density and current through which the photon interacts with the deuteron. We define the density in the state ϕ_P, associated with particle 1 at position ξ, by

$$\rho(\xi)=\int_{-\infty}^{\infty}dx_2\,|\phi_P(x_1,x_2)|^2_{x_1=\xi}\ , \tag{5.2}$$

where we used x_1 and x_2 instead of x and X. As an operator therefore we write the density as

$$\hat\rho(\xi)=\delta(x_1-\xi)\ . \tag{5.3}$$

Similarly we define the current operator at ξ by

$$\hat j(\xi)=\alpha_1\delta(x_1-\xi)\ . \tag{5.4}$$

The conservation law $[H,\hat\rho]=[P,\hat j]$ is satisfied. Because of translational invariance $\hat\rho(\xi)$ is related to $\hat\rho(0)$ by $\hat\rho(\xi)=e^{-i\hat P\xi}\hat\rho(0)e^{i\hat P\xi}$, and similarly for $\hat j$. It is easy to confirm that

$$[\hat\rho(0),K]=-i\hat j(0)\ , \tag{5.5}$$

$$[\hat j(0),K]=-i\hat\rho(0)\ , \tag{5.6}$$

which means that ρ and j form a Lorentz vector.[9]

The photon interacts with ρ and j. The relevant matrix elements can be evaluated as follows. For ρ we obtain

$$\begin{aligned}\rho_{P'P}&\equiv\int_{-\infty}^{\infty}d\xi\langle\phi_{P'}|\hat\rho(\xi)|\phi_P\rangle e^{iq\xi}\\&=2\pi\delta(P'-P-q)\frac{E'+E}{2E_0}F(Q^2)\ ,\end{aligned}\tag{5.7}$$

where

$$Q^2=q_0{}^2-q^2=(E'-E)^2-(P'-P)^2\ ,$$

and $F(Q^2)$ is what we call the form factor. Some details underlying Eq. (5.7) are given in the Appendix. For $q=0$, Eq. (5.7) is reduced to Eq. (4.24); hence $F(0)=1$. The $F(Q^2)$ can be expressed as a product of two invariant factors, i.e.,

$$F(Q^2)=\mathcal{F}(Q^2)\mathcal{G}(Q^2)\ . \tag{5.8}$$

The factor $\mathcal{F}(Q^2)$ is the Fourier transform of the (normalized) spatial overlap between $\phi_{P'}$ and ϕ_P, i.e.,

$$\begin{aligned}\mathcal{F}(Q^2)&=\tfrac{1}{2}\eta(E'+E)\\&\quad\times\int_{-\infty}^{\infty}dx\,\exp[-\eta(E'+E)|x|+iqx/2]\\&=\frac{1-\dfrac{Q^2}{4E_0{}^2}}{1-\left[1+\dfrac{1}{4\eta^2}\right]\dfrac{Q^2}{4E_0^2}}\ .\end{aligned}\tag{5.9}$$

In deriving Eq. (5.9) the following identity has been useful:

$$\frac{q^2}{(E'+E)^2}=\frac{Q^2}{Q^2-4E_0{}^2}\ . \tag{5.10}$$

The other factor $\mathcal{G}(Q^2)$ is related to the change in the spinor components of ϕ, i.e.,

$$\begin{aligned}\mathcal{G}(Q^2)&=\frac{4E_0}{\eta(E'+E)^2}(\phi_{P'}^\dagger\phi_P)_{x\to+0}\\&=\left[1-\frac{Q^2}{4E_0{}^2}\right]^{-1}\ .\end{aligned}\tag{5.11}$$

Hence, we obtain

$$F(Q^2)=\left[1-\left[1+\frac{1}{4\eta^2}\right]\frac{Q^2}{4E_0{}^2}\right]^{-1} . \quad (5.12)$$

The matrix element for j can be worked out in the same manner, with the result

$$j_{P'P}=2\pi\delta(P'-P-q)\frac{P'+P}{2E_0}F(Q^2) , \quad (5.13)$$

where $F(Q^2)$ is the same as that for $\rho_{P'P}$.

In order to see the rationale for introducing the form factor $F(Q^2)$ in the way as above, it would be useful to recall that, for a charged boson field the current is defined by

$$j^\mu=\frac{i}{2m}[\phi^\dagger\partial^\mu\phi-(\partial^\mu\phi^\dagger)\phi] , \quad (5.14)$$

where m is the mass of the boson. The matrix elements for the interaction between this j^μ and a photon take exactly the same form as Eqs. (5.7) and (5.13), except that E_0 and $F(Q^2)$ are replaced with m and unity, respectively. The deuteron is a boson, but it is a composite particle. Its internal structure gives rise to the form factor $F(Q^2)$.

In the usual naive approach one would start with the static density, i.e., the density in the rest frame with $P=0$:

$$\rho_{\rm st}\left[\frac{x}{2}\right]=\eta E_0 e^{-2\eta E_0|x|} . \quad (5.15)$$

Here $|x|/2$ is the distance from the center, which corresponds to r of Eq. (5.1).[10] Note also that this $\rho_{\rm st}$ follows from the relativistic wave function ϕ of Eq. (4.19). One then obtains the form factor as the Fourier transform of $\rho_{\rm st}$, i.e.,

$$F_{\rm st}(Q^2)=\left[1+\left[\frac{q}{4\eta E_0}\right]^2\right]^{-1} . \quad (5.16)$$

In this case it is understood that $q_0=0$, and hence $q^2=-Q^2$. Also $\mathcal{G}(Q^2)=1$ and one does not distinguish $F(Q^2)$ from $\mathcal{F}(Q^2)$. It is clear that $F_{\rm st}(Q^2)$ is different from $F(Q^2)$ of Eq. (5.12). From $\rho_{\rm st}$ the mean-square radius is obtained as

$$\left\langle\left[\frac{x}{2}\right]^2\right\rangle_{\rm st}=\frac{1}{8(\eta E_0)^2} , \quad (5.17)$$

which is related to $F_{\rm st}$ and F by

$$2\frac{\partial F_{\rm st}(Q^2)}{\partial Q^2}\bigg|_{Q^2=0}=\left\langle\left[\frac{x}{2}\right]^2\right\rangle_{\rm st} , \quad (5.18)$$

$$2\frac{\partial F(Q^2)}{\partial Q^2}\bigg|_{Q^2=0}=\left\langle\left[\frac{x}{2}\right]^2\right\rangle_{\rm st}+\frac{1}{2E_0{}^2} . \quad (5.19)$$

If this was for three dimensions, the factor 2 in the left-hand side of Eqs. (5.18) and (5.19) would be 6. Note the additional term in the RHS of Eq. (5.19).

Let us have a quantitative idea about the difference between F and $F_{\rm st}$ for the "deuteron." We denote the deuteron binding energy by B. Then $E_0=2m-B$, and

$$4\eta^2\approx B/m\approx 2\times 10^{-3} , \quad (5.20)$$

which is a measure of the difference between F and $F_{\rm st}$. To our pleasant surprise, therefore, $F_{\rm st}$ turned out to be a very good approximation. The effect of boosting is large for each of the two factors in F of Eq. (5.8), but for their product the effect is small.

There have been a few heuristic attempts at relativizing the form factor. Licht and Pagnameta[11] proposed a simple prescription. In terms of our notation, their F is related to $F_{\rm st}$ by

$$F(Q^2)=\left[1-\frac{Q^2}{4E_0{}^2}\right]^{(1-n)/2}F_{\rm st}\left[\frac{Q^2}{1-Q^2/4E_0{}^2}\right] , \quad (5.21)$$

where $n\,(=2)$ is the number of the particles in the system. The $F_{\rm st}$ in Eq. (5.21) is obtained from $F_{\rm st}(Q^2)$ by replaceing Q^2 with $Q^2/(1-Q^2/4E_0{}^2)$. Mitra and Kumari[12] proposed a modified prescription:

$$F(Q^2)=\left[1-\frac{Q^2}{4E_0{}^2}\right]^{(1-n)}F_{\rm st}\left[\frac{Q^2}{1-Q^2/4E_0{}^2}\right] . \quad (5.22)$$

This modified version is of particular interest in the sense that, when applied to our model, Eq. (5.22) leads to the correct form factor (5.12). It seems to us, however, that this agreement is fortuitous. The first factor in the RHS of Eq. (5.22) $(1-Q^2/4E_0{}^2)^{-1}$ coincides with $\mathcal{G}(Q^2)$ of Eq. (5.11). In its derivation,[11,12] this factor of Eq. (5.22) stems from the Jacobian for the transformation between the two frames representing the initial and final states of the target. On the other hand, $\mathcal{G}$ of Eq. (5.11) is due to the change in the spinor components of ϕ.

As we pointed out earlier, F is a product of two factors $\mathcal{F}$ and $\mathcal{G}$, and to each of them the relativistic correction is sizable. For their product, however, the correction becomes small. One naturally wonders as to whether this remarkable cancellation is peculiar to the interaction current that we have examined, or it is a general feature. In order to have some insight in this regard, let us consider the following problem. Instead of the vector current (ρ,j), suppose there is an interaction in which the scalar density appears. By the scalar density we mean

$$\hat{s}(\xi)=\beta_1\delta(x_1-\xi)+\beta_2\delta(x_2-\xi) . \quad (5.23)$$

As expected $\hat{s}(0)$ commutes with K.[9] The relevant matrix element is given by

$$s_{P'P}\equiv\int_{-\infty}^{\infty}d\xi\langle\phi_{P'}|\hat{s}(\xi)|\phi_P\rangle e^{iq\xi}$$
$$=2\pi\delta(P'-P-q)(E_0/m)\mathcal{F}(Q^2) . \quad (5.24)$$

This time only $\mathcal{F}(Q^2)$ appears. Since $\hat{s}$ is a scalar it is insensitive to the "spinor rotation" and hence the factor $\mathcal{G}$ does not appear. It is grossly misleading to use $F(Q^2)$ for $s_{P'P}$ in place of $\mathcal{F}(Q^2)$.

As the last such example let us mention the pseudoscalar density, defined by

$$\hat{\mathcal{P}}(\xi)=\alpha_1\beta_1\delta(x_1-\xi)+\alpha_2\beta_2\delta(x_2-\xi) . \quad (5.25)$$

Its matrix element is given by

$$\mathscr{P}_{P'P} \equiv \int_{-\infty}^{\infty} d\xi \langle \phi_{P'} | \hat{\mathscr{P}}(\xi) | \phi_P \rangle^{iq\xi}$$
$$= 2\pi\delta(P'-P-q)\frac{-E_0}{2m}\left[\frac{Q^2}{Q^2-4E_0{}^2}\right]^{1/2}\mathscr{F}(Q^2) . \quad (5.26)$$

Comparison of Eqs. (5.12), (5.24), and (5.26) shows that the different interactions probe different form factors of the bound system.

IV. SUMMARY AND DISCUSSION

By means of solvable examples in one space dimension we illustrated various features of the relativistic two-body problem. In particular, we examined how the energy and the structure of a bound system change when it is boosted. The relativistic energy-momentum relation is satisfied as expected. The size, e.g., the mean-square radius, of the bound system exhibits exact Lorentz contraction. In this connection, however, see question (ii) below. We also pointed out that the relation between the form factor $F(Q^2)$ and the "form" of the bound system is quite intriguing. Our model is obviously very simplistic. It is unfortunate that the interaction is restricted to the form of the δ function. Because of this perhaps the model is short of revealing rich dynamical features of the relativistic two-body problem which are yet to be discovered. Even with such restrictions, however, we would like to emphasize that a solvable relativistic two-body problem is a rarity.

Before ending let us mention some of the questions which came across our mind in the course of this work.

(i) We obtained the boost operator K when the interaction $f(x)$ is in the form of $\delta(x)$. Is there any other form of $f(x)$ for which K can be constructed? In that case the RHS of Eq. (3.6) would not vanish, and consequently $L\neq 0$. To be interesting the interaction has to be able to support a bound state.[3]

(ii) The bound-state wave function of our model exhibits exact Lorentz contraction. Is this a general feature of relativistic two-body problems? Lorentz contraction implies that the wave function simply scales when P is changed. How does the interaction in the instant form manage to yield such a wave function?[13]

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APPENDIX

Let us give some details regarding Eq. (5.7). There are two steps involved. The first one is

$$\rho_{P'P} = \int_{-\infty}^{\infty} d\xi \langle \phi_{P'} | e^{-i\hat{P}\xi}\hat{\rho}(0)e^{i\hat{P}\xi} | \phi_P \rangle e^{iq\xi}$$
$$= 2\pi\delta(P'-P-q)\langle \phi_{P'} | \hat{\rho}(0) | \phi_P \rangle . \quad (A1)$$

The next step is

$$\langle \phi_{P'} | \hat{\rho}(0) | \phi_P \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dX\, \phi_{P'}^{\dagger}\phi_P \delta(x_1)$$
$$= (\phi_{P'}^{\dagger}\phi_P)_{x=+0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dX\, \delta\left[X+\frac{x}{2}\right] \exp[i(P-P')X - \eta(E'+E)|x|]$$
$$= (\phi_{P'}^{\dagger}\phi_P)_{x=+0} \frac{2}{\eta(E'+E)} \mathscr{F}(Q^2) . \quad (A2)$$

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[1]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[2]N. Kemmer, Helv. Phys. Acta **10**, 48 (1937). He considered a model of the deuteron in terms of the two-body Dirac equation (in three dimensions). He noted that relativistic covariance restricts the interaction to the form of a δ function. He also found that the two-body Dirac equation in three dimensions with a δ-function interaction does not allow physically meaningful solutions. He therefore took the square-well potential, which is not exactly compatible with covariance. Unlike in three dimensions, the two-body Dirac equation in one dimension with a δ function interaction has well-behaved solutions. See, also, E. Fermi and C. N. Yang, Phys. Rev. **76**, 1739 (1949); H. M. Moseley and Nathan Rosen, *ibid.* **80**, 177 (1950); A. E. S. Green and T. Sawada, Rev. Mod. Phys. **39**, 594 (1967).

[3]If $m=0$, f_V can be any function of x. In this case, however, the two particles cannot be bound, even if f_V is a confining type.

[4]The same equation has been solved for V of $f_V(x)\propto\delta(x)$ by F. Guinea, R. E. Peierls, and R. Schrieffer, Phys. Scr. **33**, 282 (1986). We would like to thank Dr. M. Thies for bringing the work of Guinea *et al.* to our attention (after completing the

present paper). The overlap between the paper of Guinea *et al.* and ours is, however, very small. They did not discuss the boost operator nor the form factor.

[5]M. G. Calkin, D. Kiang, and Y. Nogami, Am. J. Phys. (to be published).

[6]At $|x|=a$, ϕ_2 and ϕ_4 are continuous, whereas ϕ_1 and ϕ_3 are generally discontinuous. This is why we prefer ϕ_2 over ϕ_1. The discontinuity of ϕ_1 and ϕ_3 does not give rise to any difficulty.

[7]Recall that $g>0$ is understood in Eq. (4.17). If $g<0$, Eq. (4.17) is replaced with $E_0=-2m\cos(g/4)$. In his study of the two-body Dirac equation with a square-well potential Kemmer[2] noted solutions which correspond to this and the one given by Eq. (4.26).

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[10]Sometimes in the literature, $\langle r^2\rangle$ with the relative coordinate $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ for a two-body system is called the mean-square radius. This is a misnomer; it is the diameter rather than radius.

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[12]A. N. Mitra and I. Kumari, Phys. Rev. D **15**, 261 (1977); D. P. Stanley and D. Robson, *ibid.* **26**, 223 (1982).

[13]In fact we have some reason to suspect that Lorentz contraction is not a necessary consequence of a relativistic model. This problem will be discussed in a separate paper by W. Glöckle and Y. Nogami (in preparation).