Scalar Casimir energies in $M^4 \times S^N$ for even N

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We construct a Green's-function formalism for computing vacuum-fluctuation energies of scalar fields in $4+N$ dimensions, where the extra N dimensions are compactified into a hypersphere S^N of radius a. In all cases a leading cosmological energy term $u_{\text{cosmo}} \propto a^N/b^{4+N}$ results. Here b is an ultraviolet cutoff at the Planck scale. In all cases an unambiguous Casimir energy is computed. For odd N these energies agree with those calculated by Candelas and Weinberg. For even N, the Casimir energy is logarithmically divergent: $u_{\text{Casimir}} \sim (\alpha_N/a^4)\ln(a/b)$. The coefficients α_N are computed in terms of Bernoulli numbers.

I. INTRODUCTION

The notion that the world possess a dimensionality higher than four has gained credence as a natural basis for gauge theories.[1] The original Kaluza-Klein five-dimensional model[2] unified electromagnetism and gravity, while higher-dimensional generalizations include Yang-Mills fields.[3] The Glashow-Weinberg-Salam electroweak model without fermions but including Higgs bosons can be derived from a six-dimensional Yang-Mills theory[4] and the revival of strings as fundamental entities necessitates at least ten dimensions.[5]

Where are the extra dimensions? Presumably curled up in a volume whose scale is set by the Planck length. The dynamics of this compactification is but dimly understood. It is plausible that a key role in keeping the extra dimensions small is provided by the Casimir effect,[6] the zero-point energy phenomenon that is the field-theoretic version of the van der Waals force.[7] The confinement of electromagnetic fields by ideal perfectly conducting parallel plates,[6] or by a perfectly conducting cylinder,[8] yields an attractive force, while a sphere gives rise to a repulsive force.[9] In higher dimensions it is the topology of space that provides the confining geometry.

The idea of Casimir compactification, that zero-point fluctuations stabilize the geometry of the extra dimensions, has been explored by several authors,[10–16] starting with Applequist and Chodos.[10] Candelas and Weinberg[12] showed stability could result for large numbers of scalar and Fermi fields in $4+N$ dimensions, when $N=3(\text{mod}4)$, while Chodos and Myers[15] explored graviton fluctuations and found unstable tachyonic behavior. Until recently, however, "for technical reasons" the dimensionality has been restricted to odd N. In the even-dimensional case an ultraviolet logarithmic divergence remains after all legitimate subtractions. Myers[16] has numerically computed the logarithmic term for gravity fluctuations using a ζ-function technique. Here, we will restrict our attention to scalar fields and use a Green's-function technique; the presumably more relevant gravitational fluctuations will be dealt with in a subsequent paper.

The organization of this paper is as follows. In Sec. II we will develop our Green's-function formalism and rederive the Casimir as well as cosmological energies for $M^4\times S^1$. In Sec. III we extend the calculation to general odd N. The difficulties posed by N even are surmounted in Sec. IV. Here we also find a Casimir energy proportional to $\ln(a/b)$, where b is the cutoff length (perhaps Planck) and a is the radius of the internal space (perhaps[4] the weak length $\sim 10^{-17}$ cm). We give the coefficient of this logarithm as a sum of Bernoulli numbers. The implications of this work and the prospect of extending it to calculating an effective potential including gravitational fluctuations is considered in Sec. V.

II. THE FORMALISM AND $N=1$

We compute the zero-point or Casimir energy of a massless scalar field in an $M^4\times S^N$ manifold from the vacuum expectation value of the energy-momentum tensor:

$$u(a) \equiv V_N \langle 0 \mid t^{00} \mid 0 \rangle , \tag{2.1}$$

where V_N is the volume of an N-sphere of radius a and

$$t_{AB} = \partial_A\phi\partial_B\phi - \tfrac{1}{2} g_{AB}\partial_C\phi\partial^C\phi . \tag{2.2}$$

The vacuum expectation values can be computed in terms of derivatives of the imaginary part of the Feynman Green's function

$$G(x,y;x',y') = i\langle 0 \mid T(\phi(x,y)\phi(x',y')) \mid 0 \rangle , \tag{2.3}$$

as follows:

$$\langle 0 \mid t^{00} \mid 0 \rangle = \lim_{(x,y)\to(x',y')} \partial_0\partial_0' \,\text{Im}G(x,y;x',y') . \tag{2.4}$$

The Lagrangian term in (2.2) makes no contribution to (2.1). In our approach the point-splitting limit in (2.4) is necessarily taken with a spacelike separation. We here have used x to stand for the four Minkowski coordinates and y for the coordinates on the N-sphere. Because of the translational invariance in x, we can express G as an inverse Fourier transform, by

$$G(x,y;x',y') = \int \frac{d^4k}{(2\pi)^4} e^{-ik_\mu(x^\mu - x'^\mu)} g(y,y';k^\lambda k_\lambda) , \tag{2.5}$$

and in terms of which the vacuum energy can be simply expressed as

$$u(a)=\frac{-iV_N}{2(2\pi)^4}\int d^3k\int_c d\omega\,\omega^2 g(y,y,k^\lambda k_\lambda)\,, \tag{2.6}$$

where the contour c of ω integration is both c_- and c_+ shown in Fig. 1(a) and is chosen to give the imaginary part needed in (2.4). The reduced Green's function g satisfies

$$(\nabla_N{}^2+k^\mu k_\mu)g(y,y',k^\lambda k_\lambda)=-\delta(y-y')\,, \tag{2.7}$$

where $\nabla_N{}^2$ is the Laplacian on S^N and $\delta(y-y')$ is the appropriate δ function.

The $N=1$ and 2 solutions are familiar from electrodynamic and acoustic phenomena; however, g can be found for all N by expanding in N-dimensional spherical harmonics:

$$\nabla_N{}^2 Y_l^m(y)=-\frac{M_l{}^2}{a^2}Y_l^m(y)\,, \tag{2.8}$$

whose eigenvalues and degeneracies are

$$M_l{}^2=l(l+N-1)\,,\qquad D_l=\frac{(2l+N-1)(l+N-2)!}{(N-1)!l!}\,. \tag{2.9}$$

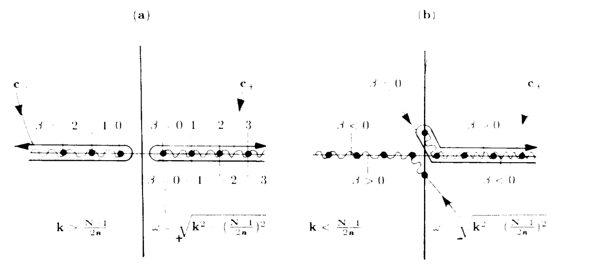

FIG. 1. The ω plane for odd N showing the Green's-function contours used and the branch cut for the two $\beta=0$ points $\{\beta^2\equiv[(N-1)/2]^2+a^2\omega^2-a^2k^2\}$. (a) is for $k>(N-1)/2a$ and (b) is for $k<(N-1)/2a$. The poles in the sum in (2.10) are only at $\beta=\pm(N-1)/2,\pm(N+1)/2,\pm(N+3)/2,\ldots$.

Use of the generalized addition formula for hyperspherical harmonics

$$\sum_m Y_l^m(y)Y_l^{m*}(y)=\frac{D_l}{V_N}$$

leads to the following expression for the energy (2.6):

$$u(a)=-\frac{i}{(2\pi)^4}\int d^3k\int_{c_+} d\omega\,\omega^2\sum_{l=0}^{\infty}\frac{D_l}{(M_l{}^2/a^2+k^2-\omega^2)}\,, \tag{2.10}$$

where the integrand's dependence on ω^2 has been used to combine the two parts of the c contour to only one on the right c_+ as shown in Fig. 1.

As is obvious from (2.10), the vacuum energy of a massless scalar in $M^4\times S^N$ is a linear sum of vacuum energies of massive scalars in 4 dimensions. The mode sum on l diverges for $N>1$ and the momentum integrals diverge for all N. To obtain finite Casimir energies we can subtract off divergences identifiable as contact or cosmological terms from the outset or we can insert cutoffs. Because the l sum is finite for the $N=1$ case we consider that case first. The masses $(M_l{}^2=l^2)$ and degeneracies $(D_0=1,\ D_{l\geq 1}=2)$ give

$$\sum_{l=0}^{\infty}\frac{D_l}{M_l{}^2/a^2+k^2-\omega^2}=\frac{a\pi}{(k^2-\omega^2)^{1/2}}\coth[\pi a(k^2-\omega^2)^{1/2}]$$
$$=\frac{a\pi}{(k^2-\omega^2)^{1/2}}\left[1+\frac{2}{e^{2\pi a(k^2-\omega^2)^{1/2}}-1}\right]. \tag{2.11}$$

The sum has been written as an asymptotic part and a remainder. The asymptotic part produces an infinite "cosmological term" in the energy and is either subtracted off completely or regulated by inserting a cutoff ($\omega_{\max}\simeq b^{-1}$, b is presumably at the Planck scale) resulting in a cosmological energy density,

$$u_{\text{cosmo}}(a)=\frac{1}{(2\pi)^4}\int d^3k\,2\int_k^{\infty}d\omega\,\omega^2\frac{a\pi}{(\omega^2-k^2)^{1/2}}$$
$$=\frac{V_1}{80\pi^2b^5}\,, \tag{2.12}$$

where $V_1=2\pi a$. Here we have taken an abrupt cutoff in ω; however, any other technique also yields $u_{\text{cosmo}}\propto V_1/b^5$. It is important to notice that the sum in (2.11) has only simple poles; however, the part that we identify as a cosmological term and subtract off has branch points at $\omega=\pm k$. For odd N, including the $N=1$ case, the branch cuts are drawn away from each other on the real ω axis out to $\pm\infty$ (see Fig. 1). The remainder of (2.11) produces the unique Casimir energy and is easily evaluated by (1) integrating over the 4π solid angle in the momentum element d^3k, (2) distorting the contour c_+ to one along the imaginary ω axis, $\omega=i\xi(-\infty\leq\xi\leq\infty)$, and (3) replacing ξ and k by plane polar coordinates $k=K\cos\theta$, $\xi=K\sin\theta$, and integrating over first θ and then K. The result is

$$u_{\text{Casimir}}=-\frac{1}{64\pi^2a^4}\int_0^\infty (aK)^4 d(aK)^2\frac{\pi}{aK}\frac{2}{e^{2\pi aK}-1}$$
$$=-\frac{3\zeta(5)}{64\pi^6 a^4}=\frac{-5.055\,807\,6\times 10^{-5}}{a^4}\,, \quad (2.13)$$

where ζ is the Riemann ζ function:

$$\zeta(s)\equiv\frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}dx\,. \quad (2.14)$$

Equation (2.13) is the result first obtained by Appelquist and Chodos.[10]

III. THE GENERAL ODD-N CASE

The Casimir energy for arbitrary odd N can be extracted similarly. We first define a new mode index m,

$$m\equiv l+(N-1)/2\,, \quad (3.1)$$

in terms of which M_l^2 and D_l of (2.9) can be written as functions of m^2:

$$M_m{}^2=m^2-[(N-1)/2]^2\,,$$
$$D_m{}'=\frac{2}{(N-1)!}\left[m^2-\left[\frac{N-3}{2}\right]^2\right]\times\left[m^2-\left[\frac{N-5}{2}\right]^2\right]\cdots(m^2-1^2)m^2\,. \quad (3.2)$$

The sum in (2.10) becomes

$$\sum_{l=0}^{\infty}\frac{D_l}{M_l{}^2/a^2+k^2-\omega^2}=a^2\sum_{m=0}^{\infty}\frac{D'_m}{m^2-\beta^2}$$
$$=a^2\sum_{m=0}^{\infty}\left[\frac{D'_\beta}{m^2-\beta^2}+\text{polynomial in } m^2 \text{ and } \beta^2\right]\,, \quad (3.3)$$

where $\beta^2\equiv[(n-1)/2]^2+a^2\omega^2-a^2k^2$. The polynomial terms make no contribution to (2.10) and can be discarded. Notice that from (3.1) the m sum should start at $(N-1)/2$; however, since $D'_m=0$ for $0\le m\le(N-3)/2$ we can start at $m=0$. The sum of pole terms in (3.3) can easily be evaluated as

$$\sum_{m=0}^{\infty}\frac{1}{m^2-\beta^2}=-\frac{\pi}{2\beta}\cot(\pi\beta)-\frac{1}{2\beta^2}\,. \quad (3.4)$$

Since D'_β/β^2 is a polynomial in β^2, the $-1/2\beta^2$ term in (3.4) makes no net contribution to (2.10) and can be discarded. The remaining $\cot(\pi\beta)$ term contains both the divergent cosmological energy and the finite Casimir energy. In Fig. 1 the branch cuts for β are shown as well as the distorted c_+ contour which avoids the $\beta=0$ branch points. As in (2.11) we write the $\cot\pi\beta$ term as an asymptotic part plus a remainder; the former yields the cosmological energy

$$u_{\text{cosmo}}\propto\frac{V_N}{b^{N+4}}\,. \quad (3.5)$$

The remainder yields the finite Casimir energy. It is best evaluated by following the three steps described after (2.12):

$$u(a)=-\frac{1}{64\pi^2a^4}\,\text{Re}\int_0^\infty (aK)^4 d(aK)^2 D'_\beta\frac{\pi i}{\beta(e^{-2\pi i\beta}-1)}\,, \quad (3.6)$$

where now

$$\beta^2=\left[\frac{N-1}{2}\right]^2-a^2K^2\,.$$

The integral in (3.6) is most easily evaluated by changing from K to β as the integration variable. The contour for β which comes from $0<K<\infty$ is not suited for extracting the Casimir energy. This energy is most easily computed by integrating β first vertically,

$$\beta=\frac{N-1}{2}+iy,\quad 0\le y\le\infty\,,$$

and then horizontally,

$$\beta=x+i\infty,\quad \frac{N-1}{2}\ge x\ge 0\,.$$

The nondivergent part of the integral on the horizontal part of the new contour is imaginary and hence does not contribute to the Casimir energy (3.6). By substituting

$$\beta=\frac{N-1}{2}+iy\,,$$
$$d(aK)^2=-\beta\,d\beta=-i\beta\,dy \quad (3.7)$$

into (3.6), we have

$$u_{\text{Casimir}}(a)=-\frac{1}{64\pi^2a^4}\,\text{Re}\int_0^\infty\left[[y^2-i(N-1)y]^2\times D_{iy}\frac{2\pi}{e^{2\pi y}-1}\right]dy\,. \quad (3.8)$$

For $N\ge 1$ this is nothing more than a sum of Riemann ζ functions at odd integer values $4+N, 4+N-2, \ldots, 3$. For example, the $N=3$ and $N=5$ expressions are

$$u_3(a)=\frac{1}{32\pi a^4}\left[\frac{\Gamma(7)\zeta(7)}{(2\pi)^7}-\frac{13\Gamma(5)\zeta(5)}{(2\pi)^5}+\frac{4\Gamma(3)\zeta(3)}{(2\pi)^3}\right]=\frac{1}{a^4}(7.568\,704\,6\ldots\times 10^{-5})\ ,$$

$$u_5(a)=-\frac{1}{384\pi a^4}\left[\frac{\Gamma(9)\zeta(9)}{(2\pi)^9}-\frac{103\Gamma(7)\zeta(7)}{(2\pi)^7}+\frac{604\Gamma(5)\zeta(5)}{(2\pi)^5}-\frac{192\Gamma(3)\zeta(3)}{(2\pi)^3}\right] \tag{3.9}$$

$$=\frac{1}{a^4}(4.283\,038\,1\ldots\times 10^{-4})\ ,$$

respectively. Results for larger N are graphed in Fig. 2 and agree with the findings of Candelas and Weinberg.[12]

IV. THE EVEN-N CASE

In the case where the internal space is a sphere S^N with N even, we find a cutoff-dependent divergent cosmological energy exactly as in the odd-N case [see (3.5)]; however, now the Casimir energy also diverges. The divergence is logarithmic in a/b and fortunately, the coefficient of that logarithm is independent of the actual cutoff technique used. We evaluate $u(a)$ of (2.10) by shifting the index mode as was done in Sec. III, with

$$m\equiv l+\frac{N-1}{2}\ ,$$
$$M_m{}^2=m^2-\left[\frac{N-1}{2}\right]^2\ , \tag{4.1}$$
$$D'_m=\frac{2}{(N-1)!}\left[m^2-\left[\frac{N-3}{2}\right]^2\right]\times\left[m^2-\left[\frac{N-5}{2}\right]^2\right]\cdots[m^2-(\tfrac{1}{2})^2]m\ .$$

The mode index m is now half-integral and, because of the vanishing of D'_m, the sum on m can start at $\frac{1}{2}$. If $D'_m/m(m^2-\beta^2)$ is again factored into a pole term plus a polynomial as in (3.3), the polynomial contributions vanish, and the remaining energy can be written as

$$u(a)=-\frac{ia^2}{(2\pi)^4}\int d^3k\,I(c_+,\Sigma)\ , \tag{4.2}$$

where the functional I in (4.2) is the contour integral in the ω plane:

$$I(c,F)\equiv\int_c d\omega\,\omega^2\frac{D'_\beta}{\beta}F(\omega)\ . \tag{4.3}$$

The function Σ in (4.2) is the infinite sum

$$\Sigma(\omega)\equiv\sum_{m=1/2}^{\infty}\frac{m}{m^2-\beta^2}\ , \tag{4.4}$$

β^2 is again

$$\beta^2=a^2\omega^2-a^2k^2+\left[\frac{N-1}{2}\right]^2\ , \tag{4.5}$$

and c_+ is the right-hand contour shown in Fig. 3(a). The m sum in (4.4) necessarily diverges unless we regulate it in some manner. Here we simply subtract a constant, $1/(m+\frac{1}{2})$, from each mode, i.e., we write

$$\Sigma(\omega)=\frac{1}{2}\sum_{m'=1}^{\infty}\left[\frac{1}{m'+(\beta-\frac{1}{2})}-\frac{1}{m'}+\frac{1}{m'-(\beta+\frac{1}{2})}-\frac{1}{m'}+\frac{2}{m'}\right]$$
$$=\frac{1}{2}[-\psi(\tfrac{1}{2}+\beta)-\psi(\tfrac{1}{2}-\beta)-2\gamma+2\zeta(1)]\ , \tag{4.6}$$

where $m'\equiv m+\frac{1}{2}$, and $\zeta(1)$ is an infinite constant. The

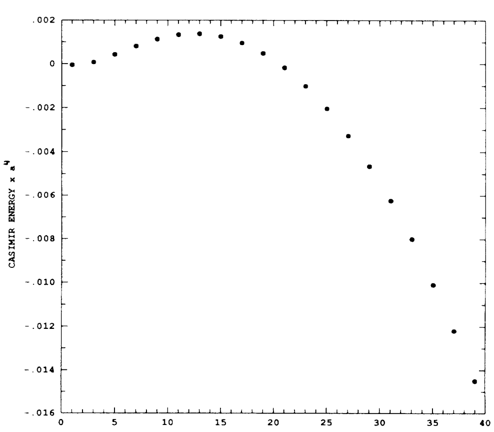

FIG. 2. Plot of $u_{\text{Casimir}}\times a^4$ for odd N.

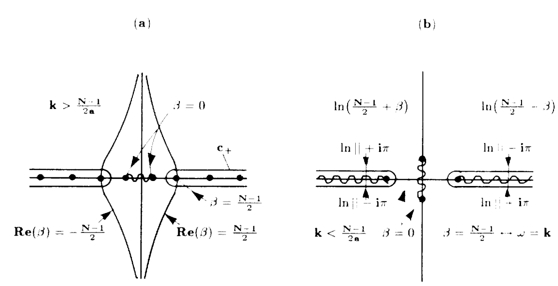

FIG. 3. The ω plane for even N, showing the Green's-function contours used and the branch cuts for $\beta=0$. (a) is for $k>(N-1)/2a$ and shows the cuts $\text{Re}(\beta)=\pm(N-1)/2$. (b) is for $k<(N-1)/2a$ and shows the branch cuts for $\ln[(N-1)/2\pm\beta]$.

digamma function $\psi(z)$ is defined by

$$\psi(z)=\frac{d}{dz}\ln\Gamma(z)\,, \tag{4.7}$$

and $\gamma=-\psi(1)\simeq 0.57721\ldots$ is Euler's constant. The digamma function $\psi(z)$ is analytic over the complex plane except at $z=0,-1,-2,\ldots,$ where it has simple poles with residues all equal to -1. To evaluate (4.2) we use the identities

$$\begin{aligned}\psi(\tfrac{1}{2}\pm\beta)&=\psi\left[\frac{N-1}{2}\pm\beta\right]-\frac{1}{(N-3)/2\pm\beta}\\&\quad-\frac{1}{(N-5)/2\pm\beta}-\cdots-\frac{1}{\frac{1}{2}\pm\beta}\,,\\ \psi(\tfrac{1}{2}-\beta)&=\psi(\tfrac{1}{2}+\beta)-\pi\tan\beta\pi\,,\end{aligned} \tag{4.8}$$

and the representation

$$\psi\left[\frac{N-1}{2}\pm\beta\right]=\ln\left[\frac{N-1}{2}\pm\beta\right]-\frac{1}{N-1\pm2\beta}-2\int_0^\infty\frac{t\,dt}{e^{2\pi t}-1}\left[\frac{1}{t^2+[(N-1)/2\pm\beta]^2}\right], \tag{4.9}$$

which is valid for $\mathrm{Re}[(N-1)/2\pm\beta]>0$. We connect the branch points $\beta=0$ by the branch cut as shown in Fig. 3. We choose the branch cut in $\ln[(N-1)/2+\beta]$ to run along the negative real ω axis starting at $\beta=-(N-1)/2$ (i.e., for $\omega=-k$) as shown in Fig. 3(b). For the $+$ sign this choice makes (4.9) valid in the ω plane to the right of the line $\mathrm{Re}(\beta)=-(N-1)/2$. The branch cut for $\ln[(N-1)/2-\beta]$ is drawn to the right from $\beta=(N-1)/2$ (i.e., for $\omega=k$) on the positive real ω axis making (4.9) valid for the $-$ case to the left of $\mathrm{Re}(\beta)=(N-1)/2$.

Using (4.8) and (4.9) we can rewrite (4.6) as

$$\begin{aligned}\Sigma(\omega)&=\frac{1}{(N-3)/2+\beta}+\frac{1}{(N-5)/2+\beta}+\cdots+\frac{1}{\frac{1}{2}+\beta}+\frac{\frac{1}{2}}{(N-1)/2+\beta}-\ln\left[\frac{N-1}{2}+\beta\right]\\&\quad+\frac{\pi}{2}\tan\pi\beta+2\int_0^\infty\frac{t\,dt}{e^{2\pi t}-1}\left[\frac{1}{t^2+[(N-1)/2+\beta]^2}\right]-\gamma+\zeta(1)\,,\end{aligned} \tag{4.10}$$

valid for $\mathrm{Re}\beta>-(N-1)/2$ (see Fig. 3). The constant $-\gamma+\zeta(1)$ makes no contribution to (4.2) and will be discarded. We rewrite the logarithm term in (4.10) as

$$-\ln\left[\frac{N-1}{2}+\beta\right]=-\tfrac{1}{2}\ln\left[\left[\frac{N-1}{2}\right]^2-\beta^2\right]+\tfrac{1}{2}\ln\left[\frac{(N-1)/2-\beta}{(N-1)/2+\beta}\right], \tag{4.11}$$

and identify the first term as producing most of the cosmological energy density:

$$\begin{aligned}u_{\log}(a)&\equiv-\frac{ia^2}{(2\pi)^4}\int d^3k\,I(c_+,-\tfrac{1}{2}\ln(a^2k^2-a^2\omega^2))\\&=\frac{a^2}{(2\pi)^2}\int_0^\infty k^2dk\int_k^\infty d\omega\,\omega^2\frac{D'_\beta}{\beta}\\&=\frac{a^2}{(2\pi)^2}\int_0^{1/b}d\omega\,\omega^2\int_0^\omega dk\,k^2\frac{D'_\beta}{\beta}\\&=\frac{1}{2\pi^2(N+4)(N+1)[(N-1)!!]^2}\frac{a^N}{b^{N+4}}+\frac{1}{24\pi^2}\frac{N(N-1)}{(N+2)[(N-1)!!]^2}\frac{a^{N-2}}{b^{N+2}}\\&\quad+\cdots+\frac{1}{72\pi^2(N-1)/2}\frac{a^2}{b^6}\,.\end{aligned} \tag{4.12}$$

The constants in (4.12) depend on the cutoff technique used. We include them because they are easy to compute and because they illustrate what kind of subtractions must be made in the final dynamical equation for a. The remaining terms of (4.7)

$$\Sigma'(\omega)\equiv\Sigma(\omega)+\gamma-\zeta(1)+\tfrac{1}{2}\ln\left[\left[\frac{N-1}{2}\right]^2-\beta^2\right], \tag{4.13}$$

contribute two more terms to the cosmological energy ($\sim 1/b^4$ and $\sim 1/a^2b^2$) as well as produce the Casimir energy ($\sim 1/a^4$). To evaluate these we integrate $\Sigma'(\omega)$ along the c_v contour and subtract its integral along the two-part $c_{+\infty}$ contour [see Fig. 4(a)]:

$$I(c_+,\Sigma')=I(c_v,\Sigma')-I(c_{+\infty},\Sigma')\,. \tag{4.14}$$

Equation (4.10) without the constants and with the cosmological logarithm term subtracted [see (4.13)] gives a valid expression for $\Sigma'(\omega)$ along $c_{+\infty}$. However, along $c_{+\infty}$ the remaining log and tan terms combine as

$$\tfrac{1}{2}\ln\left[\frac{(N-1)/2-\beta}{(N-1)/2+\beta}\right]+\frac{\pi}{2}\tan\pi\beta=\frac{\mp i\pi}{e^{\mp 2\pi i\beta}+1}\,, \tag{4.15}$$

with the $-$ ($+$) sign being valid in the first (fourth) quadrant. The combined terms in (4.15) vanish exponentially fast away from the real ω axis and contribute at most a polynomial in k^2 to (4.14). In (4.2) integrals of n powers of k^2 produce contact terms, i.e., terms $\propto(\nabla^2)^n\delta(\mathbf{x}-\mathbf{x}')$ and vanish in (2.4) before the limit is taken. The remaining terms of $\Sigma'(\omega)$ which contribute to (4.14) are

$$\Sigma_{+\infty}\equiv\frac{1}{[(N-3)/2+\beta]}+\frac{1}{[(N-5)/2+\beta]}+\cdots+\frac{1}{(\frac{1}{2}+\beta)}+\frac{\frac{1}{2}}{(N-1)/2+\beta}$$
$$+2\int_0^\infty\frac{t\,dt}{e^{2\pi t}-1}\left[\frac{1}{t^2+[(N-1)/2+\beta]^2}\right]. \tag{4.16}$$

These terms are analytic to the right of c_v and hence can simply be subtracted from $\Sigma'(\omega)$ in the c_v integral of (4.14), i.e.,

$$I(c_v,\Sigma'-\Sigma_{+\infty})=I(c_v,\Sigma')-I(c_{+\infty},\Sigma_{+\infty})\,. \tag{4.17}$$

Using (4.6), (4.8), (4.9), and (4.16), we can express

$$\Sigma_v\equiv\Sigma'-\Sigma_{+\infty}=\frac{\beta}{[(N-3)/2]^2-\beta^2}+\frac{\beta}{[(N-5)/2]^2-\beta^2}+\cdots+\frac{\beta}{(\frac{1}{2})^2-\beta^2}+\frac{\beta/2}{[(N-1)/2)^2-\beta^2}$$
$$-\frac{i}{2}\int_0^\infty\frac{dt}{e^{2\pi t}-1}\left[\frac{1}{\beta+(N-1)/2+it}-\frac{1}{\beta+(N-1)/2-it}+\frac{1}{\beta-(N-1)/2-it}-\frac{1}{\beta-(N-1)/2+it}\right] \tag{4.18}$$

for $|\mathrm{Re}\beta|<(N-1)/2$.

The part of $u(a)$ in (4.2) not given by (4.12) is

$$u(a)-u_{\log}(a)=-\frac{ia^2}{(2\pi)^4}\int d^3k\,I(c_v,\Sigma_v)\,. \tag{4.19}$$

Because Σ_v is odd in β the only contribution to $I(c_v,\Sigma_v)$ comes when c_v skirts the branch cut in $\beta(\omega)$ [see Fig. 4(a)]. If we replace c_v by a contour completely encircling the cut we get twice (4.19), i.e.,

$$u(a)-u_{\log}(a)=-\frac{ia^2}{2(2\pi)^4}\int d^3k\,I(c_0,\Sigma_v)\,, \tag{4.20}$$

where c_0 is shown in Fig. 4(b). This integral is evaluated by distorting c_0 to c_∞ (a complete circle at $\omega=\infty$), and picking up residues at the poles of $(D'_\beta/\beta)\Sigma_v(\omega)$. All terms in Σ_v when integrated around c_∞ produce polynomials in k^2 and hence contribute only contact terms to (4.20). There are only six poles in Σ_v which are not canceled by zeros in D'_β/β and they are (with independent $\pm$ signs)

$$\beta=\pm(N-1)/2\,,$$

and (4.21)

$$\beta=\pm(N-1)/2\pm it\,,$$

giving

$$u(a)-u_{\log}(a)=-\frac{1}{(2\pi)^2}\int_0^\infty dk\,k^2\left[-\tfrac{1}{2}D_0k+2\,\mathrm{Im}\int_0^\infty\frac{dt}{e^{2\pi t}-1}D_{it}\{k^2+[i(N-1)t-t^2]/a^2\}^{1/2}\right]$$
$$=\frac{1}{32\pi^2b^4}-\frac{1}{8\pi^2}\int_0^\infty\frac{dt}{e^{2\pi t}-1}\,\mathrm{Im}\left\{\frac{1}{b^4}D_{it}+\frac{1}{a^2b^2}D_{it}[i(N-1)t-t^2]\right.$$
$$-\frac{1}{2a^4}\left[\ln\left[\frac{a}{b}\right]-\tfrac{1}{2}\ln\left[\frac{i(N-1)t-t^2}{4}\right]-\frac{1}{4}\right]$$
$$\left.\times D_{it}[i(N-1)t-t^2]^2\right\}. \tag{4.22}$$

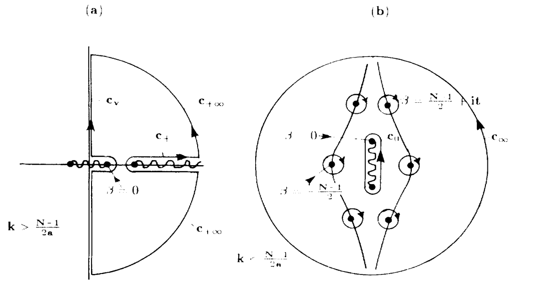

FIG. 4. Contours in the ω plane used in evaluating $I(c,\Sigma)$. (a) is for $k>(N-1)/2a$ and shows the contours c_v and $c_{+\infty}$. (b) is for $k<(N-1)/2a$ and shows the six poles in Σ_v as given in (4.21).

TABLE I. For even N the Casimir energy has the form $u_{\text{Casimir}}=(1/a^4)[\alpha_N \ln(a/b)+\beta_N]$, where a is the radius of S^N and b is a cutoff at the Planck length. The coefficient α_N, given here, is independent of cutoff technique.

N	α_N
2	$-8.041\,363\,7\times10^{-5}$
4	$-4.992\,346\,6\times10^{-4}$
6	$-1.314\,488\,8\times10^{-3}$
8	$-2.505\,290\,3\times10^{-3}$
10	$-4.035\,553\,5\times10^{-3}$
12	$-5.873\,420\,2\times10^{-3}$
14	$-7.993\,120\,1\times10^{-3}$
16	$-1.037\,396\,7\times10^{-2}$
18	$-1.299\,918\,0\times10^{-2}$
20	$-1.585\,493\,3\times10^{-2}$

The first three terms are additional contributions to the cosmological energy density and the latter the Casimir energy:

$$u_{\text{Casimir}}=\frac{1}{a^4}[\alpha_N \ln(a/b)+\beta_N]\ , \tag{4.23}$$

where

$$\alpha_N=\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{e^{2\pi t}-1}\,\text{Im}\{D_{it}[i(N-1)t-t^2]^2\}\ , \tag{4.24a}$$

and

$$\beta_N=-\frac{1}{32\pi^2}\int_0^\infty \frac{dt}{e^{2\pi t}-1}\left[\frac{1}{2}\,\text{Im}\{D_{it}[i(N-1)t-t^2]^2\}\left[\ln\frac{(N-1)^2t^2+t^4}{16}+1\right]\right.$$
$$\left.+\,\text{Re}\{D_{it}[i(N-1)t-t^2]^2\}\arctan\frac{N-1}{t}\right]\ . \tag{4.24b}$$

Here we have given meaning to the divergent logarithms by cutting the frequency off at $1/b$, where b is presumably at the Planck scale. The coefficient α_N in (4.2) is independent of the cutoff technique, e.g., the same result is obtained by a ζ-function regularization method. The coefficient of the log is easily expressed in terms of Bernoulli numbers since

$$\int_0^\infty \frac{dt\,t^{2k-1}}{e^{2\pi t}-1}=\frac{|B_{2k}|}{4k}\ . \tag{4.25}$$

The first few values are

$$\alpha_2=\frac{1}{8\pi^2}\left(\tfrac{1}{12}|B_6|-\tfrac{1}{4}|B_4|\right)$$
$$=-\frac{1}{1260\pi^2}=-8.041\,363\,7\times10^{-5}\ ,$$
$$\alpha_4=\frac{-1}{3!8\pi^2}\left(\tfrac{1}{16}|B_8|-\tfrac{85}{24}|B_6|+\tfrac{153}{16}|B_4|\right)$$
$$=-\frac{149}{50\,240\pi^2}=-4.992\,346\,6\times10^{-4}\ , \tag{4.26}$$
$$\alpha_6=\frac{1}{5!8\pi^2}\left(\tfrac{1}{20}|B_{10}|-\tfrac{210}{16}|B_8|+\tfrac{3024}{12}|B_6|-\tfrac{4325}{8}|B_4|\right)$$
$$=-\frac{411}{31\,680\pi^2}=-1.314\,488\,8\times10^{-3}\ .$$

Values for larger N are tabulated in Table I and graphed in Fig. 5. On the other hand, the values of β_N are dependent on the cutoff technique employed. We merely note

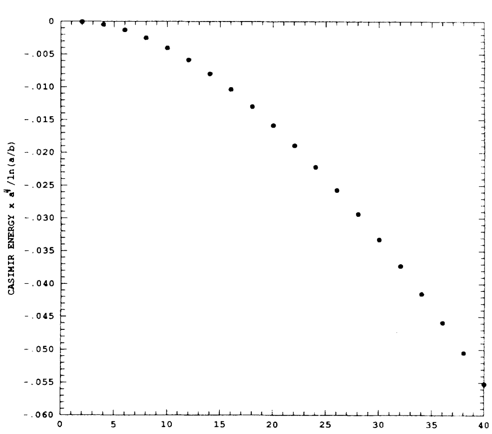

FIG. 5. Plot of α_N for even N, where $u_{\text{Casimir}}\sim\alpha_N \ln(a/b)/a^4$.

that with the form given by (4.24b), the values for β_N are smaller in magnitude than α_N by a factor of 2 or more, typically, as well as opposite in sign for $N>4$. In any case, for the model we have in mind (Ref. 4) $a/b \approx 10^{16}$ and the unique logarithm term dominates.

V. CONCLUSIONS

We have developed a simple and physical scheme for computing the Casimir energy of a scalar field in $4+N$ dimensions, where the extra N dimensions are compactified into a hypersphere of radius a. The known results[12] for odd N are confirmed, while for even N, the Casimir energy contains an irreducible logarithm divergence.[16] This weak infinity is assumed cut off by the Planck scale b, that is, interpreted as $\ln(a/b)$.

In this paper we have been primarily concerned with developing a careful Green's-function treatment of vacuum fluctuation phenomena, rather than using the formalistic ζ-function prescription. This work is thus complementary to previous efforts on the electrodynamic and chromodynamic Casimir effect.[17] In subsequent papers we will treat the gauge-field and graviton fluctuations as well as consider other geometries for the internal space. In principle, such extensions of the Green's-function technique should pose no serious difficulties. We will then be able to make contact with realistic gauge/gravity models.

However, at this intermediate stage we point out that we have now brought the theory of even-dimensional curved-space Casimir forces to about the stage the theory of such forces has reached for the bag model of QCD: there a logarithm divergence persists after all contact terms have been removed, and which is also estimated by physical arguments.[17] In both cases, the idealized geometry used is too extreme.

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