Finite-temperature quantum field theory in curved spacetime: Quasilocal effective Lagrangians

B. L. Hu, R. Critchley,* and Aris Stylianopoulos
Department of Physics and Astronomy, Univeristy of Maryland, College Park, Maryland 20742
(Received 18 August 1986)

We use momentum-space techniques and a quasilocal expansion to derive the imaginary-time thermal Green's functions and the one-loop finite-temperature effective Lagrangians for $\lambda\phi^4$ fields in curved spacetimes. These approximations are useful for treating quasiequilibrium conditions associated with gradual changes in the background fields and the background spacetimes. For problems in spacetimes with small curvature, we use a Riemann normal coordinate for the background metric, a derivative expansion for the background field, and a small-proper-time Schwinger-DeWitt expansion to derive the finite-temperature effective Lagrangians. For problems in homogeneous cosmology we consider conformally related fields and the Robertson-Walker universe as background to carry out finite-temperature perturbation calculations. We study a massless conformal $\lambda\phi^4$ theory in a Bianchi type-I universe and derive the finite-temperature effective Lagrangian in orders of small anisotropy. The quasilocal method presented here is related to the adiabatic method in finite-temperature quantum field theory presented earlier in similar settings. These results are useful for the study of quantum thermal processes in the early Universe.

I. INTRODUCTION

Finite-temperature quantum field theory in curved spacetime is an important tool for the study of thermal quantum processes near black holes[1] and in the early Universe.[2] Development of these theories has spanned about a decade in history, trailing almost directly after theories in flat space.[3] One may say that it began with the work of Hartle and Hawking and Gibbons and Perry[4] on applying the imaginary-time thermal Green's function for the description of Hawking radiation in black-hole spacetimes. This technique is particularly suited for spacetimes possessing Euclidean sections such as Rindler, Schwarzschild, and de Sitter spacetimes. The nature of thermal radiation in these spacetimes associated with vacuum fluctuations around event horizons is very different from that associated with the quanta of ambient radiation in general cosmological spacetimes such as the Robertson-Walker universe. For curved but static or stationary spacetimes, finite-temperature field theory is well defined, as the existence of a global timelike Killing vector permits a thermal equilibrium state to persist. Generalization of flat-space thermal Green's-function techniques has been carried out in the work of Dowker, Critchley, Kennedy, and Altaire.[5] These techniques are applicable for spacetimes which are conformally static, such as the Robertson-Walker universes, as seen in the work of Kennedy,[6] Drummond,[7] and Critchley, Davies, and Kennedy.[8] The basic notions and the conditions underlying a viable finite-temperature field theory in general dynamical spacetimes were first discussed by Hu.[9] In this work, the conformal properties of fields and spacetimes are invoked as conditions for maintaining thermal equilibrium for free fields. By way of a quasiadiabatic expansion the notion of an adiabatic n-particle state was introduced and the finite-temperature energy density was calculated for near-conformal fields and for spacetimes nearly conformally static. The adiabatic method was also applied to the derivation of effective quasipotentials for self-interacting fields in Robertson-Walker[10] and Bianchi type-I universes.[11] The finite-temperature method and results obtained in this work and generalizations thereof are useful for the discussion of finite-temperature symmetry breaking,[12] particle production and back reaction,[13–16] and entropy generation[17–20] problems, although proper treatment of these processes necessitates the development of nonequilibrium methods, such as real-time Green's function[21,22] and Wigner function techniques.[23,24]

An approach parallel to the adiabatic expansion method in finite-temperature field theory (as exemplified in Ref. 8 and discussed in Ref. 10) is by way of the Schwinger-DeWitt proper-time expansion[25] in the effective Lagrangian. The two approaches have been shown to yield identical results in the treatment of ultraviolet divergences in zero-temperature field theories.[26] The physical meaning behind these two expansions are however different. The proper time s can be viewed as a scaling parameter on the curvature-induced mass α [$\alpha^2=m^2+(1-\xi)\xi_d R+\lambda\hat{\phi}^2/2$, see Eq. (2.5)] in a $\lambda\phi^4$ theory, which determines the energy or length scale of the relevant physical processes. Small s gives the short-distance (local), high-frequency (UV) behavior while large s gives the long-distance (global), low-frequency (IR) behavior. This is why a small-proper-time expansion is usually used to identify the ultraviolet divergences. By contrast, in the adiabatic expansion the relevant parameter is the nonadiabaticity parameter[9] $\bar{\Omega}\equiv\Omega'/\Omega^2$, which measures the time rate of change of the natural frequency Ω of each normal mode of the system compared to the background dynamics. It measures the *rate* rather than the *scale* of variation. Since it is a criterion for measuring particle production ($\bar{\Omega}\geq 1$), the adiabatic expansion method can be used to define adiabatic vacuum or n-particle states in zero[26] and finite-temperature theories[9] in dynamical spacetimes. For the same reason, the physical meaning of these theories are better defined and understood by the adiabatic method. On the other hand, the proper-time effective action method has many advantages. Not only is it covariant (note however that finite-temperature theory in dynamic spacetimes necessarily breaks covariance, as a timelike Killing vector is singled out for the definition of thermal equilibrium states), but a variety of techniques has been developed in zero-temperature field theory which can readily be adopted for finite-temperature considerations. This includes background-field methods,[27] momentum-space techniques,[28] and quasilocal expansion, all of these are useful for our present work.[29] The quasilocal approximation is an expansion of the effective action Γ in orders of the derivatives of the background field $\hat{\phi}$, which can have temporal or spatial dependence. The first term in this expansion (constant background field) gives the well-known effective potential V, while higher-order approximations take into account the "kinetic terms" in the Lagrangian. We have earlier called this an effective "quasipotential." This expansion is useful for constructing approximate field theories in cosmological spacetimes, as the background field is invariably dynamical. Earlier[30] we have derived an exact form of the zero-temperature effective Lagrangian for $\lambda\phi^4$ fields in curved spacetime by quasilocal expansion techniques up to second order in $\hat{\phi}$. Here, we want to generalize these results to finite temperature. It is of interst to note that derivative-expansion techniques have recently been developed for zero- and finite-temperature theories in flat space[31] for the study of soft-gluon processes in quantum chromodynamics,[32] Skyrmion stability problems,[33] and locally supersymmetric theories.[34]

In this work we use the momentum-space technique and quasilocal expansion to derive the imaginary-time thermal Green's functions and the one-loop finite-temperature effective Lagrangians for a $\lambda\phi^4$ field in curved spacetimes. Restrictions on the techniques used naturally limit the validity of our results to gradual changes in the background spacetime and in the background field so that a condition of quasiequilibrium is maintained in each successive interval. (These conditions can be made precise by identifying and grouping the successive terms in the effective Lagrangian in accordance to their adiabatic order.) For the benefit of both theoretical and practical inquiries, we have derived the finite-temperature effective Lagrangian for a general curved spacetimes (in a local coordinate patch) and for two cosmological spacetimes, the Robertson-Walker and the Bianchi type-I universes. They are discussed in Secs. II and III, respectively.

In Sec. II we carry out a Riemann-normal-coordinate expansion[28] (up to fourth order) for the background metric and Minkowski space and a quasilocal expansion (up to second order) for the background field. We then introduce a momentum-space representation of the Green's function and a proper-time representation of the effective Lagrangian, which for second-order variation in the fields can be written in a closed form.[30] Upon imposing a periodic condition on the Euclideanized time and carrying out a small-proper-time expansion and integration one obtains the finite-temperature effective Lagrangian in orders of deviation from the flat space. The Bose-Einstein-type sums contained therein can be implemented by taking the high-temperature limit. The result is in the form of a double series in inverse temperature and small proper time (or inverse frequency) orders (they can also be grouped according to their adiabatic orders). In Sec. III we study $\lambda\phi^4$ theory in homogeneous cosmologies, starting with the Robertson-Walker (RW) spacetime. Because RW space is conformally flat, simplification can be achieved by working with conformal time and conformally related fields. A finite-temperature theory is well defined in such cases. The transformed scalar wave operator has the same kinetic part as in flat space. Using this as the background we extend our discussion to massless $\lambda\phi^4$ theory in a Bianchi type-I spacetime with small anisotropy. This is a spatially flat universe with different expansion rates in the three directions. It is of interest both for the study of quantum anisotropy dissipation processes in conventional $(3+1)$ Bianchi cosmologies[35] and as a model of cosmological compactification in Kaluza-Klein cosmologies.[15,16] Here we follow the approach of Ref. 35 in expanding the wave operator in orders of small anistropy off the RW background. Because the massless $\lambda\phi^4$ field in the RW universe permits a condition of thermal equilibrium, the thermal effect of anisotropy can be clearly identified. In Sec. IV we conclude with a discussion on the range of validity of the approximations used in this work. In a companion paper[36] we will discuss two additional aspects: the relationship of the proper time and the adiabatic methods and the infrared behavior and higher-loop effects. The latter problems arise, for example, in symmetry-breaking considerations, where terms in odd powers of the effective mass in the one-loop calculation become imaginary and have to be corrected by higher-loop contributions.[3,37] Finite-temperature theory in the imaginary-time formulation has topology $S^1\times R^3$, with the radius of the circle equal to the inverse temperature β. In the high-temperature limit finite-size effect can be important in influencing its infrared behavior. The methods and results from an earlier study of symmetry breaking in curved spacetime[38] can be used to tackle this problem. These are topics which require further studies.

II. CURVED SPACETIME IN RIEMANN NORMAL COORDINATES

Consider a massive (m) self-interacting (λ) scalar field Φ coupled (ξ) to a general curved spacetime with metric $g_{\mu\nu}$ and scalar curvature R. It is described by the Lagrangian density

$$L[\Phi,g_{\mu\nu}]=-\tfrac{1}{2}\Phi[\Box+(1-\xi)\xi_d R+m^2]\Phi -\lambda\Phi^4/4!\,, \tag{2.1}$$

where $\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the Laplace-Beltrami operator and $\xi_d=\frac{1}{4}(d-2)/(d-1)$ is the conformal coefficient in d dimensions ($\xi_d=\frac{1}{6}$ in 4-dimensions) and $\xi=0,1$ denote conformal and minimal couplings, respectively. The action has a minimum at $\Phi=\hat{\phi}$, which satisfies the classical equation of motion

$$A_2\hat{\phi}(x)=(\Box+M_2)\hat{\phi}(x)=0\ , \tag{2.2}$$

where

$$M_2=m^2+(1-\xi)\xi_d R+\lambda\hat{\phi}^2/6\ . \tag{2.3}$$

Fluctuations $\phi=\Phi-\hat{\phi}$ around the classical background $\hat{\phi}$ satisfies an equation (to lowest order)

$$A_1\phi(x)=(\Box+M_1)\phi(x)=0\ , \tag{2.4}$$

where

$$M_1=m^2+(1-\xi)\xi_d R+\lambda\hat{\phi}^2/2 \tag{2.5}$$

is an effective mass which depends on the coupling ξ, background curvature R, and the background field $\hat{\phi}$. Contributions of the fluctuation field to the equation of motion for $\hat{\phi}$ in (2.2) enter through the vacuum expectation value and the thermal average of its variance $\lambda\langle\phi^2\rangle/2$, which acts as additional terms in the effective mass M_2 in (2.3). The effective action Γ related to $L_{\rm eff}$ by

$$\Gamma[\hat{\phi},g_{\mu\nu}]=\int d^dx\sqrt{-g}\ L_{\rm eff}\ , \tag{2.6}$$

is expanded perturbatively in powers of $\hbar$ as

$$\Gamma[\hat{\phi}]=S[\hat{\phi}]+\Gamma^{(1)}+\Gamma'\ , \tag{2.7}$$

where $S[\hat{\phi}]$ is the classical action,

$$S=\int d^dx\sqrt{-g}\ L_{\phi}^{(0)},\quad L_{\phi}^{(0)}=L[\hat{\phi},g_{\mu\nu}]\ , \tag{2.8}$$

and $\Gamma^{(1)}$ and Γ' are the one-loop and higher-loop effective actions:

$$\Gamma^{(1)}=\int d^dx\sqrt{-g}\ L^{(1)}=-\frac{i\hbar}{2}\ln(\det G)\ . \tag{2.9}$$

Here G is the bare Feynman Green's function satisfying

$$A_1G(x,x')=(-g)^{-1/2}\delta(x,x')$$

or (2.10)

$$A_1\bar{G}(x,x')=\delta(x,x')\ ,$$

where

$$\bar{G}(x,x')=(-g)^{-1/4}(x)G(x,x')(-g)^{-1/4}(x')\ . \tag{2.11}$$

In a static homogeneous spacetime $\hat{\phi}$ is a constant field, in which case one can define an effective potential V as

$$V(\hat{\phi})=-(\mathrm{vol})^{-1}\Gamma(\hat{\phi})\ , \tag{2.12}$$

where (vol) denotes the spacetime volume. In general $\hat{\phi}$ has temporal and spatial dependences, which render $V(\hat{\phi})$ ill-defined. Under circumstances where the background spacetime and the background field change only gradually compared to the characteristic scales of change of the system one can carry out a quasilocal expansion of the field and the metric around any spacetime point x^μ, including their derivatives up to a certain order. Thus for the background field up to second-derivative order

$$\hat{\phi}^2(x')=\hat{\phi}^2(x)+\hat{\phi}^2{}_{,\mu}(x)(x'-x)^\mu+\tfrac{1}{2}\hat{\phi}^2{}_{,\mu\nu}(x'-x)^\mu(x'-x)^\nu+\cdots\ . \tag{2.13}$$

The effective Lagrangian will then be a functional of $\hat{\phi}$ and its derivatives. Likewise when the spacetime curvature is small we can expand the metric $g_{\mu\nu}$ around x^μ in a local coordinate patch. In a Riemann-normal-coordinate expansion up to fourth order in the variation of $g_{\mu\nu}$:

$$g_{\mu\nu}=\eta_{\mu\nu}+\tfrac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta+\tfrac{1}{6}R_{\mu\alpha\nu\beta;\gamma}y^\alpha y^\beta y^\gamma+(\tfrac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta}+\tfrac{2}{45}R_{\alpha\mu\beta\lambda}R^\lambda{}_{\gamma\nu\delta})y^\alpha y^\beta y^\gamma y^\delta+\cdots\ , \tag{2.14}$$

where $\eta_{\mu\nu}$ is the Minkowski metric $(+1,-1,-1,-1)$, $y=x'-x$, and the coefficients in this expansion are evaluated at $y=0$. Using this expansion for the wave operator A_1 in (2.4) we get an equation for the Green's function $\bar{G}(x,x')$ in (2.11) up to quadratic order in y (Ref. 28)

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu+\alpha^2+\beta_\mu y^\mu+\tfrac{1}{4}\gamma^2_{\mu\nu}y^\mu y^\nu)\bar{G}(y)=\delta^d(y)\ , \tag{2.15}$$

where

$$\alpha^2=m^2+(1-\xi)\xi_d R+\tfrac{1}{2}\lambda\hat{\phi}^2\equiv m^2+U\ , \tag{2.16a}$$

$$\beta_\mu=U_{;\mu}\ , \tag{2.16b}$$

$$\tfrac{1}{4}\gamma^2_{\mu\nu}=\tfrac{1}{2}U_{;\mu;\nu}+a_{\mu\nu}\ , \tag{2.16c}$$

$$a_{\mu\nu}\equiv\tfrac{1}{120}R_{;\mu\nu}-\tfrac{1}{40}\Box R_{\mu\nu}-\tfrac{1}{30}R_\mu{}^\lambda R_{\lambda\nu}+\tfrac{1}{60}R^\kappa{}_\mu{}^\lambda{}_\nu R_{\kappa\lambda}+\tfrac{1}{60}R^{\lambda\rho\kappa}{}_\mu R_{\lambda\rho\kappa\nu}\ . \tag{2.17}$$

Except for the $a_{\mu\nu}$ tensor (whose trace gives the a_2 coefficient in a Schwinger-DeWitt expansion related to the trace anomaly), all the expansion coefficients α, β, and γ are proportional to the derivatives of the generalized mass $U=\frac{1}{2}\lambda\hat{\phi}^2+(1-\xi)\xi_d R$. Equation (2.15) for the Green's function $\bar{G}$ will be the same as that in Minkowski space if U were a constant. This is the case for constant background fields $\hat{\phi}$ and for either conformal coupling ($\xi=0$) or for spacetimes of constant four-curvature (e.g., de Sitter universe). For these cases a global timelike Killing vector field exists (with respect to t time), which allows a global thermal equilibrium state to exist and to be maintained.[9] Let us first consider these cases.

A. Constant generalized mass

A convenient formulation of finite-temperature theory under these special conditions for the fields and the geometry when $\alpha=$const is via the Euclideanized space obtained from a Wick rotation to imaginary time $t\to -i\bar{\tau}$. The Euclidean Green's function G_E defined by

$$\bar{G}(t,\mathbf{x},t',\mathbf{x}')=\bar{G}_E(-i\bar{\tau},\mathbf{x},-i\bar{\tau}',\mathbf{x}') \tag{2.18}$$

satisfies the equation

$$(\partial_E{}^2+\partial_x{}^2+\alpha^2+\bar{\beta}_\mu y^\mu+\tfrac{1}{4}\bar{\gamma}^2_{\mu\nu}y^\mu y^\nu)\bar{G}_E=\delta(\bar{\tau},\mathbf{x},\bar{\tau}',\mathbf{x}')\ , \tag{2.19}$$

where

$$\bar{\beta}_0=i\beta_0,\quad \bar{\beta}_i=\beta_i\quad (i=1,2,3) \tag{2.20}$$

$$\bar{\gamma}^2_{00}=-\gamma^2_{00},\quad \bar{\gamma}^2_{0i}=-i\gamma^2_{0i},\quad \bar{\gamma}^2_{ij}=\gamma^2_{ij}\ .$$

In the coordinate patch defined in (2.14) one can introduce a momentum space centered at point x and define the Fourier transform $G_E(p)$ by

$$\overline{G}_E(y)=\frac{1}{(2\pi)^d}\int d^dp\, e^{ipy}\overline{G}_E(p)\ , \tag{2.21}$$

where $py \equiv p_\mu y^\mu = \eta^{\mu\nu}p_\mu y_\nu$. The momentum-space Green's function $G_E(p)$ satisfies

$$(p^2+\alpha^2-i\hat{\beta}_\mu\partial^\mu-\tfrac{1}{4}\hat{\gamma}^{\,2}_{\mu\nu}\partial^\mu\partial^\nu)\overline{G}_E(p)=1\ , \tag{2.22}$$

where $\partial^\mu\overline{G}_E(p)\equiv\partial\overline{G}_E/\partial p_\mu$. Henceforth we will drop the subscript E and the overbar on G_E. In the aforementioned cases $\hat{\beta}=\hat{\gamma}=0$, the Green's function is given by

$$G(p)=(p^2+\alpha^2)^{-1}\equiv\int_0^\infty ds\, e^{-\alpha^2 s}e^{-p^2 s}\ . \tag{2.23}$$

We have written the second equality in a proper-time (s) integral representation. Note that the only change from flat space is in the mass term α^2, which now contains constant background field and curvature contributions. The configuration space Green's function $G(x,x')$ in the coincidence limit ($x\to x'$ or $y\to 0$) becomes

$$G(x,x)=\frac{1}{(2\pi)^d}\int d^dp\int_0^\infty ds\, e^{-\alpha^2 s}e^{-p^2 s}\ . \tag{2.24}$$

A finite-temperature theory is constructed by imposing a periodicity condition on the imaginary time y^0 in (2.24), i.e., $\tau\to\tau+n\beta$ and summing over n [do not confuse the inverse temperature $\beta=1/k_BT$ and the first-order coefficients β_μ in (2.16)].

Expressing G_β as an image sum over $G\,(=G_{T=0})$:

$$G_\beta(x,x')=\sum_{n=-\infty}^{\infty}G(x+n\beta u,x'),\quad u=(1,0,\ldots,0)$$

and noting that

$$\sum_{n=-\infty}^{\infty}e^{ip_0 n\beta}=\frac{2\pi}{\beta}\sum_{n=-\infty}^{\infty}\delta\left[p_0-\frac{2\pi n}{\beta}\right],\quad p_0=\frac{2\pi n}{\beta} \tag{2.25}$$

one obtains the thermal Green's function G^0_β for these special cases in proper-time representation

$$G^0_\beta(x,x)=\frac{1}{(2\pi)^{d-1}\beta}\int d^{d-1}p\int_0^\infty ds\sum_{n=-\infty}^{\infty}e^{-\alpha^2 s}e^{-(2\pi n/\beta)^2 s}e^{-|\mathbf{p}|^2 s}\ . \tag{2.26}$$

The $(d-1)$-dimensional integral is easy to evaluate:

$$\int_{-\infty}^{\infty}d^{d-1}p\, e^{-|\mathbf{p}|^2 s}=\left[\frac{\pi}{s}\right]^{(d-1)/2}\ . \tag{2.27}$$

For the proper-time integration after this, we can use the relation

$$\int_0^\infty ds\, s^{(l-d+1)/2}e^{-(\alpha^2+p_0{}^2)s}=\Gamma\left[\frac{l-d+3}{2}\right](\alpha^2+p_0{}^2)^{(d-l-3)/2}\ . \tag{2.28}$$

We have left $l\neq 0$ in (2.28) in anticipation of similar proper-time expansion in more general cases. The thermal Green's function for the present case $l=0$

$$G^0_\beta(x,x')=\frac{\Gamma\left[\frac{l-d+3}{2}\right]}{(4\pi)^{(d-1)/2}\beta}\sum_{n=-\infty}^{\infty}\left[\alpha^2+\left[\frac{2\pi n}{\beta}\right]^2\right]^{(d-l-3)/2}\ . \tag{2.29}$$

The one-loop effective Lagrangian is related to the Green's function by

$$\frac{\partial L^{(1)}}{\partial\alpha^2}=\frac{\hbar}{2}G(x,x)\ . \tag{2.30}$$

Upon integrating (2.29) with respect to α^2, we obtain the finite-temperature effective Lagrangian for these special cases

$$L^{(1)}_\beta=-\frac{\hbar}{\beta}\frac{\Gamma\left[\frac{1-d}{2}\right]}{2(2\pi)^{d-1}}\sum_{n=-\infty}^{\infty}\left[\alpha^2+\left[\frac{2\pi n}{\beta}\right]^2\right]^{(d-1)/2}\ . \tag{2.31}$$

The series in (2.29) or (2.31) cannot in general be summed to an analytic form. But for certain ranges of α and β, in particular if $\alpha^2\beta^2\ll 1$, one can perform a high-temperature expansion of (2.29) or (2.31) using

$$\sum_{n=-\infty}^{\infty}\left[\alpha^2+\left[\frac{2\pi n}{\beta}\right]^2\right]^b=\left[\frac{2\pi}{\beta}\right]^{2b}\sum n^{2b}(1+n^{-2}a^2)^b=\left[\frac{2\pi}{\beta}\right]^{2b}\sum n^{2b}(1+bn^{-2}a^2+\cdots)\ . \tag{2.32}$$

Here we have defined $a=\alpha\beta/2\pi$ and $b=(d-l-3)/2$ in (2.29) or $(d-1)/2$ in (2.31). Inspection of (2.32) indicates that the thermal Green's function contains a finite part and a part with poles:

$$G^0_\beta=G^0_\beta(\text{finite})+G^0_\beta(\text{pole})\ , \tag{2.33}$$

where

$$G^0_\beta(\text{finite})=-\frac{i}{4\pi\beta}\left[\alpha-\frac{\pi}{3\beta}-\frac{\beta^3\alpha^4}{32\pi^3}\zeta(3)+\cdots\right] \tag{2.33a}$$

and

$$G^0_\beta(\text{pole}) = \frac{i\alpha^2}{8\pi^2}\left[\frac{1}{d-4}+\cdots\right]. \tag{2.33b}$$

These ultraviolet-divergent terms are to be combined with those arising from the zero-temperature theory. Note that one can get the $T=0$ result from L, but not from high-temperature expansion. Renormalization in $T=0$ theory was treated in detail in Ref. 30 (and other work referred therein). We have for a general curved spacetime the one-loop ultraviolet-divergent terms

$$L^{(1)}_{\text{div}} = \frac{\alpha^4}{32\pi^2}\left[\frac{1}{d-4}+\frac{1}{2}\ln\frac{\alpha^2}{4\pi\mu^2}+\frac{\gamma}{2}-\frac{3}{4}\right] + \frac{a_2}{16\pi^2}\left[\frac{1}{d-4}+\frac{1}{2}\ln\frac{\alpha^2}{4\pi\mu^2}+\frac{\gamma}{2}\right],$$

$$a_2 = -\tfrac{1}{12}\hat{\gamma}^2. \tag{2.34}$$

Note that certain terms in (2.33b) and (2.34) combine and as in flat space all the divergences in a finite-temperature theory are already contained in the zero-temperature theory. The divergence-free part is given by (2.33a). It has the same form as in flat space except for α^2 now taking the place of M^2.

When $\alpha\beta \to 0$, corresponding to very small effective mass or at very high temperatures, infrared divergence will appear in β^{-n} terms in (2.33). The one-loop approximation is no longer valid and one would need to take into account higher-loop contributions. One can obtain the leading higher-loop contribution from the large-N limit of an N component field, as has been done in flat-space theories.[3,37] It is helpful to view the finite-temperature theory in R^4 as a zero-temperature theory in a space with topology $S^1\times R^3$, where β is the radius of S^1. Recent results[38] on the infrared behavior of curved spacetime can easily be applied to the finite-temperature case. Let us now come to the general case where α is not a constant.

B. Variable generalized mass

Permitting $\hat{\phi}(x)$ and $R(x)$ to have space and time dependence, we want to seek solutions to the Green's function G or G_E in (2.15) or (2.19) with $\beta_\mu, \gamma_{\mu\nu}$ or $\hat{\beta}_\mu, \hat{\gamma}_{\mu\nu} \neq 0$. If we include variations in $\hat{\phi}(x)$ only up to the second derivative order, we can write $G(p)$ in the same form as (2.23) but with p^2 replaced by a general quadratic polynomial in p^μ, i.e.,

$$G(p) = \int_0^\infty ds\, e^{-\alpha^2 s}\exp[-p^\mu A_{\mu\nu}(s)p^\nu + iB_\mu(s)p^\mu + C(s)], \tag{2.35}$$

where A, B, and C are functions of $\hat{\beta}$ and $\hat{\gamma}$ with constraints that they reduce to $A_{\mu\nu}(s)\to\delta_{\mu\nu}s$, $B_\mu(s)\to 0$ and $C(s)\to 0$ in the constant-field limit. The configuration space Euclidean Green's function $G(x,x')$ now assumes the general form

$$G(x,x') = \frac{1}{(2\pi)^d}\int_0^\infty ds\, e^{-\alpha^2 s + C}\int d^dp\, e^{ip\cdot y}e^{-(p\cdot A\cdot p - iB\cdot p)}. \tag{2.36}$$

Under a change of variables $p \to p' = -\frac{1}{2}A^{-1}p + B$, the integral becomes Gaussian, which can easily be evaluated. Taking the concidence limit $y\to 0$ of $G(x,x')$ gives

$$G(x,x) = \int_0^\infty \frac{ds}{(4\pi s)^{d/2}}\exp[-\alpha^2 s + C - \tfrac{1}{4}BA^{-1}B - \tfrac{1}{2}\operatorname{tr}\ln(As^{-1})], \tag{2.37}$$

where

$$A = \hat{\gamma}^{-1}\tanh\hat{\gamma}s, \tag{2.37a}$$

$$B = 2\hat{\gamma}^{-2}(1-\operatorname{sech}\hat{\gamma}s)\hat{\beta}, \tag{2.37b}$$

$$C = -\tfrac{1}{2}\operatorname{tr}\ln(\cosh\hat{\gamma}s) - \hat{\beta}\hat{\gamma}^{-3}(\tanh\hat{\gamma}s - \hat{\gamma}s)\hat{\beta}. \tag{2.37c}$$

These results were obtained earlier in Ref. 30. To extend them to finite-temperature considerations, one should first make sure that there is a well-defined finite-temperature theory in the unperturbed background like those in case 1 discussed above, i.e., cases where $\alpha = m^2 + (\lambda/2)\hat{\phi}^2 + (1-\xi)\xi_d R$ is a constant. Under conditions that the background field and the background curvature change sufficiently gradually (this can be measured by the nonadiabaticity parameter[9]) we can then use the finite-temperature theory associated with the unperturbed background obtained in the adiabatic limit at every sequential interval of time. Note the difference between the adiabatic and instantaneous definitions of finite-temperature theory. The former is better defined as it allows for sufficient time for spontaneous and induced) particle production and interaction between the intervals. The adiabatic approach[9] gives a more precise notion of quasiequilibrium, as is assumed in the present approach. Technically, the finite-temperature theory associated with the unperturbed background can be obtained by imposing periodic boundary conditions on the imaginary time at every instant. The finite-temperature theory associated with the full theory wherein the background field and spacetime undergo gradual change is defined with respect to the timelike Killing vector of the unperturbed background in every interval. The interval is chosen so that the increase of entropy from produced particles and interactions are exponentially small (this making the adiabatic n-particle state well defined at each instance). Physically the above conditions mean that deviation from the equilibrium value of physical observable in that interval is weighted with a thermal distribution established in the corresponding unperturbed theory. There are different ways of accounting for the variation arising from a slowly changing background: instead of working with the adiabatic state and imposing the imaginary time at each interval, one can, for instance, work with the instantaneous thermal state but relax the periodicity condition on the imaginary time [e.g., integrating from $-i\beta/2$ to $+i\beta/2$ (Ref. 7)]. We prefer the adiabatic approach as the adiabatic state is better defined and has a more direct physically meaning.

In the manner described above, we now seek to generalize the quasilocal Lagrangian to finite temperature. Define the momentum integral in (2.36) as

$$I=\int d^dp\, e^{-p\cdot A\cdot p+iB\cdot p}=e^{-BA^{-1}B/4}I' \,, \tag{2.38a}$$

where

$$I'=\int d^dp\, e^{-p\cdot A\cdot p} \,. \tag{2.38b}$$

Split the $p_\mu=(p_0,\bar{p})$ into time and space parts (where an overbar denotes the $d-1$ spatial components) and impose periodic boundary conditions on the (Euclidean) time. This changes the integral $\int dp_0$ to a summation $(2\pi/\beta)\sum_n$,

$$I'\rightarrow\frac{2\pi}{\beta}\sum_{n=-\infty}^{\infty}e^{-A_{00}p_0^2}\int d^{d-1}\bar{p}e^{-\bar{p}\bar{A}\bar{p}-\bar{p}\bar{D}p_0}$$
$$=\frac{2\pi}{\beta}(\pi)^{(d-1)/2}e^{-\mathrm{tr}\ln\bar{A}/2}\sum_n e^{Fp_0^2} \,, \tag{2.39a}$$

where

$$F=-A_{00}+\tfrac{1}{4}\bar{D}\bar{A}^{-1}\bar{D},\quad D_i=A_{0i}+A_{i0} \,. \tag{2.39b}$$

The thermal Green's function from (2.37) becomes, after taking the coincidence limit,

$$G_\beta(x,x)=\frac{1}{(4\pi)^{(d-1)/2}\beta}\int_0^\beta\frac{ds}{s^{(d-1)/2}}\exp[-\alpha^2s+C-\tfrac{1}{4}BA^{-1}B-\tfrac{1}{2}\mathrm{tr}\ln(\bar{A}s^{-1})]\sum e^{Fp_0^2} \,. \tag{2.40}$$

The one-loop finite-temperature effective Lagrangian is obtained from (2.30):

$$L_\beta^{(1)}=\frac{\hbar}{(4\pi)^{(d-1)/2}\beta}\int_0^\infty\frac{ds}{s^{(d+1)/2}}e^{-\alpha^2s-\bar{f}(s)}\sum_n e^{Fp_0^2} \,, \tag{2.41}$$

where

$$\bar{f}(s)=-C+\tfrac{1}{4}BA^{-1}B+\tfrac{1}{2}\mathrm{tr}\ln(\bar{A}s^{-1}) \,.$$

In Eqs. (2.40) or (2.41), the quasithermal Green's function and effective Lagrangian are given in proper-time representation. In the Schwinger-DeWitt formulation the proper time s does not have any direct physical meaning. But from the form of the integrand in (2.40) s can be regarded as a scaling parameter on the characteristic mass α or natural frequency of the theory: small s yields the local or ultraviolet behavior whereas large s yields global or infrared behavior. A small proper-time expansion gives the contribution of the high-frequency modes of the system, which is also the domain where adiabatic methods are applicable and the finite-temperature theories are well defined. Thus, expanding all quantities in (2.41) depending on s in a power series in s up to $O(s^3)$, i.e.,

$$C\simeq-\tfrac{1}{4}s^2\mathrm{tr}\hat{\gamma}^2+\tfrac{1}{3}s^3\hat{\beta}^2,\quad BA^{-1}B\simeq s^3\hat{\beta}^2 \,,$$
$$\tfrac{1}{2}\mathrm{tr}\ln(\bar{A}s^{-1})\simeq\tfrac{1}{6}s^2\mathrm{tr}'\hat{\gamma}^2\ \ [\mathrm{tr}'\text{ denotes trace over }(d-1)\text{-dimensional indices}], F\simeq-s+\tfrac{1}{3}s^3\hat{\gamma}_{00}^2 \,, \tag{2.42}$$

we get $(p_0=2\pi n/\beta)$

$$L_\beta^{(1)}=-\frac{\hbar}{(4\pi)^{(d-1)/2}\beta}\sum_n\int_0^\infty ds\, e^{-(\alpha^2+p_0^2)s}s^{-(d+1)/2}[1+Ds^2+(E+Fp_0^2)s^3+\cdots] \,, \tag{2.43}$$

$$D=-\tfrac{1}{4}\mathrm{tr}\hat{\gamma}^2+\tfrac{1}{6}\mathrm{tr}'\hat{\gamma}^2 \,,$$

where

$$E=\tfrac{1}{12}\hat{\beta}^2,\quad F=\tfrac{1}{3}\hat{\gamma}_{00}^2 \,.$$

The proper-time integrals can be related to Γ functions by (2.28), whereby we obtain

$$L_\beta^{(1)}=L_{\beta0}^{(1)}+L_{\beta2}^{(1)}+L_{\beta3}^{(1)}+\cdots \,, \tag{2.44}$$

$$L_{\beta\sigma}^{(1)}=-\frac{\hbar A_\sigma}{(4\pi)^{(d-1)/2}\beta}\sum_n\frac{\Gamma\left[\dfrac{2\sigma+1-d}{2}\right]}{(\alpha^2+p_0^2)^{(2\sigma+1-d)/2}} \,, \tag{2.45}$$

where

$$A_0=1,\quad A_2=D,\quad A_3=E+Fp_0^2 \,.$$

Here σ denotes the order of proper time. The coefficients D,E,F defined in (2.43) are functions of $\hat{\phi}$, a, R, and their derivatives. We recognize that the leading zero σ-order terms gives us the result (2.29) for the constant mass background theory (case 1), where the finite-temperature theory is well-defined globally.

At this point we can perform a high-temperature expansion $\alpha\beta\ll1$ for the sums in (2.45). Using some of the series expressions given in Appendix A, we obtain finally [the coefficients D,E,F are defined in (2.43)] $(\hbar=1)$

$$L_{\beta 0}^{(1)} = -\left[\frac{\pi^2}{90}\frac{1}{\beta^4} - \frac{1}{24}\frac{\alpha^2}{\beta^2} + \frac{1}{12\pi}\frac{\alpha^3}{\beta} + \frac{1}{64\pi^2}\alpha^4 \ln\alpha^2\beta^2 + \frac{1}{64\pi^2}\alpha^4(\gamma - \ln 4\pi) - \frac{\zeta(3)}{8(32)\pi^4}\alpha^4\beta^2 - \frac{1}{32\pi^2}\alpha^4\left[\frac{1}{d-4}\right]\right]$$
$$+O(\alpha^3\beta^3)\,, \tag{2.46a}$$

$$L_{\beta 2}^{(1)} = -\frac{D}{16\pi}\left[\frac{1}{\alpha\beta} + \frac{1}{2\pi}\ln\alpha^2\beta^2 - \frac{2\zeta(3)}{\pi^2}\alpha^2\beta^2 - \frac{1}{2}(1-\ln 4\pi) - \frac{1}{\pi}\left[\frac{1}{d-4}\right]\right], \tag{2.46b}$$

$$L_{\beta 3}^{(1)} = -\frac{1}{16\pi}\left[\frac{E}{2}\frac{1}{\alpha^3\beta} + \frac{F}{4\pi}\ln\alpha^2\beta^2 + \frac{\zeta(3)}{8\pi^3}\left[E - \frac{3}{2}F\alpha^2\right]\beta^2 + \frac{F}{4\pi}(2+\gamma-\ln 4\pi) - \frac{F}{2\pi}\left[\frac{1}{d-4}\right]\right]. \tag{2.46c}$$

Note that when $\alpha^2 < 0$, all odd power terms of α are complex. These terms come from the zero mode of the fluctuation field operator and they are modified by higher-loop contributions, as will be discussed above in.[36]

From Ref. 30 we can obtain the zero-temperature one-loop effective Lagrangian (up to second-derivative order)

$$L_{(T=0)}^{(1)} = \frac{\alpha^4}{32\pi^2}\left[\frac{1}{2}\ln\frac{\alpha^2}{4\pi\mu^2} + \frac{1}{2}\left[\gamma - \frac{3}{2}\right] + \frac{1}{d-4}\right] + \frac{a_2}{16\pi^2}\left[\frac{1}{2}\ln\frac{\alpha^2}{4\pi\mu^2} + \frac{\gamma}{2} + \frac{1}{d-4}\right],\quad a_2 = -\tfrac{1}{12}\hat{\gamma}^2\,. \tag{2.47}$$

Choosing the renormalization scale $\mu = 1/\beta$ we have $L^{(1)} = L_{(T=0)}^{(1)} + L_{\beta}^{(1)}$ (the total effective Lagrangian):

$$L^{(1)} = \frac{\pi^2}{90}\frac{1}{\beta^4} - \frac{1}{24}\frac{\alpha^2}{\beta^2} + \frac{1}{12\pi}\frac{\alpha^3}{\beta} - \frac{\zeta(3)}{8(32)\pi^2}\alpha^6\beta^2 + \frac{\alpha^4}{16\pi^2}\left[\frac{1}{d-4}\right] - \frac{1}{16\pi}\alpha^{-1}\left[\frac{D}{\beta} + \frac{E}{2}\frac{1}{\alpha^2\beta}\right]$$
$$+\frac{J(3)}{8\pi^3}\left[D + \frac{3F}{32\pi} - \frac{E\alpha^{-2}}{16}\right]\alpha^2\beta^2 - \frac{1}{32\pi^2}\left[D + F\left[1 + \frac{\gamma}{2}\right]\right] + \frac{a_2}{8\pi^2}\left[\frac{1}{d-4}\right]. \tag{2.48}$$

We see that the divergent terms are of the same form as in the $T=0$ theory.

III. COSMOLOGICAL SPACETIMES: ANISOTROPIC PERTURBATIONS

In the previous section we considered finite-temperature theories in static or constant curvature spacetimes, where thermal equilibrium can be defined and maintained. We then considered an approximate finite-temperature theory for more general conditions where the background field and curvature vary sufficiently slowly to allow for quasiequilibrium conditions to be maintained. We used a quasilocal expansion for the background field (2.13) and a Riemann normal coordinate for the background metric (2.14). In this section we will consider as background spatially homogeneous cosmological spacetimes. The Robertson-Walker universe constitutes a rather unique class because they are conformally static, i.e., they can all be related to a static metric by means of conformal transformations (this property defined by the Weyl tensor $C_{\alpha\beta\gamma\delta} = 0$ is sometimes called "conformally flat"). The existence of a global conformal Killing vector in these spacetimes permits a thermal equilibrium condition to be maintained for conformal fields. A finite-temperature theory can then be defined with respect to the conformal time throughout the cosmological history. This point has been discussed earlier.

Using these background spacetimes, one can then consider approximate finite-temperature theories for more general conditions where the background field varies with conformal time or the background metric departs from conformal staticity. These are the theoretical basis for the discussion of finite-temperature theories in homogeneous cosmology.[10,11] In this section we will use the proper time and quasilocal approximation to treat the Robertson-Walker universe and then extend to the Bianchi type-I universes with small anisotropy. Here since we are only interested in homogeneous backgrounds, the use of Riemann normal coordinate is not appropriate. Rather we will use a perturbation expansion on the metric in powers of the small anisotropy parameter, and calculate the finite-temperature effective Lagrangian with thermal Green's functions defined in the background Robertson-Walker universe.

A. Robertson-Walker universe

Let us consider for simplicity the spatially flat Robertson-Walker (RW) universe whose metric is given by

$$ds^2 = dt^2 - a^2(t)\sum_{i=1}^{3}(dx^i)^2$$
$$= a^2(\eta)\left[d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right], \tag{3.1}$$

where η is the conformal time defined by $dt = a\,d\eta$. The scalar 4-curvature is given by

$$R = 6a''/a^3 \text{ (where primes denote } d/d\eta)\,. \tag{3.2}$$

We consider a massive $\lambda\phi^4$ field coupled to the RW background. After a background-field decomposition, the background field and the fluctuation field satisfy, respectively, Eqs. (2.3) and (2.4). The scalar Green's function in

RW space $\overline{G}$ is related to that of flat space $\widetilde{G}$ (2.11) by conformal factors, i.e.,

$$\overline{G}(\eta,\mathbf{x},\eta',\mathbf{x})=a^{1-d/2}(\eta)\widetilde{G}(\eta,\mathbf{x},\eta',\mathbf{x}')a^{1-d/2}(\eta') , \quad (3.3)$$

where d is the dimension of spacetime. $\widetilde{G}$ satisfies the wave equation

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu+\widetilde{\alpha}^2)\widetilde{G}(x,x')=\delta^d(x,x') . \quad (3.4)$$

Throughout this section $x^\mu=(\eta,\mathbf{x})$ and we will use a tilde to denote the conformally related quantities, e.g.,

$$\widetilde{\alpha}^2(\eta)=\alpha^2(\eta)a^2(\eta)=ma^2(\eta)+\widetilde{U}(\eta), \text{ etc.,}$$

where (3.5)

$$\widetilde{U}=[(1-\xi)\xi_d R(\eta)+\tfrac{1}{2}\lambda\hat{\phi}^2(\eta)]a^2(\eta) .$$

We see that in the special case of free massless conformal fields $\widetilde{\alpha}=0$, there is no particle production and entropy generation to disrupt any existing thermal equilibrium. For interacting fields, entropy could be generated both from particle production and particle interaction or decay. If the interaction time of the system is short compared with the Hubble time, one generally assumes that an equilibrium condition can be reached at each successive interval in the evolutionary history. These equilibrium states are of course different from one instant to another as additional entropy is added to the system. One needs to use interacting quantum field theory and statistical mechanics to analyze these processes to get the full picture. For our purpose here we shall assume that the quantity $\widetilde{\alpha}^2$ varies sufficiently slowly in η time that whatever thermal equilibrium condition previously established is least affected (again in an adiabatic sense as described previously[9]) so that an approximate finite-temperature theory is well defined. Note a case of special interest, i.e, for a radiation-dominated RW universe, $a\propto\eta$, $R=0$, in which case thermal equilibrium can be established for massless particles irrespective of the type of field coupling. Our treatment of the RW universe is parallel to that of Sec. II. One can work with the conformally related Euclidean-time ($\eta\to-i\overline{\eta}$) Green's function $\widetilde{G}_E$ which satisfies

$$(\partial_{\overline{\eta}}{}^2+\partial_{\mathbf{x}}{}^2+\widetilde{\alpha}^2)\widetilde{G}_E(\overline{\eta},\mathbf{x},\overline{\eta}',\mathbf{x}')=\delta^d(\overline{\eta},\mathbf{x},\overline{\eta}',\mathbf{x}') . \quad (3.6)$$

Assuming that $\widetilde{\alpha}^2$ varies slowly with η and $\mathbf{x}$, one can carry out a derivative expansion of $\widetilde{\alpha}(x)$ around x' analogous to that for $\hat{\phi}$ in (2.13):

$$\widetilde{\alpha}^2(x')=\widetilde{\alpha}^2(x)+\widetilde{\alpha}^2{}_{,\mu}(x)(x'-x)^\mu + \tfrac{1}{2}\widetilde{\alpha}^2{}_{,\mu\nu}(x)(x'-x)^\mu(x'-x)^\nu+\cdots . \quad (3.7)$$

Since the background is conformally flat, expansion such as (2.14) on the metric is not necesary. The variations of the field are, in the cases of homogeneous cosmology, coupled only with the inhomogeneity in time. This gives us

$$[\partial_{\overline{\eta}}{}^2+\partial_{\mathbf{x}}{}^2+\widetilde{\alpha}^2(x)+\widetilde{\beta}_\mu y^\mu+\tfrac{1}{4}\widetilde{\gamma}^2{}_{\mu\nu}y^\mu y^\nu] \widetilde{G}_E(\overline{\eta},\mathbf{x},\overline{\eta}',\mathbf{x}')=\delta^d(\overline{\eta},\mathbf{x},\overline{\eta}',\mathbf{x}') , \quad (3.8)$$

where

$$\widetilde{\beta}_\mu=\widetilde{\alpha}^2{}_{;\mu}, \quad \tfrac{1}{4}\widetilde{\gamma}^2{}_{\mu\nu}=\tfrac{1}{2}\widetilde{\alpha}^2{}_{;\mu;\nu}$$

and the Euclideanized expressions $\hat{\beta}_\mu$ and $\hat{\gamma}_{\mu\nu}$ are related to $\widetilde{\beta}_\mu$ and $\widetilde{\gamma}_{\mu\nu}$ by (2.20). Here the coefficients are all evaluated at $y=0$. Note that $\widetilde{\gamma}^2_{\mu\nu}$ does not contain $a_{\mu\nu}$ because of the spatial homogeneity of the background. However, terms of the form $a_{\mu\nu}$ does arise in the regularized energy-momentum tensor in the zero-temperature theory, giving rise to the conformal anomaly. This can be obtained by carrying out a scale transformation in the effective Lagrangian, as illustrated in Ref. 39. Since the trace anomaly enters only in the zero-temperature theory we do not need the details at this point. Once the problem is cast into the standard form, the ensuing discussion is identical to that encountered in Sec. II.

In particular, the thermal Green's function in RW universe expanded up to second derivative order is given by [cf. Eq. (2.36)]

$$\widetilde{G}\equiv G^0_\beta(x,x')=\frac{1}{(2\pi)^{d-1}\beta}\sum_n\int d^{d-1}p\, e^{ip(x-x')}G^0_\beta(p)$$

and

$$G^0_\beta(p)=\int_0^\infty ds\, e^{-\widetilde{\alpha}^2 s} \times e^{-p^2 s}e^{-p_0{}^2 s}[1+\widetilde{D}s+(\widetilde{E}+p_0{}^2\widetilde{F})s^3+\cdots]$$

with

$$\widetilde{D}=M^2\gamma, \quad \widetilde{E}=-\tfrac{1}{3}M^4\delta^2, \quad \widetilde{F}=-\tfrac{4}{3}M^2\gamma . \quad (3.9)$$

Here

$$\gamma=\left[\frac{a''}{a}-\left[\frac{a'}{a}\right]\frac{\theta}{M^2}+\frac{K}{M^2}\right], \quad \delta=\left[\frac{a'}{a}+\frac{\theta}{M^2}\right],$$

and

$$M^2=(m^2+\tfrac{1}{2}\lambda\hat{\phi}^2)a^2, \quad \theta=\tfrac{1}{2}\lambda\hat{\phi}\hat{\phi}'a^2 , \quad (3.10a)$$

$$K=\tfrac{1}{2}\lambda\hat{\phi}^2a^2\left[\frac{\hat{\phi}''}{\hat{\phi}}+\left[\frac{\hat{\phi}'}{\hat{\phi}}\right]^2+\frac{4\hat{\phi}'a'}{\hat{\phi}a}\right] . \quad (3.10b)$$

The one-loop finite-temperature effective Lagrangian $L^{(1)}_{\beta(\mathrm{RW})}$ is given by $1/\alpha^4\widetilde{L}^{(1)}_\beta$ where $\widetilde{L}^{(1)}_\beta$ is the Lagrangian in (2.48), with α^2D,E,F replaced by the corresponding tilded quantities. The finite-temperature effective action is

$$\Gamma^{(1)}_{\beta(\mathrm{RW})}=\int d^4x\, a^4L^{(1)}_{\beta(\mathrm{RW})} .$$

There are no geometric terms other than a and its derivatives and R (derivatives of R enter at third order and higher but are absent in our approximation).

B. Bianchi type-I universes

We now consider a Bianchi type-I universe with metric[35]

$$ds^2=a^2(\eta)[d\eta^2-(e^{2\beta})_{ij}dx^idx^j], \quad i,j=1,2,3 , \quad (3.11)$$

where $\beta_{ij}(\eta)$ is the anisotropy matrix which is symmetric and traceless. Its rate of change measures the shear $\sigma^2=\beta_{ij}\beta^{ij}$ [do not confuse with the coefficient of first-order derivative expansion β_μ in (2.16b) and the inverse temperature β]. We shall consider cases where β_{ij} is small and treat the background as a small perturbation off the spatially flat Robertson-Walker universe with scale factor $a(\eta)$. One expects that in addition to the conformally related generalized mass $\tilde{\alpha}^2$ in the potential part of the Lagrangian, there will also be additional kinetic terms from expanding the Laplace-Beltrami operator in orders of the small anisotropy parameter. Indeed, using the results of Ref. 35 we get up to $O(\beta_{ij}{}^2)$

$$\Box=\frac{1}{a^2}\left[\partial_\eta{}^2+2\left[\frac{a'}{a}\right]\partial_\eta-(e^{-2\beta})_{ij}\partial_i\partial_j\right]$$

and (3.12)

$$^{(4)}R=\frac{1}{a^2}\left[6\frac{a''}{a}+\beta'_{ij}\beta'_{ij}+\cdots\right].$$

The zero-temperature Green's function has the form to $O(\beta_{ij}{}^2)$

$$\overline{G}=(A_1+V)^{-1}=\overline{G}_0+\overline{G}_0V_1\overline{G}_0+\overline{G}_0V_2\overline{G}_0+\tfrac{1}{2}\overline{G}_0V_1\overline{G}_0V_1\overline{G}_0+\cdots, \quad (3.13)$$

where A_1 is defined in (2.4), G_0 is the RW zero-temperature Green's function defined in (2.10) and V_i are the interactions terms defined in (3.12) from the anisotropy

$$V_1=2a^{-2}\beta^{ij}\partial_i\partial_j, \quad (3.14a)$$

$$V_2=\frac{a^{-2}}{6}\beta'_{ij}\beta^{ij'}-2a^{-2}\beta^{ik}\beta_k{}^j\partial_i\partial_j. \quad (3.14b)$$

The one-loop effective action is given by

$$L=-\frac{i}{2}\operatorname{tr}\ln\overline{G}. \quad (3.15)$$

As discussed earlier a finite-temperature field theory is well defined for massless conformal $\lambda\phi^4$ fields in the Robertson-Walker universe. The thermal conformal Green's function G^0_β in a RW universe where the background field and curvature vary slowly is given by (tilde is dropped below $G^0_\beta\equiv\tilde{G}_\beta$)

$$G^0_\beta(x,x')=\frac{1}{(2\pi)^{d-1}\beta}\sum_n\int d^{d-1}p\,e^{ip\cdot(x-x')}G(p) \quad (3.16)$$

and

$$G^0_\beta(p)=\int_0^\infty ds\,e^{-\alpha^2 s}\exp(-p^\mu A_{\mu\nu}p^\nu+iB_\mu p^\mu+C),$$

where $A_{\mu\nu},B_\mu,C$ are slowly varying functions of the background field and curvature. In a path-integral approach, we can assume that the in and out regions are described by thermal n-particle states of the RW spacetime where a finite-temperature theory is well defined. During the evolution if the anisotropy β_{ij} of spacetime remains small, an approximate finite-temperature theory can then be defined in the quasiadiabatic case. We can expand the metric and the wave operator in orders of β_{ij} and use the RW thermal Green's function to compute the finite-temperature one-loop effective action. This is given by (the left-hand superscript on L or Γ denotes the order in β_{ij})

$$\Gamma_\beta=-\frac{i}{2}\operatorname{tr}\ln\overline{G}_\beta={}^0\Gamma_\beta+{}^1\Gamma_\beta+{}^2\Gamma_\beta, \quad (3.17)$$

where

$${}^0\Gamma_\beta=-\frac{i}{2}\operatorname{tr}\ln\overline{G}{}^0_\beta, \quad (3.18)$$

$${}^1\Gamma_\beta=-\frac{i}{2}\operatorname{tr}(V_1\overline{G}{}^0_\beta), \quad (3.19)$$

$${}^2\Gamma_\beta=-\frac{i}{2}\operatorname{tr}(V_2\overline{G}{}^0_\beta)-\frac{i}{4}\operatorname{tr}(V_1\overline{G}{}^0_\beta V_1\overline{G}{}^0_\beta). \quad (3.20)$$

${}^0\Gamma_\beta$ is the effective action of the RW universe derived in Sec. III A. We will show that ${}^1\Gamma_\beta$ vanishes and the only contribution comes from ${}^2\Gamma_\beta$. Now in (3.19)

$${}^1\Gamma_\beta=-i\operatorname{tr}[a^{-2}\beta^{ij}\partial_i\partial_j\overline{G}{}^0_\beta(x,x')]$$
$$=-i\operatorname{tr}\beta^{ij}\sum\int ds\int d^{d-1}p\partial_i\partial_je^{-p^2s}e^{-p_0{}^2s}\times e^{-\alpha^2s}e^{ip\cdot(x-x')}f(p_0,s), \quad (3.21)$$

where $f(p_0,s)$ can be expanded in powers of the proper time. For $d=4$, the three-dimensional integral in (3.21) after taking the coincidence limit gives

$$\int d^3p\,p_ip_je^{-p^isp^j}=0 \text{ unless } i=j.$$

For $i=j$, this yields

$${}^1\Gamma_\beta=-\tfrac{1}{2}(\operatorname{tr}\beta)\sum\int\frac{ds}{s}\left[\frac{\pi}{s}\right]^{(d-1)/2}. \quad (3.22)$$

However, since $\operatorname{tr}\beta=0$, this implies ${}^1\Gamma_\beta=0$. There are two second-order terms in ${}^2\Gamma_\beta$. Denote the first term in (3.20) as

$${}^2\Gamma_\beta(V_2)=-\frac{i}{2}\operatorname{tr}(V_2\overline{G}{}^0_\beta)$$
$$=-\frac{i}{2}\operatorname{tr}(\tfrac{1}{6}\beta'_{ij}\beta^{ij'}-\beta^{ik}\beta_k{}^j\partial_i\partial_j)\overline{G}{}^0_\beta \quad (3.23)$$

and the second term as

$${}^2\Gamma_\beta(V_1{}^2)=-\frac{i}{4}\operatorname{tr}(V_1\overline{G}{}^0_\beta V_1\overline{G}{}^0_\beta). \quad (3.24)$$

The first term in (3.23) is of the same form as (2.29) and can be treated accordingly. The second term in (3.23) has an integral over p which is again nonzero only when $i=j$. Integration over p gives a factor of $[(d-1)/2s](\pi/s)^{(d-1)/2}$. The remaining integral over proper time s is in the form

$$J\equiv\frac{1}{(2\pi)^{d-1}\beta}\sum_n\int\frac{ds}{s}\left[\frac{\pi}{s}\right]^{(d-1)/2}e^{-\alpha^2s}e^{-p_0{}^2s}f(p_0,s). \quad (3.25a)$$

Expanding J in a proper-time series $J=\sum_i J_i$, where J_i are of the ith proper-time order, we have

$$J_i=\frac{1}{(4\pi)^{(d-1)/2}\beta}\sum_n\int ds\, s^{-(d+1)/2}e^{-\alpha^2 s}e^{-p_0^2}(c_i s^i)$$
(no sum on i) . (3.25b)

Here the coefficients c_i in the proper-time expansion are functions of R, $\hat{\phi}$, and their derivatives. The integrals J_i ($i=0,2,3$) under high-temperature expansion are given in Appendix B as [Eqs. (B13)–(B15)]. Collecting the result for both terms in (3.23) from Appendix B we get

$$
\begin{aligned}
{}^2L_\beta(V_2)=-\frac{i}{\alpha^4}\Bigg\{&\frac{\pi^2}{90}\frac{B}{\beta^4}-\frac{1}{12}\left[\frac{\pi}{3}+\frac{\alpha^2}{2}B\right]\frac{1}{\beta^2}+\frac{1}{12}\left[Q+\frac{\alpha^2}{\pi}B\left[1+\frac{3}{4}\frac{D}{\alpha^4}-\frac{3}{8}\frac{E}{\alpha^6}\right]\right]\frac{\alpha}{\beta}\\
&+\frac{1}{64\pi^2}\left[-\frac{\alpha^2}{3}Q+\alpha^4B\left[1+\frac{2}{3}\frac{D}{\alpha^4}\right]\right]\ln\alpha^2\beta^2-\frac{\zeta(3)}{8(16\pi^3)}\left[\frac{Q}{3}+\frac{\alpha^2}{2\pi}B\left[1+\frac{4(8\pi-1)}{\alpha^4}D+\frac{2}{\alpha^6}E\right]\right]\alpha^4\beta^2\\
&+\frac{1}{32\pi}\left[-\frac{\alpha^2}{3\pi}\left[4+\frac{\gamma}{2}-\frac{1}{2}\ln 4\pi\right]Q+\frac{\alpha^4}{2\pi}(\gamma-\ln4\pi)B+D(1-\ln4\pi)B-\frac{2D}{3\pi}(2+\gamma-\ln4\pi)B\right]\\
&+\frac{\alpha^2}{32\pi^2}\left[\frac{Q}{3}-\alpha^2B\left[1+\frac{10}{3}\frac{D}{\alpha^4}\right]\right]\left[\frac{1}{d-4}\right]\Bigg\}\ ,
\end{aligned}
$$

where

$$B=\beta^{ij}\beta_{ij},\quad Q=\beta'_{ij}\beta^{ij'}\ (\text{shear}),\quad \text{and } D=M^2\gamma=\tfrac{3}{4}F\ . \tag{3.26}$$

We now come to the second term in ${}^2\Gamma_\beta$ in (3.20):

$${}^2\Gamma_\beta(V_1{}^2)=-\frac{i}{4}\mathrm{tr}(V_1\bar{G}{}^0_\beta V_1\bar{G}{}^0_\beta)\ . \tag{3.27}$$

Written explicitly,

$${}^2\Gamma_\beta(V_1{}^2)=-i\int d^4x\,\beta^{ij}(\eta)\int d^dx'\beta^{kl}(\eta')\partial_{l'}\partial_{k'}G^0_\beta(x',x)\partial_i\partial_jG^0_\beta(x,x')\ . \tag{3.28}$$

Now using the thermal Green's function in momentum space one can reduce this to an integral: $\int d^{d-1}p_ip_jG(p)G(k)p_kp_l$. The only nonzero terms are those containing even numbers of identical indices. Defining

$$H(\eta,\eta')=K_4\left[\sum_i(\beta_i{}^i)^2\right]+K_2\left[2\sum_{i<j}(\beta_i{}^i\beta_j{}^j)+4\sum_{i<j}(\beta^{ij}\beta_{ij})\right]\equiv Z(\eta,\eta')K_4\ , \tag{3.29}$$

where

$$K_4=\int dp_1p_1{}^4\int dp_2\int dp_3\,{}^0G_\beta{}^2(p^2)$$

and (3.30)

$$K_2=\int dp_1p_1{}^2\int dp_2p_2{}^2\int dp_3\,{}^0G_\beta{}^2(p^2)\ .$$

As the two Green's functions $G(p)$ and $G(k)$ in the integral are of two different arguments p,k (although $\mathbf{p}=\mathbf{k}, p_0\neq k_0$ this reflects the spatially homogeneous but temporally nonlocal nature of particle production) one has to perform a double integration and summation.

The integrals are treated in Appendix C, where it is shown that $K_4=3K_2$. After a proper-time (σ) expansion of the integrand we find

$${}^2\Gamma_\beta(V_1{}^2)=\frac{3}{4}\frac{(-i)}{(4\pi)^{(d-1)/2}\beta}\int d^dx\int d\eta' Z\sum_m\sum_{k=-\infty}^{\infty}e^{i(p_0-q_0)(\eta-\eta')}\sum_{\sigma=0}^{3}(A_\sigma I_\sigma+B_\sigma I_\sigma) \tag{3.31}$$

with

$$p_0=2\pi m/\beta,\quad q_0=2\pi k/\beta\ .$$

Here

$$I_0(a,b)=-\Gamma\left[-\frac{(d+1)}{2}\right]\left[\frac{a^{d+1}-b^{d+d}}{a^2-b^2}\right],$$

where

$$a^2=\alpha^2(\eta)+p_0{}^2,\quad b^2=\alpha^2(\eta')+q_0{}^2\ , \tag{3.32}$$

$$I_\sigma(a,b)=\frac{\Gamma\left[\dfrac{1+2\sigma-d}{2}\right]}{(\sigma+1)a^{1+2\sigma-d}}F\left[1,\frac{1+2\sigma-d}{2};\sigma+2;\frac{a^2-b^2}{2}\right]$$

and

$$J_\sigma=I_\sigma(b,a)\quad(\sigma=2,3)\ ,$$

where F denotes the hypergeometric function ${}_2F_1$ (the $\sigma=1$ term vanishes) and A,B are geometric and temperature-dependent coefficients. Its structure and meaning are similar to that of ${}^2\Gamma(V_1{}^2)$ which is the dominant term in the zero-temperature effective Lagrangian.[35] ${}^2\Gamma_\beta(V_1{}^2)$ being nonlocal and complex describes particle production at finite temperature and the red-shifting of produced particles. Since we have used a quasilocal approximation for the Green's function we can only discuss cases when η is close to η' for consistency. Imposing a local approximation ($\eta=\eta'$) for the exponential factor we can calculate the diagonal ($k=m$) term in ${}^2\Gamma_\beta(V_1{}^2)$ explicitly. This is given at the end of Appendix C. This real quantity gives the dominant contribution to ${}^2\Gamma_\beta(V_1{}^2)$ at high temperatures ($\beta\to 0$) since the fast oscillations of the exponential term force all other contributions toward zero. Results from using local approximation depict particle production and red-shifting in the high-frequency ranges and is consistent with the high-temperature small-proper-time expansion used here or the quasiadiabatic expansion used earlier.[11] The statistical thermodynamic properties of this system under such conditions are discussed in Ref. 12.

IV. DISCUSSION

In this paper we have derived the finite-temperature effective Lagrangian for $\lambda\phi^4$ fields in the following spacetimes: (1) general curved spacetime perturbed from flat space, (2) The Robertson-Walker universe, and (3) the Bianchi type-I universe with small anistropy. In closing, it may help to review the conceptual and technical basis we have assumed in our calculation.

Conceptually, we have drawn upon our earlier studies on the conditions for thermal equilibrium in dynamical spacetimes to define a finite-temperature theory. It can be argued that if particles in the system can interact at a rate faster than the expansion rate of the universe one can assume that thermal equilibrium can be maintained throughout. If this were the case, then the requirements (conformal invariance of the field, including interacting fields such as $\lambda\phi^4$, and conformal staticity of spacetime[9]) for a well-defined finite-temperature theory can be lifted and the results obtained here based on quasiequilibrium are unconditionally valid. However, the above assumption may not always be valid for weakly interacting fields (such as gravitons, neutrinos, axions) at certain energy scales (the Planck time) or during periods when particles are distributed at distances greater than the horizon (as in certain epochs in quantum or inflationary cosmologies). Under these conditions the particles are essentially free and questions concerning finite-temperature theory would have to be addressed in the conditions set forth in Refs. 9 and 10. This is because even in the absence of interaction, entropy generated from particle production[18] (at spacelike separations) can disturb any equilibrium state present. In this sense, the conditions we assumed are the weakest of all. We have used these conditions as a guide to constructing finite-temperature field theory. In Sec. II where we discussed curved-spacetime effects using the local Riemann coordinate expansion off flat space, the condition for maintaining global equilibrium translates to the generalized mass α being a constant [Eq. (2.16a)]. In Sec. III where we discussed homogeneous cosmology, the condition is for the conformally related mass α to be constant [Eq. (3.5)], which of course is strictly satisfied only for massless conformal fields in RW spacetimes.

Technically we used perturbation theories to consider more general situations where a finite-temperature theory is approximately well defined in a quasiadiabatic sense. These are cases where the background field and curvature vary slowly. Thus, in Sec. II B we used the quasilocal expansion on the field and the Riemann normal expansion on the metric. In Sec. III B with the RW universe as the background spacetime we performed an anisotropic expansion on the metric and a corresponding derivative expansion on the wave operator in orders of the anisotropy parameter, while using the thermal Green's functions in RW space to calculate the effective Lagrangian.

The imaginary-time thermal Green's function approach used here, i.e., the use of adiabatic n-particle states and the imposition of periodic conditions on the imaginary time at each interval, is only valid in a perturbative sense described above.

In addition to the quasilocal and Riemann expansion on the background field and spacetimes, we have also used small proper time and high-temperature expansions. It may help to understand the physical meaning of these approximations by considering the parameters used to characterize the system. A quasilocal expansion is in the derivatives of the field $\partial_\mu\hat{\phi}$ or metric $\partial_\mu g$. Let us denote by L and T the length and time scales at which significant variation in the spatial (inhomogeneity) and temporal (nonadiabaticity) dependence of $\hat{\phi}$ and g occurs. This is to be compared with the frequency ω or period τ of the normal mode of the fluctuation field. The background field decomposition and the quasilocal expansion are valid only for $T\gg\tau$. This is true for slowly varying backgrounds or for high-frequency modes (thus the quasiadiabatic condition). The proper time s is a scaling parameter on the generalized mass α. Small s gives the local behavior from the high-frequency modes while large s gives the global behavior or that of the low-frequency modes. Thus a small proper-time expansion is consistent with the quasilocal approximation. Finally, the temperature factor β. A finite temperature theory (in flat space) obtained by compactifying the imaginary time dimension has topology $S^1\times R^3$. The radius of S^1 is equal to β which is small at high temperature while large at low temperature. As long as $L\gg\beta$, the quasilocal variation will remain as small perturbations in the high temperature domain, which is consistent. On the other hand, the infrared behavior of the finite-temperature theory will become important at small β due to finite-size effects associated with the zero mode (band) of S^1. This effect needs separate consideration and is the topic of a later paper. For the present work we are assured that the quasilocal, small proper-time and high-temperature approximations are all consistent with each other and with the premises of a finite-temperature theory in curved spacetime. Departures from these conditions would require treatment using nonequilibrium quantum kinetic theory or statistical field theories in curved space, subjects which are currently under development.[24]

APPENDIX A: HIGH TEMPERATURE EXPANSION: DERIVATION OF EQ. (2.46)

We have for $\sigma=0$ in (2.45): ($\hbar=1$)

$$L_{\beta 0}^{(1)}=-\tfrac{1}{2}f_0(d)\frac{1}{\beta}\sum_{m=-\infty}^{\infty}(\alpha^2+p_0{}^2)^{(d-1)/2},$$

where

$$f_0(d)=\frac{\Gamma\left[\dfrac{1-d}{2}\right]}{(4\pi)^{(d-1)/2}},\quad p_0=\frac{2\pi m}{\beta}. \tag{A1}$$

Expressing the sum as

$$\sum_{-\infty}^{\infty}=(m=0)+2\sum_{m=1}^{\infty}$$

and defining

$$\Sigma_0=\sum_{k=1}^{\infty}(\alpha^2+p_0{}^2)^{(d-1)/2}$$

we have

$$\Sigma_0=\alpha^{d-1}\sum_k(\nu k)^{d-1}\left[1+\left[\frac{1}{\nu k}\right]^2\right]^{(d-1)/2}, \tag{A2}$$

where $\nu=2n/\alpha\beta$.

Performing a high-temperature expansion ($\beta\to 0$) keeping up to terms $O(\alpha^3\beta^3)$:

$$\Sigma_0\simeq\alpha^{d-1}\nu^{\epsilon}[\nu^3\zeta(1-d)+\tfrac{1}{2}(d-1)\nu\zeta(3-d)+\tfrac{1}{8}(d-1)(d-3)\nu^{-1}\zeta(1-\epsilon)+\tfrac{1}{16}(d-1)(d-3)(d-5)\nu^{-3}\zeta(7-d)] \tag{A3}$$

with $\epsilon=d-4$ and $\zeta(n)=\sum_{k=1}^{\infty}1/k^n$ is the Riemann ζ function. Using dimensional regularization and expanding the coefficient of the pole term $\zeta(1-\epsilon)$ to first order in ϵ: ($\epsilon\to 0$)

$$F_0(d)=(d-1)(d-3)f_0(d),\quad F_0(4+\epsilon)=F_0(4)+\left.\frac{\partial F_0}{\partial d}\right|_{d=4}\epsilon, \tag{A4}$$

we have

$$F_0(4)=\frac{1}{2\pi},\quad \left.\frac{\partial F_0}{\partial d}\right|_{d=4}=\frac{1}{4\pi}(\gamma-\ln\pi).$$

Also $\nu^{\epsilon}=1+\epsilon\ln\nu$ and $\zeta(1-\epsilon)=\gamma-1/\epsilon$. Collecting the finite terms plus the pole terms we have, for $d=4$,

$$L_{\beta 0}^{(1)}=-\left[\frac{\pi^2}{90}\frac{1}{\beta^4}-\frac{1}{24}\frac{\alpha^2}{\beta^2}+\frac{1}{12\pi}\frac{\alpha^3}{\beta}+\frac{1}{64\pi^2}\alpha^4\ln\alpha^2\beta^2+\frac{1}{64\pi^2}\alpha^4(\gamma-\ln 4\pi)-\frac{\zeta(3)}{8(32)\pi^4}\alpha^4\beta^2-\frac{1}{32\pi^2}\alpha^4\left[\frac{1}{d-4}\right]\right]$$
$$+O(\alpha^3\beta^3). \tag{A5}$$

The term in second order in proper time ($\sigma=2$) in (2.45) is

$$L_{\beta 2}^{(1)}=-\tfrac{1}{2}A_2f_2(d)\frac{1}{\beta}\sum_{k=-\infty}^{\infty}(\alpha^2+p_0{}^2)^{(d-5)/2}, \tag{A6}$$

where

$$f_2(d)=\frac{\Gamma\left[\dfrac{5-d}{2}\right]}{(4\pi)^{(d-1)/2}},\quad A_2=D$$

Now

$$\Sigma_2=\alpha^{d-5}\sum_{k=1}^{\infty}(\nu k)^{d-5}\left[1+\tfrac{1}{2}(d-5)\left[\frac{1}{\nu k}\right]^2\right]$$
$$+O(\alpha^5\beta^5). \tag{A7}$$

Repeating the procedure used for $L_{\beta 0}^{(1)}$ we find

$$L_{\beta 2}^{(1)}=-\frac{D}{16\pi}\left[\frac{1}{\alpha\beta}+\frac{1}{2\pi}\ln\alpha^2\beta^2-\frac{2\zeta(3)}{\pi^2}\alpha^2\beta^2-\tfrac{1}{2}(1-\ln 4\pi)-\frac{1}{\pi}\left[\frac{1}{d-4}\right]\right]$$
$$+O(\beta^3). \tag{A8}$$

The term in third proper-time order ($\sigma=3$) in (2.45) expanded up to second-derivative order is

$$L_{\beta 3}^{(1)}=-\tfrac{1}{2}f_3(d)\sum_{k=-\infty}^{\infty}(E+p_0{}^2F)(\alpha^2+p_0{}^2)^{(d-7)/2}, \tag{A9}$$

where

$$f_3(d)=\frac{\Gamma\left[\frac{7-d}{2}\right]}{(4\pi)^{(d-1)/2}} .$$

Expanding

$$\Sigma_3=\alpha^{d-7}\sum_{k=1}^{\infty}(\nu k)^{d-7}\left[1+\tfrac{1}{2}(d-7)\left[\frac{1}{\nu k}\right]^2\right]+O(\beta^6) \tag{A10}$$

we get

$$L_{\beta 3}^{(1)}=-\frac{1}{16\pi}\left[\frac{E}{2}\frac{1}{\alpha^3\beta}+\frac{F}{4\pi}\ln\alpha^2\beta^2 +\frac{\zeta(3)}{8\pi^3}(E-\tfrac{3}{2}F\alpha^2)\beta^2 +\frac{F}{4\pi}(2+\gamma-\ln 4\pi)-\frac{F}{2\pi}\left[\frac{1}{d-4}\right]\right] +O(\beta^3) . \tag{A11}$$

APPENDIX B: EVALUATION OF $^2\Gamma_\beta(V_2)$ IN EQ. (3.23)

We now evaluate the first term in Eq. (3.20):

$$^2\Gamma_\beta(V_2)=-\frac{i}{2}\mathrm{tr}(V_2\bar{G}_\beta^0) , \tag{B1}$$

where

$$V_2=\tfrac{1}{6}a^{-2}\beta'_{ij}\beta^{ij'}-2a^{-2}\beta^{ik}\beta_k{}^j\partial_i\partial_j .$$

Write

$$^2\Gamma_\beta(V_2)={}^{21}\Gamma_\beta+{}^{22}\Gamma_\beta$$

with

$$\begin{aligned} ^{21}\Gamma_\beta &\equiv \int d^dx\, a^4\, {}^{21}L_\beta \\ &=-\frac{i}{12}\mathrm{tr}(a^{-2}\beta'_{ij}\beta'^{ij}\bar{G}_\beta^0) \\ &=-\frac{i}{12}\int d^dx\, a^d(\eta)[\beta'_{ij}(\eta)\beta'^{ij}(\eta)] \\ &\quad\times a^{-2}(\eta)\bar{G}_\beta^0(x,x) . \end{aligned} \tag{B2}$$

Using the form for G_β^0 given by (2.29), we get

$$^{21}L_\beta=-\frac{i}{12}\frac{(\beta'_{ij}\beta'^{ij})}{a^4}\frac{\Gamma\left[\frac{3-d}{2}\right]}{(4\pi)^{(d-1)/2}}\frac{1}{\beta}\sum_{k=-\infty}^{\infty}(\alpha^2+p_0{}^2)^{(d-3)/2} \tag{B3}$$

carrying out a high-temperature expansion in a manner similar to Appendix A we find

$$^{21}L_\beta=-\frac{i}{12}\frac{(\beta'_{ij}\beta'^{ij})}{a^4}\left[-\frac{\pi}{3}\frac{1}{\beta^2}+\frac{\alpha}{\beta}-\frac{J(3)}{4(2\pi)^3}\alpha^4\beta^2-\frac{1}{16\pi^2}\alpha^2\ln\alpha^2\beta^2-\frac{1}{8\pi^2}\alpha^2(4+\gamma/2-\tfrac{1}{2}\ln 4\pi)+\frac{1}{8\pi^2}\alpha^2\left[\frac{1}{d-4}\right]\right] . \tag{B4}$$

Now consider

$$^{22}\Gamma_\beta=i\,\mathrm{tr}a^{-2}\beta^{ik}\beta_k{}^j\partial_i\partial_j\bar{G}_\beta^0 . \tag{B5}$$

We first evaluate $\partial_i\partial_j\bar{G}_\beta^0(x,x')$ and then apply the trace defined as $\mathrm{tr}F(x,x')=\int d^dx\, d^d(x)F(x,x)$. In the proper-time representation

$$^{22}\Gamma_\beta\equiv\int d^4x\, a^4({}^{22}L_\beta)=i\int d^dx\,\beta^{ik}\beta_k{}^j\int ds\frac{1}{\beta}\sum\int\frac{d^{d-1}p}{(2\pi)^{d-1}}(-p_ip_j)e^{-p^2s}e^{-p_0{}^2s}e^{-\alpha^2s}f(p_0,s) . \tag{B6}$$

We can integrate out the second moment of momenta

$$\beta^{ik}\beta^j{}_k\int d^{d-1}p(p_ip_j)e^{-p^2s}=\frac{1}{d-1}(\beta^{ik}\beta_{ki})\int d^{d-1}p\,p^2e^{-p^2s}$$

and

$$\int d^{d-1}p\,p^2e^{-p^2s}=\frac{d-1}{2s}\left[\frac{\pi}{s}\right]^{(d-1)/2} \tag{B7}$$

and get the effective Lagrangian

$$^{22}L_\beta=-\frac{i}{2}\frac{(\beta^{ik}\beta_{ki})}{a^4}\frac{1}{(4\pi)^{(d-1)/2}}\frac{1}{\beta}\sum\int\frac{ds}{s}\left[\frac{1}{s}\right]^{(d-1)/2}e^{-\alpha^2s}e^{-p_0{}^2s}f(p_0,s) . \tag{B8}$$

Define proper-time integral:

$$J=\frac{1}{(4\pi)^{(d-1)/2}}\frac{1}{\beta}\sum\int\frac{ds}{s}\left[\frac{1}{s}\right]^{(d-1)/2}e^{-\alpha^2 s}e^{-p_0{}^2 s}f(p_0,s)\ , \tag{B9}$$

where

$$\alpha^2=a^2(\eta)[m^2+(1-\xi)\xi_d R(\eta)+\tfrac{1}{2}\lambda\hat{\phi}^2(\eta)]$$

and

$$f(p_0,s)=1+s^2M^2\gamma-\tfrac{4}{3}s^3(\tfrac{1}{4}M^4\delta^2+p_0{}^2M^2\gamma)+O(s^4) \tag{B10}$$

expanded up to second derivative order with M^2,γ,δ defined in (3.10). Expanding J in a proper time series:

$$J=\sum_i J_i,\quad J_i=\frac{1}{(4\pi)^{(d-1)/2}}\frac{1}{\beta}\sum\int ds\, s^{-(d+1)/2}e^{-\alpha^2 s}e^{-p_0{}^2 s}c_i s^i\ \ (\text{no sum on } i)\ , \tag{B11}$$

where the coefficients c_i are functions of R, $\hat{\phi}$, and their derivatives. Performing the s integration:

$$\int ds\, s^{(2p-1-d)/2}e^{-\alpha^2 s}e^{-p_0{}^2 s}=\Gamma\left[\frac{2p+1-d}{2}\right](\alpha^2+p_0{}^2)^{(d-2p-1)/2}\ ,$$

with p the proper time order, we get

$$J_0=\frac{\Gamma\left[\frac{1-d}{2}\right]}{(4\pi)^{(d-1)/2}}\frac{1}{\beta}\sum_{k=-\infty}^{\infty}(\alpha^2+p_0{}^2)^{(d-1)/2}\ . \tag{B12}$$

This has the same form as $L^{(1)}_{\beta 0}$ in (A1), with high-temperature expansion given by

$$^{22}L_{\beta 0}=(-i)\frac{\beta^{ik}\beta_{ki}}{a^4}\left[\frac{\pi^2}{90}\frac{1}{\beta^4}-\frac{1}{24}\frac{\alpha^2}{\beta^2}+\frac{1}{12\pi}\frac{\alpha^3}{\beta}+\frac{1}{64\pi^2}\alpha^4\ln\alpha^2\beta^2+\frac{1}{64\pi^2}\alpha^4(\gamma-\ln 4\pi)-\frac{\zeta(3)}{8(32)\pi^4}\alpha^6\beta^2 -\frac{1}{32\pi^2}\alpha^4\left[\frac{1}{d-4}\right]\right]\ . \tag{B13}$$

Likewise J_2 is of the same form as $L^{(1)}_{\beta 2}$ in (A6). Thus

$$^{22}L_{\beta 2}=(-i)(\beta^{ik}\beta_{ki})\frac{M^2\gamma}{a^4}\left[\frac{1}{16\pi}\frac{1}{\alpha\beta}+\frac{1}{32\pi^2}\ln\alpha^2\beta^2-\frac{\zeta(3)}{8\pi^3}\alpha^2\beta^2+\frac{1}{32\pi}(1-\ln 4\pi)-\frac{1}{16\pi^2}\left[\frac{1}{d-4}\right]\right]\ . \tag{B14}$$

Similarly for J_3 which is similar to $L^{(1)}_{\beta 3}$ in (A11), with $E=\frac{1}{3}M^4\delta^2$ and $F=\frac{4}{3}M^2\gamma$ we have

$$^{22}L_{\beta 3}=(i)\frac{\beta^{ik}\beta_{ki}}{16\pi a^4}\left[\frac{E}{2}\frac{1}{a^3\beta}+\frac{F}{4\pi}\ln\alpha^2\beta^2+\frac{\zeta(3)}{8\pi^3}(E-\tfrac{3}{2}F\alpha^2)\beta^2+\frac{F}{4\pi}(2+\gamma-\ln 4\pi)-\frac{F}{2\pi}\left[\frac{1}{d-4}\right]\right]\ . \tag{B15}$$

Finally collecting terms in

$$^2\Gamma_\beta(V_2)=\int d^4x\, a^4(^2L_\beta(V_2))$$

and using the notation

$$B=\beta^{ik}\beta_{ki},\quad Q=\beta'_{ij}\beta'^{ij}(\text{shear}), D=M^2\gamma=\tfrac{3}{4}F\ ,$$

we get for the term in $^2L_\beta$ containing V_2 in (3.20), Eq. (3.26).

APPENDIX C: EVALUATION OF $^2\Gamma_\beta(V_1{}^2)$ IN EQ. (3.24)

The second term in $^2\Gamma_\beta$:

$$^2\Gamma_\beta(V_1{}^2)=-\frac{i}{4}\text{tr}(V_1\bar{G}^0_\beta V_1\bar{G}^0_\beta)$$

where tr acts on $(\eta,x_1,\ldots,x_{d-1})$ through $\int d^dx\, a^d(\eta)[\ ]$. Written out explicitly, (3.24) is

$$^2\Gamma_\beta(V_1{}^2)=-\frac{i}{4}\int a^dx\, d^d(\eta)\int d^dx' a^d(\eta')[V_1(\eta)\bar{G}^0_\beta(x,x')][V_1(\eta')\bar{G}^0_\beta(x',x)]\ , \tag{C1}$$

where

$$V_1=2a^{-2}\beta_{ij}\partial_i\partial_j \ .$$

Substituting

$$\bar{G}^0_\beta(x-x')=[a(\eta)]^{1-d/2}G^0_\beta(x-x')[a(\eta')]^{1-d/2}$$

and

$$G^0_\beta(x-x')=\frac{1}{(2\pi)^{d-1}}\frac{1}{\beta}\sum\int d^{d-1}p\, e^{ip(x-x')}G_\beta(p)_x$$

[subscript x means that coefficients in $G_\beta(p)$ are evaluated at x (here they depend only on η)]. One gets

$$\begin{aligned}{}^2\Gamma_\beta(V_1{}^2)&=-i\int d^dx\int d^dx'[\beta_{ij}(\eta)\partial_i\partial_j G^0_\beta(x-x')][\beta_{kl}(\eta')\partial'_k\partial'_l G^0_\beta(x'-x)]\\ &=\frac{-i}{(2\pi)^{2d-2}}\int d^dx\int d^dx'\left[\frac{1}{\beta^2}\right]\beta_{ij}(\eta)\sum\int d^{d-1}p(-p_ip_j)e^{ip(x-x')}G^0_\beta(p)_\eta\\ &\qquad\times\beta_{kl}(\eta')\sum\int d^{d-1}q(-q_kq_l)e^{-iq(x-x')}G^0_\beta(q)_{\eta'} \ .\end{aligned} \tag{C2}$$

Using

$$\int d^{d-1}x'e^{i(\bar{p}-\bar{q})(x-x')}=(2\pi)^{d-1}\delta^{d-1}(\bar{p}-\bar{q})$$

(where $\bar{x}$ denotes $d-1$ dim vector) this reduces to

$${}^2\Gamma_\beta(V_1{}^2)=\frac{-i}{(2\pi)^{d-1}}\int d^dx\int d\eta'\left[\frac{1}{\beta^2}\right]\beta_{ij}(\eta)\beta_{kl}(\eta')\sum\sum\int d^{d-1}p(p_ip_jp_kp_l)e^{i(p_0-q_0)(\eta-\eta')}G^0_\beta(\bar{p},p_0)_\eta G^0_\beta(\bar{p},q_0)_{\eta'} \ . \tag{C3}$$

In the momentum integrals, only terms containing even numbers of identical indices are nonzero. These are of two types:

$$K_4=\int dp_1p_1{}^4\int d^{d-2}p\,({}^0G_\beta{}^2),\quad K_2=\int dp_1p_1{}^2\int dp_2p_2{}^2\int d^{d-3}p({}^0G_\beta{}^2) \ , \tag{C4a}$$

where

$${}^0G_\beta{}^2\equiv G^0_\beta(\bar{p},p_0)_\eta G^0_\beta(\bar{p},q_0)_{\eta'} \ . \tag{C4b}$$

Performing the summation over the indices:

$${}^2\Gamma_\beta(V_1{}^2)=\frac{-i}{(2\pi)^{d-1}}\int d^dx\int d\eta'\left[\frac{1}{\beta^2}\right]\sum\sum e^{i(p_0-q_0)(\eta-\eta')}H(\eta,\eta') \ , \tag{C5}$$

where

$$H(\eta,\eta')=K_4\left[\sum_i(\beta_i{}^i)^2\right]+K_2\left[2\sum_{i<j}(\beta_i{}^i\beta_j{}^j)+4\sum_{i<j}(\beta^{ij}\beta_{ij})\right] \ . \tag{C6}$$

The dependence of $G_\beta(p,p_0)$ on p is only through e^{-p^2s}. Thus [from now on p is a $(d-1)$-dimensional vector]

$$\int dp_1p_1{}^4\int d^{d-2}\bar{p}\,e^{-p^2s}=\tfrac{3}{4}(\pi)^{(d-1)/2}\left[\frac{1}{s}\right]^{(d+3)/2} , \tag{C7a}$$

$$\int dp_1p_1{}^2\int dp_2p_2{}^2\int d^{d-3}\bar{p}\,e^{-p^2s}=\tfrac{1}{4}(\pi)^{(d-1)/2}\left[\frac{1}{s}\right]^{(d+3)/2} , \tag{C7b}$$

$$\Longrightarrow K_4=3K_2 \quad\text{and}\quad H(\eta,\eta')=Z(\eta,\eta')K_4 \tag{C8}$$

with

$$Z(\eta,\eta')=\left[\sum_i(\beta_i{}^i)^2\right]+\tfrac{1}{3}\left[2\sum_{i<j}(\beta_i{}^i\beta_j{}^j)+4\sum_{i<j}(\beta^{ij}\beta_{ij})\right]$$

which depends only on η and η'. Now the expansion of $G^0_\beta(p,p_0)$ up to second-derivative order is

$$G_\beta^0(p,p_0)\simeq\int ds\, e^{-\alpha^2 s}e^{-p^2 s}e^{-p_0{}^2 s}(1+s^2M^2\gamma-\tfrac{1}{3}s^3M^4\delta^2-\tfrac{4}{3}s^3p_0{}^2M^2\gamma)\,, \tag{C9}$$

where γ and δ are geometric factors defined in Appendix B and

$$\alpha^2(\eta)=a^2(\eta)[m^2+(1-\xi)\xi_d R(\eta)+\tfrac{1}{2}\lambda\hat{\phi}^2(\eta)]\,,$$

$$G_\beta^0(p,p_0)_\eta G_\beta^0(p,q_0)_{\eta'}=\int\int ds\,dr\, e^{-\alpha^2(s+r)}e^{-p^2(s+r)}e^{-p_0{}^2 s}e^{-q_0{}^2 r}\,.$$

$$\times[1+(M^2\gamma)_\eta s^2+(M^2\gamma)_{\eta'}r^2-\tfrac{1}{3}(M^4\delta^2)_\eta s^3-\tfrac{1}{3}(M^4\delta^2)_{\eta'}r^3-\tfrac{4}{3}(M^2\gamma)_\eta p_0{}^2s^3$$

$$-\tfrac{4}{3}(M^2\gamma)_{\eta'}q_0{}^2r^3]\,. \tag{C10}$$

In evaluating K_4 after performing the momentum integration using (C7a) we obtain integrals of the form

$$I_\sigma(a,b)=\int\int\frac{ds\,dr\,e^{-a^2 s}e^{-b^2 r}}{(r+s)^{(d+3)/2}}s^\sigma,\quad \sigma=0,2,3\;\begin{cases}a^2=\alpha^2(\eta)+p_0{}^2\,,\\ b^2=\alpha^2(\eta')+q_0{}^2\,,\end{cases}$$

$$I_0=-\Gamma\left[-\frac{(d+1)}{2}\right]\left[\frac{a^{d+1}-b^{d+1}}{a^2-b^2}\right], \tag{C11}$$

$$I_\sigma=\frac{\Gamma\left[\dfrac{1+2\sigma-d}{2}\right]}{(\sigma+1)a^{1+2\sigma-d}}F\left[1,\frac{1+2\sigma-d}{2};\sigma+2;\frac{a^2-b^2}{a^2}\right],\quad \sigma=2,3\,.$$

F is the hypergeometric function ${}_2F_1$. Putting everything together

$${}^2\Gamma_\beta(V_1{}^2)=(-i)\frac{3\pi^{(d-1)/2}}{4(2\pi)^{d-1}}\int d^dx\int d\eta' Z(\eta,\eta')\left[\frac{1}{\beta^2}\right]\sum_m\sum_{k=-\infty}^{\infty}e^{i(p_0-q_0)(\eta-\eta')}\sum_{\sigma=0}^{3}(A_\sigma I_\sigma+B_\sigma J_\sigma)\,, \tag{C12}$$

where A_σ,B_σ are the geometric and temperature-dependent factors of proper-time power σ in (C10). Also

$$p_0=\frac{2\pi m}{\beta},\quad q_0=\frac{2\pi k}{\beta},\quad\text{and}\quad J_\sigma=I_\sigma(b,a)\,.$$

Taking a local approximation ($\eta=\eta'$) for the factor $e^{i2\pi/\beta(m-k)(\eta-\eta')}$ we can calculate explicitly the diagonal ($k=m$) term in ${}^2\Gamma_\beta(V_1{}^2)$:

$$I_0(m\to k)_{\eta=\eta'}=\Gamma\left[\frac{1-d}{2}\right]a^{d-1},\quad I_\sigma(m=k)_{\eta=\eta'}=\frac{\Gamma\left[\dfrac{1+2\sigma-d}{2}\right]}{(\sigma+1)a^{1+2\sigma-d}},\quad \sigma=2,3\,. \tag{C13}$$

Define $Q(\eta)=\int d\eta' Z(\eta,\eta')$. Assuming in the quasilocal approximation η' is close to η [this is the approximation for $G(x,x')$ we used] and expressing the effective action in a proper-time series

$${}^2\Gamma_\beta(V_1{}^2)=\sum_{i=0}^{3}\Gamma_i=\Gamma_0+\Gamma_2+\Gamma_3,\quad \Gamma=\int d^4x\,a^4L$$

we get for

$$L_0=\frac{(-i)}{a^4}\frac{3}{4(4\pi)^{(d-1)/2}}Q\left[\frac{1}{\beta^2}\right]\sum_{m=-\infty}^{\infty}I_0\,. \tag{C14}$$

Carrying out a high-temperature expansion for Σ_0 as given in Appendix A up to $O(\alpha^2\beta^2)$ we have

$$L_0=-\frac{3i}{2}\frac{Q}{a^4}\left[\frac{\pi}{90}\frac{1}{\beta^5}-\frac{1}{24}\frac{\alpha^2}{\beta^3}+\frac{1}{12\pi}\frac{\alpha^3}{\beta^2}+\frac{1}{16\pi^2}\frac{\alpha^4}{\beta}(\gamma-\ln 4\pi)+\frac{1}{64\pi^2}\frac{\alpha^4}{\beta}\ln\alpha^2\beta^2-\frac{\zeta(3)}{8(32)\pi^4}\alpha^4\beta-\frac{1}{32\pi^2}\frac{\alpha^4}{\beta}\left[\frac{1}{d-4}\right]\right]. \tag{C15}$$

Similarly for

$$L_2=\frac{3(-i)}{4(4\pi)^{(d-1)/2}}\frac{Q}{a^4}\left[\frac{1}{\beta^2}\right]\left[\frac{2A_2}{3}\right]\Gamma\left[\frac{5-d}{2}\right]\sum_m a^{d-5},\quad\text{where}\quad A_2=M^2\gamma,\quad a^2=\alpha^2+\left[\frac{2\pi m}{\beta}\right]^2.$$

Using the results for Σ_2 in Appendix A we get

$$L_2=(-i)Q\frac{M^2\gamma}{a^4}\left[\frac{1}{16\pi}\left[\frac{1}{\alpha\beta^2}\right]+\frac{1}{32\pi^2}\frac{1}{\beta}\ln\alpha^2\beta^2-\frac{\zeta(3)}{8\pi^3}\alpha^2\beta+\frac{1}{32\pi}(1-\ln 4\pi)\frac{1}{\beta}-\frac{1}{16\pi^2}\frac{1}{\beta}\left[\frac{1}{d-4}\right]\right]+O(\beta^3)\ . \tag{C16}$$

Finally

$$L_3=\frac{3(-i)}{4(4\pi)^{(d-1)/2}}\frac{Q}{a^4}\left[\frac{1}{\beta^2}\right]\left[\frac{1}{2}A_3\right]\Gamma\left[\frac{7-d}{2}\right]\sum_m a^{d-7}\ ,$$

where

$$A_3=-\frac{4}{3}(M^4\delta^2+M^2\gamma p_0{}^2)\ .$$

Using Σ_3 from Appendix A we get

$$L_3=(i)\frac{Q}{a^4}\left[\frac{3E}{8}\frac{1}{\alpha^3\beta^2}+\frac{3F}{16\pi}\frac{1}{\beta}\ln\alpha^2\beta^2+\frac{3F}{16\pi}(2+\gamma-\ln 4\pi)\frac{1}{\beta}+\frac{3E\zeta(3)}{32\pi^3}\beta-\frac{9F\zeta(3)}{64\pi^3}\alpha^2\beta-\frac{3F}{8\pi}\frac{1}{\beta}\left[\frac{1}{d-4}\right]\right]+O(\beta^3)\ , \tag{C17}$$

where

$$E=\tfrac{1}{3}M^4\delta^2,\quad F=\tfrac{4}{3}M^2\gamma\ .$$

Collecting these terms we have, with ${}^2\Gamma_\beta(V_1{}^2)=\int d^4x\, a^4[{}^2L_\beta(V_1{}^2)]$

$$\begin{aligned}{}^2L_\beta(V_1{}^2)=(-i)\frac{Q}{a^4}\Bigg\{&\frac{\pi^2}{60}\frac{1}{\beta^5}-\frac{1}{16}\frac{\alpha^2}{\beta^3}+\frac{1}{8}\left[\frac{1}{\pi}\alpha^3+\frac{1}{2\pi}\frac{D}{\alpha}-\frac{3E}{\alpha^3}\right]\frac{1}{\beta^2}+\frac{1}{16\pi}\left[\frac{3}{8\pi}\alpha^4+\frac{(1-8\pi)}{2\pi}D\right]\frac{1}{\beta}\ln\alpha^2\beta^2\\&-\frac{\zeta(3)}{8\pi^3}\left[\frac{3}{64\pi}\alpha^2-\frac{D}{2}\right]\alpha^2\beta+\frac{1}{16\pi}\left[\frac{3}{8\pi}\alpha^4(\gamma-\ln 4\pi)+\frac{D}{2}(1-\ln 4\pi)-4D(2+\gamma-\ln 4\pi)\right]\frac{1}{\beta}\\&-\frac{1}{16\pi}\left[\frac{3}{4\pi}\alpha^4-\left[\frac{1}{\pi}+\frac{1}{2}\right]D\right]\frac{1}{\beta}\left[\frac{1}{d-4}\right]\Bigg\}\ ,\end{aligned} \tag{C18}$$

where $D=M^2\gamma$, and $Q=\int d\eta' z(\eta,\eta')$. The temperature-dependent ultraviolet divergence in the last term which does not have a counterpart in zero-temperature theory arises from the nonlocal kernel in ${}^2L_\beta(V_1{}^2)$ at finite temperature. We will discuss this point in Ref. 36.

---

*Permanent address.

[1]P. C. W. Davies, Rep. Prog. Phys. **41**, 1313 (1978).
[2]B. L. Hu, in *Proceedings of the Third Marcel Grossmann Meeting on General Relativity,* edited by Hu Ning (Science Press, Beijing and North-Holland, Amsterdam, 1983).
[3]D. A. Kirzhnits and A. D. Linde, Phys. Lett. **42B**, 471 (1972); C. W. Bernard, Phys. Rev. D **9**, 3312 (1974); L. Dolan and R. Jackiw, *ibid.* **9**, 3320 (1974); S. Weinberg, *ibid.* **9**, 3357 (1974).
[4]J. B. Hartle and S. W. Hawking, Phys. Rev. D **13**, 2188 (1976); G. Gibbons and M. J. Perry, Proc. R. Soc. London **A358**, 467 (1978).
[5]J. S. Dowker and R. Critchley, Phys. Rev. D **15**, 1484 (1977); J. S. Dowker and G. Kennedy, J. Phys. A **11**, 895 (1978); M. B. Altaire and J. S. Dowker, Phys. Rev. D **18**, 3557 (1978).
[6]G. Kennedy, J. Phys. A **11**, 177 (1978).
[7]I. Drummond, Nucl. Phys. **B190** [FS3], 93 (1981).
[8]R. Critchley, P. C. W. Davies, and G. Kennedy, Phys. Lett. **112B**, 331 (1982).
[9]B. L. Hu, Phys. Lett. **108B**, 19 (1982).
[10]B. L. Hu, Phys. Lett. **123B**, 189 (1983).
[11]L. F. Chen and B. L. Hu, Phys. Lett. **160B**, 36 (1985).
[12]B. L. Hu, in *The Very Early Universe,* edited by G. Gibbons, S. W. Hawking, and S. Siklos (Cambridge University Press, Cambridge, 1983).
[13]B. L. Hu, Phys. Lett. **103B**, 331 (1981).
[14]A. D. Linde, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 398 (1982) [JETP Lett. **35**, 492 (1982)].
[15]M. Yoshimura, Phys. Rev. D **30**, 344 (1984).
[16]K. Maeda, Phys. Rev. D **30**, 2482 (1984).
[17]B. L. Hu, Phys. Lett. **90A**, 375 (1982); **97A**, 368 (1983).

[18]B. L. Hu, in *Cosmology of the Early Universe,* edited by L. Z. Fang and R. Ruffini (World Scientific, Singapore, 1984).

[19]B. L. Hu and H. Kandrup (unpublished); B. L. Hu and D. Pavon, Phys. Lett. **180B**, 329 (1986).

[20]H. Kodama, Prog. Theor. Phys. **67**, 1415 (1982); M. Sasaki and M. Morikawa, Phys. Lett. **165B**, 59 (1985).

[21]H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermal Field Dynamics and Condensed States* (North-Holland, Amsterdam, 1982); A. S. Niemi and G. W. Semenoff, Ann. Phys. (N.Y.) **152**, 105 (1984); Nucl. Phys. **B230** [FS10], 181 (1984); G. Semenoff and D. Weiss, Phys. Rev. D **31**, 689 (1985); N. Nakazawa and T. Fukuyama, Nucl. Phys. **B252**, 621 (1985); Y. Fugimoto, Class. Quantum Grav. **3**, 921 (1986).

[22]G. Zhou, Z. B. Su, B. L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985); E. Calzetta and B. L. Hu (unpublished); E. Calzetta, B. L. Hu, and A. Stylianopoulos (unpublished).

[23]J. Winter, Phys. Rev. D **32**, 1871 (1985).

[24]E. Calzetta and B. L. Hu, in Proceedings of the First International Conference on Phase Space, College Park, 1986; E. Calzetta, S. Habib, and B. L. Hu (unpublished).

[25]J. Schwinger, Phys. Rev. **82**, 664 (1951); B. S. DeWitt, *The Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).

[26]See, e.g, N. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[27]See, e.g, B. S. DeWitt, Phys. Rev. **160**, 1112 (1967).

[28]T. S. Bunch and L. Parker, Phys. Rev. D **20**, 2499 (1979).

[29]M. R. Brown and M. J. Duff, Phys. Rev. D **11**, 2114 (1975); J. Iliopoulos, C. Itzykson, and A. Martin, Rev. Mod. Phys. **47**, 165 (1975).

[30]B. L. Hu and D. J. O'Connor, Phys. Rev. D **30**, 743 (1984).

[31]L. H. Chan, Phys. Rev. Lett. **54**, 1222 (1985); C. M. Fraser, Z. Phys. C **28**, 101 (1985); J. A. Zuk, J. Phys. A **18**, 1795 (1985); I. G. Moss, D. J. Toms, and W. A. Wright (unpublished).

[32]K. Shizuya, Nucl. Phys. **B227**, 134 (1983).

[33]I. J. R. Aitchison and C. M. Fraser, Phys. Lett. **146B**, 63 (1985); Phys. Rev. D **32**, 2190 (1985); R. Mackenzie, F. Wilczek, and A. Zee, Phys. Rev. Lett. **53**, 2203 (1984).

[34]P. Binetruy and M. K. Gaillard, Phys. Rev. D **32**, 931 (1985).

[35]J. B. Hartle and B. L. Hu, Phys. Rev. D **20**, 1772 (1979); **21**, 2756 (1980).

[36]B. L. Hu, D. J. O'Connor, and A. Stylianopoulos (unpublished).

[37]C. Callan, S. Coleman, and R. Jackiw, Phys. Rev. D **10**, 2491 (1974); J. Cornwall, R. Jackiw, and E. Tomboulis, *ibid.* **10**, 2428 (1974).

[38]D. J. O'Connor, Ph.D. thesis, University of Maryland, 1985; B. L. Hu, in *Proceedings of the 4th Marcel Grossmann Meeting on General Relativity,* edited by R. Ruffini (North-Holland, Amsterdam, 1987); B. L. Hu and D. J. O'Connor, Phys. Rev. Lett. **56**, 1613 (1986); Phys. Rev. D **34**, 2535 (1986), and (unpublished).

[39]S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977).