# Closed-time-path functional formalism in curved spacetime: Application to cosmological back-reaction problems

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We discuss the generalization to curved spacetime of a path-integral formalism of quantum field theory based on the sum over paths first going forward in time in the presence of one external source from an in vacuum to a state defined on a hypersurface of constant time in the future, and then backwards in time in the presence of a different source to the same in vacuum. This closed-time-path formalism which generalizes the conventional method based on in-out vacuum persistence amplitudes yields real and causal effective actions, field equations, and expectation values. We apply this method to two problems in semiclassical cosmology. First we study the back reaction of particle production in a radiation-filled Bianchi type-I universe with a conformal scalar field. Unlike the in-out formalism which yields complex geometries the real and causal effective action here yields equations for real effective geometries, with more readily interpretable results. It also provides a clear identification of particle production as a dissipative process in semiclassical theories. In the second problem we calculate the vacuum expectation value of the stress-energy tensor for a nonconformal massive $\lambda\phi^4$ theory in a Robertson-Walker universe. This study serves to illustrate the use of Feynman diagrams and higher-loop calculations in this formalism. It also demonstrates the economy of this method in the calculation of expectation values over the mode-sum Bogolubov transformation methods ordinarily applied to matrix elements calculated in the conventional in-out approach. The capability of the closed-time-path formalism of dealing with Feynman, causal, and correlation functions on the same footing makes it a potentially powerful and versatile technique for treating nonequilibrium statistical properties of dynamical systems as in early-Universe quantum processes.

## I. INTRODUCTION

The effective-action formalism[1–3] based on the vacuum persistence amplitude has proven to be a powerful method for analyzing the behavior of classical gravitational fields interacting with quantum matter. In this semiclassical approach one studies a generalized Einstein's equation (containing terms of higher derivatives of the metric tensor) with a source given by the vacuum expectation value of the energy-momentum tensor of matter-field operators.[4–8] This effective field equation is derivable from the effective action $\Gamma$, which is the Legendre transform of the generating functional $W$ related to the vacuum persistence amplitude $\langle 0_+ \mid 0_- \rangle$ by

$$\langle 0_+ \mid 0_- \rangle_J = e^{iW[J]} = \int D\Phi \exp\{ i(S[g,\Phi]+J\Phi)\} \;, \tag{1.1}$$

where $\mid 0_\pm \rangle$ denote the vacuum states at $t=\pm\infty$. Here $J$ is an external source, $S$ is the action, $g$ is the metric of a classical spacetime, and $D\Phi$ is the measure of the functional integral over the scalar field $\Phi$. As spacetime evolves, the out vacuum $\mid 0_+ \rangle$ can in general be different from the in vacuum $\mid 0_- \rangle$ due to particle production. The effective action is in general complex, its imaginary part measures the total probability $P$ to produce a particle pair over the entire history of the Universe:

$$P = 2\,\mathrm{Im}\,W \;. \tag{1.2}$$

Since the $\mid 0_\pm \rangle$ are different in this conventional approach (which we will call the in-out formulation) one can only calculate the *matrix elements* $\langle 0_+ \mid T \mid 0_- \rangle$ of, say, a tensor operator $T$ between the in $\mid 0_- \rangle$ and out $\mid 0_+ \rangle$ states, which are complex, rather than the physically relevant *expectation values* of an observable $\langle 0_- \mid T \mid 0_- \rangle$ taken with respect to the same state. In order to translate, say, the in-out matrix element to the in-in expectation value, one needs to sum over a set of intermediate complete states, which involves knowing the Bogolubov coefficients.[9] For spacetimes with high symmetry where mode decomposition is readily available this is a feasible but often cumbersome task. One can also use functional transforms[10] to relate these quantities but these are more likely to be of formal than practical value. In the conventional formulation since both the in and out states need to be specified in the asymptotic regions, solution of the effective geometry and fields is by nature a boundary-value problem rather than an initial-value problem. If one can devise a functional integral method formulated with respect to the same initial state, one may hope to be able to calculate the physically relevant quantities like the expectation values in a more convenient way by solving the equations of motion with given initial Cauchy data. Such a formalism indeed exists. It was first proposed by Schwinger[11] and later developed by Keldysh,[12] Korenman,[13] and many others. The nonrelativistic formulation has since been applied to problems in statistical mechanics and condensed-matter physics. A relativistic fieldtheoretical formalism was developed lately by Zhou *et al.*[14] (We refer the reader to their review on this subject and references to previous work.) Extension of this formalism to curved-spacetime problems has recently been carried out by Jordan.[15]

In the in-in formalism we let the in vacuum evolve independently under two different external sources $J^+(x)$ and $J^-(x)$ and compare the results with a common state $|\psi\rangle$ in the future. The generating functional $W[J^+,J^-]$ is defined by

$$e^{iW[J^+,J^-]}=\sum_{\psi}\langle 0_- \,|\, \psi\rangle_{J^-}\langle \psi \,|\, 0_-\rangle_{J^+} \; , \tag{1.3}$$

where $|\psi\rangle$ is a common eigenvector of the field operator $\Phi_H$ at some large time $t^*$, i.e.,

$$\Phi_H(\mathbf{x},t^*)\,|\,\psi\rangle=\psi(\mathbf{x})\,|\,\psi\rangle \; . \tag{1.4}$$

The set $|\psi\rangle$ is complete and orthonormal. In the path integral representation, this can be thought of as a sum over paths which go forward in time in the presence of $J^+$ from the in vacuum $|0_-\rangle$ to a state $|\psi\rangle$ defined on a hypersurface $\Sigma$ of constant time $t^*$ and then backwards in time in the presence of $J^-$ to the in vacuum. Thus it has acquired the name "closed-time-path" formalism. We will use the abbreviation "CTP" or "in-in" to denote this formulation as distinct from the "conventional" or "in-out" formalism. The in-in effective action can be defined as the Legendre transform of $W$:

$$\Gamma[\hat{\phi}^+,\hat{\phi}^-]=W[J^+,J^-]-J^+\hat{\phi}^+ +J^-\hat{\phi}^- \; , \tag{1.5}$$

where $\hat{\phi}^{\pm}\equiv\pm\delta W[J^+,J^-]/\delta J^{\pm}$. When $J^+=J^-=0$, they are the expectation values of $\Phi_H$ with respect to $|0_-\rangle$, i.e.,

$$\hat{\phi}^+(x)=\hat{\phi}^-(x)=\langle 0_-\,|\,\Phi_H(x)\,|\,0_-\rangle \; . \tag{1.6}$$

The field equations satisfied by the expectation values are given by

$$\delta\Gamma[\hat{\phi}^+,\hat{\phi}^-]/\delta\hat{\phi}^+\,|_{\hat{\phi}^+=\hat{\phi}^-}=0 \; . \tag{1.7}$$

The effective field equation in curved space has been shown by Jordan[15] to be real and causal up to two-loop order. He also described the relation among the different Green's functions and the modified Feynman rules in perturbation calculations following closely the flat-space counterparts.

As one can easily see, in the in-in formalism, doubling the sources and fields increases the number of Feynman diagrams which need to be included. This excess baggage adds some technical complexity, which may be the primary reason why this formalism has not been in wider use. Nevertheless the advantage one gains makes it worthwhile. As was pointed out earlier since the effective action and field equations are now real and causal, the results are more easily interpretable physically. The other advantage is that the different Green's functions, i.e., the Feynman and Wightman functions or the causal and correlation functions are now treated on the same footing (as different elements of a $2\times2$ matrix) obeying the same set of (matrix) equations.[14] These features make the in-in formalism particularly useful in tackling problems in statistical physics, where the interest usually lies in the causal and correlational properties of a system as a function of time. Whereas in problems in particle physics and quantum cosmology one may be more interested in the transition amplitude between the initial and final states, in statistical physics one is more likely to be interested in the correlation between states and the causal properties of the system, whereby the present formalism is better adapted. Indeed our interest in this formalism has been mainly directed at its potential in the treatment of statistical properties of quantum processes in curved space. These include, for example, statistical distribution of particle production and their back reaction, quantum dissipative processes, real-time finite-temperature theory, dynamical critical phenomena in the early Universe, and quantum radiance and thermodynamic suitability of black holes. As for problems in quantum cosmology (or more precisely in the semiclassical approximation) like those treated in Refs. 3–8, we feel that the in-in and the in-out formalism each can address a different aspect of the problem better than the other. The suitability of these methods depends on the question one asks. If one is interested in, say, finding the rate of particle production with back reaction, the in-out effective action is sufficient. The same is true for studies of quantum tunneling and vacuum stability, as in these cases it is the transition amplitude between two vacuum states which is relevant. As for regarding the capability of the in-in formalism in yielding an equation of motion of the initial-value type as an advantage, we feel that again it depends on what one knows and what one wants to find out in the problem: When one asks questions in cosmology concerning the conditions of the Universe near the Planck time, the out state or the Friedmann universe is certainly a better understood and well-defined state than the in state, and it is not clear physically whether the initial-value problem is really a better-posed problem than that of the boundary-value problem.[4] On the other hand, if one is interested in the evolution of the background geometry as a result of back reaction of particle production, the complex effective geometry in the in-out formalism is not so easy to interpret. The fact that the in-in formalism yields a real and causal equation is then certainly an advantage. As mentioned earlier, there have been attempts[9,10] to translate the complex geometry to real quantities (via Bogolubov coefficients or functional transforms), so even for these purposes the in-in formalism is not completely indispensable. Despite this it is perhaps the most direct and convenient way of achieving these goals. In addition, the real and causal results of the in-in effective action make it easy to identify and discuss the dissipative properties of semiclassical theories (e.g., resistive versus reactive effects as coming from the imaginary and real parts of the equation of motion in the frequency domain). The in-in formalism is also a good starting point for the study of nonequilibrium quantum statistical processes.

In this paper we briefly introduce the in-in formalism and apply it to two problems in curved spacetime: one is the dissipation of anisotropy through quantum effects in a Bianchi type-I universe, the other is a self-interacting field theory in the Robertson-Walker universe. The purpose of this study is twofold. As most of the earlier discussions on the in-in formalism have been quite formal, we want to make it more approachable by applying it to some already familiar physical problems, thereby demonstrating its versatility, power and wieldiness. Secondly, we want to use it to get results which would not easily be reached by other methods. In Sec. II we give a brief summary of the methods first in flat space and then in curved space. We include only those parts which are necessary for our latter discussion, although we will try to make it self-contained. In Sec. III we study the back reaction of particle production in a Bianchi type-I universe with small anisotropy. We derive a real and causal field equation for the anisotropy. Our analysis based on the in-in effective action closely parallels the method and reproduces all the major results of the in-out formalism obtained in Ref. 5. In Sec. IV we study a $(m)$ massive $\lambda\phi^4$ theory in a Robertson-Walker spacetime with arbitrary coupling $\xi$ to the curvature $R$. This example serves to illustrate the use of Feynman diagram techniques in this new approach. We calculate the expectation value of the energy-momentum tensor $\langle T_{\mu\nu}\rangle$ to second order in $m,\xi$ and first order in $\lambda$. The free-field problem has been studied by Hartle[3,7] using the in-out effective-action approach, and by Davies and Unruh[16] who computed $\langle T_{\mu\nu}\rangle$ with explicit particle modes. The interacting field problem has been considered by Bunch, Panangaden, and Parker[17] and Birrell, Davies, and Ford[18] using $S$-matrix theory and adiabatic wave functions. This example shows how the above results can be obtained in a unified manner via the in-in formalism. For example, it confirms that corrections to particle production due to self-interaction appear already to first order in $\lambda$. In Sec. IV we end with a few general remarks, indicating also how this formalism could be usefully applied to the consideration of related problems in field theory and statistical mechanics in curved spacetime.

## II. BASIC FORMALISM

In this section we will give a short introduction to the in-in formalism first in flat spacetime and then its extension to curved spacetime. To avoid overburdening the discussion we will go into details only in those parts which differ from the in-out formulation, with which we assume the reader to have some familiarity. Consider the Heisenberg representation for the field operator $\Phi_H$ and states for an interacting field theory such as the $\lambda\phi^4$ theory with classical action

$$S[\Phi]=\int d^4x\left[\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi-\tfrac{1}{2}m^2\Phi^2-\frac{\lambda}{4!}\Phi^4\right] . \quad (2.1)$$

(The metric signature is $+---$ throughout.) As the (self-)coupling constant $\lambda$ is adiabatically switched off at $t\to\pm\infty$, the theory becomes free in the distant past and future. The two special Heisenberg states, the in vacuum $|0_-\rangle$, and the out vacuum $|0_+\rangle$ coincide with the free field vacuum in the past and future asymptotic regions, respectively. (In flat-space theory, usually $|0_-\rangle=|0_+\rangle$, but we do not need to assume this; in curved spaces, often $|0_-\rangle\neq|0_+\rangle$.) If the field is coupled to an external, $c$-number source $J(x)$, the Heisenberg states will evolve in time according to

$$|0_-\rangle_J(t)=T\exp\left[i\int_{-\infty}^{t}dt'\int d^3xJ(\mathbf{x},t')\Phi_H(\mathbf{x},t')\right]|0_-\rangle , \quad (2.2)$$

where $T$ means temporal order. The vacuum persistence amplitude defined as

$$Z[J]=\langle 0_+|0_-\rangle_J=\left\langle 0_+\left|T\exp\left[i\int d^4xJ(x)\Phi_H(x)\right]\right|0_-\right\rangle \quad (2.3)$$

contains all the dynamical information about the theory. It has a path-integral representation

$$Z[J]=\int D\Phi\exp\{i(S[\Phi]+J\Phi)\} , \quad (2.4)$$

where the integral is over all field configurations. For massive field theories like (2.1) one usually adds an imaginary part $(-i\varepsilon)$ to the mass to account for the boundary conditions. $Z[J]$ generates the time-ordered matrix elements of a product of $n$ Heisenberg fields between the in and out states

$$\langle 0_+|T[\Phi_H(x_1)\cdots\Phi_H(x_n)]|0_-\rangle=(-i)^n\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}Z[J]\bigg|_{J=0} . \quad (2.5)$$

As per earlier discussion, what one would like to find is a functional which generates expectation values with respect to the in state. Following Schwinger,[11] we introduce two external sources $J^+(x)$ and $J^-(x)$ and consider the quantity

$$Z[J^+,J^-]={}_{J^-}\langle 0_-|0_-\rangle_{J^+} . \quad (2.6)$$

In contrast with the in-out formalism, where one lets the in vacuum evolve under the influence of an external source and compares the result with the out vacuum, in the in-in formalism, one lets the in vacuum evolve independently under two sources $J^+$ and $J^-$, and compare the results in the future. We may rewrite (2.6) as

$$Z[J^+,J^-]=\int D\psi\left\langle 0_-\left|\tilde{T}\exp\left[-i\int_{-\infty}^{t^*}dt\int d^3xJ^-(x)\Phi_H(x)\right]\right|\psi\right\rangle \times\left\langle\psi\left|T\exp\left[i\int_{-\infty}^{t^*}dt\int d^3xJ^+(x)\Phi_H(x)\right]\right|0_-\right\rangle , \quad (2.7)$$

where $\tilde{T}$ denotes antitemporal order. Here $|\psi\rangle$ is an element of a complete, orthonormal set of common eigenvectors of the field operators at some late time $t^*$

$$\Phi_H(\mathbf{x},t^*)\,|\,\psi\rangle=\psi(\mathbf{x})\,|\,\psi\rangle\ . \tag{2.8}$$

We will assume $t^*=+\infty$ for all practical purposes.

From the definition (2.6) and (2.7) we obtain the following useful relations:

$$Z[J,J]=1,\quad Z[J^+,J^-]=(Z[J^-,J^+])^*\ , \tag{2.9}$$

$$(-i)^n\left.\frac{\partial^n Z[J^+,J^-]}{\partial J^+(x_1)\cdots\partial J^+(x_n)}\right|_{J^+=J^-=0}=\langle 0_-\,|\,T[\Phi_H(x_1)\cdots\Phi_H(x_n)]\,|\,0_-\rangle\ , \tag{2.10}$$

$$(i)^m\left.\frac{\partial^m Z[J^+,J^-]}{\partial J^-(x_1)\cdots\partial J^-(x_m)}\right|_{J^+=J^-=0}=\langle 0_-\,|\,\tilde{T}[\Phi_H(x_1)\cdots\Phi_H(x_m)]\,|\,0_-\rangle\ , \tag{2.11}$$

$$(-i)^{n-m}\left.\frac{\partial^{n+m}Z[J^+,J^-]}{\partial J^-(x_1)\cdots\partial J^-(x_m)\partial J^+(y_1)\cdots\partial J^+(y_n)}\right|_{J^+,J^-=0}=\langle 0_-\,|\,\tilde{T}[\Phi_H(x_1)\cdots\Phi_H(x_m)]\times T[\Phi_H(y_1)\cdots\Phi_H(y_n)]\,|\,0_-\rangle\ . \tag{2.12}$$

Observe that $Z$ generates expectation values other than the time-ordered ones. $Z[J^+,J^-]$ has a functional-integral representation

$$Z[J^+,J^-]=\int D\Phi^+(x)D\Phi^-(x)\exp\{i(S[\Phi^+]+J^+\Phi^+-S^*[\Phi^-]-J^-\Phi^-)\}\ , \tag{2.13}$$

where $J\Phi$ denotes $\int d^4x\,J(x)\Phi(x)$. $S^*$ indicates that in this functional, $m^2$ carries an $+i\epsilon$ term. The integral in (2.13) is over all field configurations which coincide at $t=t^*$ (in practice, $t^*=+\infty$). We do not require the fields to go to zero as $t\to\infty$ (eventually we may consider that $i\varepsilon$ is switched off in this region), nor do we require $\partial_t\Phi^+|_{t=t^*}=\partial_t\Phi^-|_{t=t^*}$. As in the in-out formalism, it is easier to work with the generating functional defined as

$$W[J^+,J^-]=-i\ln Z[J^+,J^-] \tag{2.14}$$

which generates normalized expectation values. Let us consider the classical fields defined by

$$\hat{\phi}^+(x)=\frac{\partial W[J^+,J^-]}{\partial J^+(x)}\quad\text{and}\quad\hat{\phi}^-(x)=-\frac{\partial W[J^+,J^-]}{\partial J^-(x)}\ . \tag{2.15}$$

If $J^+=J^-=J$, then $\hat{\phi}^+=\hat{\phi}^-=\hat{\phi}$ is the expectation value of the Heisenberg field with respect to the state which evolved from $|0_-\rangle$ under the influence of the source $J$. The in-in effective action $\Gamma$ is defined as the Legendre transform of $W$, i.e.,

$$\Gamma[\hat{\phi}^+,\hat{\phi}^-]=W[J^{*+},J^{*-}]-J^{*+}\hat{\phi}^++J^{*-}\hat{\phi}^-\ , \tag{2.16}$$

where $J^{*+},J^{*-}$ are the solutions of

$$\hat{\phi}^{\pm}=\pm\frac{\partial}{\partial J^{\pm}}W[J^{*+},J^{*-}]\ . \tag{2.17}$$

We then have

$$\frac{\partial\Gamma[\hat{\phi}^+,\hat{\phi}^-]}{\partial\hat{\phi}^{\pm}}=\mp J^{*\pm}\ . \tag{2.18}$$

Equations (2.15) or their equivalent (2.18) are the equations of motion for $\hat{\phi}$. As different from the in-out formalism, $\hat{\phi}$, the common value of $\hat{\phi}^+$ and $\hat{\phi}^-$, is real and depends causally on $J$, the common value of $J^+$ and $J^-$. The reality of $\hat{\phi}$ follows directly from $\hat{\phi}$ being the expectation value of the Heisenberg fields with respect to some state. As for causality, let us compute the functional derivative $\partial\hat{\phi}(x)/\partial J(x')$ using (2.9):

$$\frac{\partial\hat{\phi}(x)}{\partial J(x')}=(-i)\left.\left[\frac{\partial^2 Z}{\partial J^+(x')\partial J^+(x)}+\frac{\partial^2 Z}{\partial J^-(x')\partial J^+(x)}\right]\right|_{(J^+=J^-=J)}\ . \tag{2.19}$$

From the definition (2.7) it is easy to show that this is zero whenever $t'>t$. Thus $\hat{\phi}(x)$ depends only on values of $J$ in the past of $x$ (from Lorentz invariance, we may say "in the past light cone of $x$"). Jordan[15] has demonstrated these properties to the two-loop order. Observe that in practice only one of the equations (2.18) need be solved. Since $\Gamma[\hat{\phi},\hat{\phi}]$ is identically zero, we have the identity

$$\frac{\partial\Gamma}{\partial\hat{\phi}^+}(\hat{\phi}^+=\hat{\phi}^-=\hat{\phi})=-\frac{\partial\Gamma}{\partial\hat{\phi}^-}(\hat{\phi}^+=\hat{\phi}^-=\hat{\phi})\ . \tag{2.20}$$

So if $J^+=J^-=J$, (2.18) always has a solution with $\hat{\phi}^+=\hat{\phi}^-=\hat{\phi}$ and each of the equations in (2.18) implies the other.

As in the in-out approach, exact computations of $Z$, $W$, or $\Gamma$ are usually impossible except for some special cases. In general one needs to resort to perturbative methods. Observe that the path-integral representation (2.13) of $Z$ has the same form as the in-out path integral for a theory of two scalar fields, and therefore the perturbative evaluation of $Z$ may proceed exactly as in the conventional case. For example, $W$ will be the sum of all connected graphs of the two-component field theory, while $\Gamma$ will be the sum of the one-particle-irreducible (1PI) graphs. Likewise background-field techniques can also be carried over to this case. However, in these calculations it is important to note that $\Phi^+$ and $\Phi^-$ are not independent integration variables, but are linked through the boundary conditions on the hypersurface at a common time $t^*$ in the future.

Let us consider the specific example of a $\lambda\phi^4$ field (2.1). In the free field theory the path integral (2.13) is Gaussian and $Z$ takes on the form

$$Z_{\rm free}[J^+,J^-]=\exp\left[\frac{i}{2}\int d^4x\, d^4x'[J^+(x)K_{++}(x,x')J^-(x')-J^+(x)K_{+-}(x,x')J^-(x')\right.$$
$$\left.-J^-(x)K_{-+}(x,x')J^+(x')+J^-(x)K_{--}(x,x')J^-(x')]\right] .$$

The kernels $K_{++}$ and $K_{--}$ are symmetric, and $K_{+-}(x,x')=K_{-+}(x',x)$. They are related to the classical fields $\hat\phi^{\pm}(x)$ of Eq. (2.15) by

$$\begin{aligned}\hat\phi^+(x)&=\int d^4x'[K_{++}(x,x')J^+(x')-K_{+-}(x,x')J^-(x')]\,,\\ \hat\phi^-(x)&=\int d^4x'[K_{-+}(x,x')J^+(x')-K_{--}(x,x')J^-(x')]\,.\end{aligned} \tag{2.21}$$

One can introduce an alternate representation for the fields as

$$\begin{aligned}\hat\phi^+(x)=(Z[J^+,J^-])^{-1}\Big\langle 0_-\Big|&\left[\tilde T\exp\left[-i\int d^4x'J^-\Phi_H\right]\right]\left[T\exp\left[i\int_{t'>t}d^4x'J^+\Phi_H\right]\right]\\ &\times\Phi_H(x)\left[T\exp\left[i\int_{t'<t}d^4x'J^+\Phi_H\right]\right]\Big|0_-\Big\rangle\,,\\ \hat\phi^-(x)=(Z[J^+,J^-])^{-1}\Big\langle 0_-\Big|&\left[\tilde T\exp\left[-i\int_{t'<t}d^4x'J^-\Phi_H\right]\right]\Phi_H(x)\\ &\times\left[\tilde T\exp\left[-i\int_{t'>t}d^4x^1J^-\hat\phi_H\right]\right]\left[T\exp\left[i\int d^4x'J^+\Phi_H\right]\right]\Big|0_-\Big\rangle\,.\end{aligned} \tag{2.22}$$

From the Heisenberg equations of motion or a free field, the canonical commutation relations, and (2.22), it is easy to show that

$$(\Box+m^2)\hat\phi^{\pm}(x)=J^{\pm}(x)\,. \tag{2.23}$$

As $t\to-\infty$, $\hat\phi^+$ ($\hat\phi^-$) contains only negative (positive) frequencies, and as $t\to+\infty$, $\hat\phi^+(x)$ goes into $\hat\phi^-(x)$ and $\partial_t\hat\phi^+(x)$ goes into $\partial_t\hat\phi^-$. A positive-frequency wave is defined as possessing the form $\exp(-i\omega t)$, $\omega>0$. The solution of (2.23) is

$$\hat\phi^+(x)=-\int d^4x'[\Delta_F(x-x')J^+(x')+\Delta^+(x-x')J^-(x')]\,, \tag{2.24a}$$

$$\hat\phi^-(x)=-\int d^4x'[\Delta^-(x-x')J^+(x')+\Delta_D(x-x')J^-(x')]\,, \tag{2.24b}$$

where

$$\Delta_F(x-x')=\int\frac{d^4p}{(2\pi)^4}e^{ip(x-x')}(p^2-m^2+i\epsilon)^{-1}\,, \tag{2.25a}$$

$$\Delta_D(x-x')=\int\frac{d^4p}{(2\pi)^4}e^{ip(x-x')}(p^2-m^2-i\epsilon)^{-1}\,, \tag{2.25b}$$

$$\Delta^+(x-x')=\int\frac{d^4p}{(2\pi)^4}e^{ip(x-x')}2\pi i\delta(p^2-m^2)\theta(p^0)\,, \tag{2.25c}$$

$$\Delta^-(x-x')=\int\frac{d^4p}{(2\pi)^4}e^{ip(x-x')}2\pi i\delta(p^2-m^2)\theta(-p^0)\,, \tag{2.25d}$$

are the Feynman, Dyson, and positive- and negative-frequency Wightman functions, respectively. Comparing (2.21) and (2.24) we may write

$$\begin{aligned}Z_{\rm free}[J^+,J^-]=\exp\Bigg[\frac{-i}{2}\int d^4x\, d^4x'[&J^+(x)\Delta_F(x-x')J^+(x')+J^+(x)\Delta^+(x-x')J^-(x')\\ &-J^-(x)\Delta^-(x-x')J^+(x')-J^-(x)\Delta_D(x-x')J^-(x')]\Bigg]\,.\end{aligned} \tag{2.26}$$

From (2.26), (2.9), (2.10), and (2.11) we find, for a free field,

$$\langle 0_- \mid T[\Phi_H(x)\Phi_H(x')] \mid 0_- \rangle = i\Delta_F(x-x') , \tag{2.27a}$$

$$\langle 0_- \mid \tilde{T}[\Phi_H(x)\Phi_H(x')] \mid 0_- \rangle = -i\Delta_D(x-x') , \tag{2.27b}$$

$$\langle 0_- \mid \Phi_H(x')\Phi_H(x) \mid 0_- \rangle = -i\Delta^+(x-x') . \tag{2.27c}$$

For interacting fields, we write

$$Z[J^+,J^-] = \exp\left[\frac{-i\lambda}{4!}\int d^4x \left[\frac{\partial^4}{\partial J^+(x)^4} - \frac{\partial^4}{\partial J^-(x)^4}\right]\right] Z_{\text{free}}[J^+,J^-] . \tag{2.28}$$

With $Z_{\text{free}}$ given by (2.26) and (2.25). From (2.28); the evaluation of $Z$, $W$, or $\Gamma$ proceeds as usual. There will be two kinds of vertices, "+" vertices coming from the $\partial^4/\partial J^{+4}$ term, and "−" vertices coming from the $\partial^4/\partial J^{-4}$ term. There are three kinds of internal legs, "+ +", "− −," or "+ −", distinguished by the signs of the vertices, the external source or the background fields to which they are attached. The corresponding Feynman rules are given by Eqs. (2.27a)–(2.27c) as can be seen by writing them as derivatives of $Z_{\text{free}}$. Observe that, according to Eqs. (2.27c) and (2.25c), the momentum flowing through a "+ −" line is always on the mass shell. Also observe that "+ −" lines never appear as external lines in the effective action, because they are eliminated by the Klein-Gordon operators which remove the propagators from these external legs.

To adopt the background field methods it is convenient to think of the pair $(\Phi^+,\Phi^-)$ as a field doublet in some internal space. In particular, for the one-loop effective action we will find

$$\Gamma[\hat{\phi}^+,\hat{\phi}^-] = S[\hat{\phi}^+] - S[\hat{\phi}^-] - \frac{i}{2}\ln\mathrm{Det}\begin{bmatrix} \Box + m^2 - i\epsilon + \frac{\lambda}{2}(\hat{\phi}^+)^2 & 0 \\ 0 & -\left[\Box + m^2 + i\epsilon + \frac{\lambda}{2}(\hat{\phi}^-)^2\right] \end{bmatrix}^{-1} . \tag{2.29}$$

In computing the inverse, however, it is important to keep in sight the right boundary conditions. $\hat{\phi}^+$ and $\hat{\phi}^-$ must satisfy the conditions stated following Eq. (2.23). As in the conventional approach infinities appear in the theory. The in-in effective action can be rendered finite by adding to the classical action the same set of counterterms as in the conventional formalism.[14] The crucial observation here is that a primitively divergent graph in the in-in effective action contains only vertices of the same sign. If there were vertices of different signs, at least two internal lines would be of "+ −" type (the graph is 1PI), the corresponding momenta would be on shell, the corresponding loop integral would be finite, and the graph would not be primitively divergent. Now the graphs of the in-in effective action with all vertices of the same sign are just the graphs of the in-out theory plus their complex conjugates, so the primitive divergences must be the same. Once the primitive divergences are controlled, it is only a matter of combinatorics to show that the overlapping divergences disappear as well.

An important particular case of the in-in vacuum formalism is that when either $J^+$ or $J^-$ does not induce any transition from the vacuum to $n$-particle states. For example, if $J^-$ does not create particles, the in vacuum evolves only by acquiring a phase, which is given by the vacuum persistence amplitude $Z[J]$ of Eq. (2.3). So we have, in this case only,

$$Z[J^+,J^-] = Z[J^-]^* Z[J^+] , \tag{2.30a}$$

$$W[J^+,J^-] = W[J^+] - W^*[J^-] , \tag{2.30b}$$

$$\Gamma[\hat{\phi}^+\hat{\phi}^-] = \Gamma[\hat{\phi}^+] - \Gamma^*[\hat{\phi}^-] . \tag{2.30c}$$

Here, functionals of a single argument refer to the conventional in-out approach. Equation (2.30c) shows that the conventional effective action may be considered as a particular case of the in-in one. For a massive $\lambda\phi^4$ theory, a constant source (and in particular, a zero source) does not cause transitions from the vacuum.

The extension of the in-in formalism to curved background geometry does not pose any new problem over and above those encountered in the conventional in-out approach.[15,19] Of course for a general geometry there will not be a preferred "time," but still one can define in and out states on Cauchy surfaces $\Sigma^{\pm}$ in the far past and future, and use an evolution operator of the form

$$U[\Sigma^+,\Sigma^-] = P\exp\left[i\int_V d^4x\sqrt{-g}\,J(x)\Phi_H(x)\right] , \tag{2.31}$$

where $P$ stands for path ordering and $V$ is the spacetime volume sandwiched in between these Cauchy surfaces. Surfaces of constant time now should be thought of as Cauchy surfaces, and all spacetime integrals should refer to the invariant measure $d^4x(-g)^{1/2}$. In particular, the same boundary conditions on the background (classical) fields hold, although now the definition of positive-frequency must be decided on a case-by-case basis in accordance with the appropriate normal-mode decomposition. As in the usual approach, the definition of the vacuum states depends on whether there exists positive-frequency modes in the asymptotic regions. The form of the propagators in (2.25) alters, but Eqs. (2.24), (2.26), and (2.27) still hold.

The graphs which express the perturbative expansion of the in-in functionals are the same as in flat-spacetime theory, although one loses the convenience of a globally defined momentum representation. One observes that for the few well-known field theories studied in curved space, those which are renormalizable in flat spacetime are also renormalizable in curved background, and that the counterterms for the in-in effective action are the same as those of the in-out formalism.

The semiclassical theory of gravity may be thought of as an approximation to a fully quantized theory.[2] The in-in effective action for quantum gravity is given by

$$\Gamma[g^+_{\mu\nu},\hat\phi^+,g^-_{\mu\nu},\hat\phi^-]=-i\ln\int Dh^+_{\mu\nu}Dh^-_{\mu\nu}D\phi^+D\phi^-D(\text{ghost fields})$$
$$\times\exp(i\{S[g^+_{\mu\nu}+h^+_{\mu\nu},\hat\phi^++\phi^+]-S[g^-_{\mu\nu}+h^-_{\mu\nu},\hat\phi^-+\phi^-]$$
$$+(\text{gauge-fixing, ghost, and tadpole terms})\})\ , \tag{2.32}$$

where $g^\pm_{\mu\nu}$ is the gravitational background field, $h^\pm_{\mu\nu}$ the fluctuation field, and $\hat\phi^\pm,\phi^\pm$ represent the matter background and fluctuation fields. The field equations for gravity have the form

$$\frac{\partial\Gamma}{\partial g^\pm_{\mu\nu}}=\mp J^{\mu\nu\pm}\ . \tag{2.33}$$

The physical gravitational field $g_{\mu\nu}$, in the presence of a source $J^{\mu\nu}$ and background matter fields $\hat\phi$, is obtained under the conditions $J^{\mu\nu+}=J^{\mu\nu-}=J^{\mu\nu}$, $\hat\phi^+=\hat\phi^-=\hat\phi$, and $g^+_{\mu\nu}=g^-_{\mu\nu}=g_{\mu\nu}$. Write $S[g_{\mu\nu},\hat\phi]=S_g[g_{\mu\nu}]+S_f[g_{\mu\nu},\hat\phi]$ where subscripts $g$ and $f$ denote gravitational and matter fields, respectively. The semiclassical approximation amounts to neglecting the fluctuations of the gravitational fields and ghosts in (2.32), whereby one obtains

$$\Gamma[g^+_{\mu\nu},\hat\phi^+,g^-_{\mu\nu},\hat\phi^-]=S_g[g^+_{\mu\nu}]-S_g[g^-_{\mu\nu}]+\Gamma_f[g^+_{\mu\nu},\hat\phi^+,g^-_{\mu\nu},\hat\phi^-]\ , \tag{2.34}$$

$$\Gamma_f[g^+_{\mu\nu},\hat\phi^+,g^-_{\mu\nu},\hat\phi^-]=(-i)\ln\int D\psi^+D\phi^-\exp\{i(S_f[g^+_{\mu\nu},\hat\phi^++\phi^+]-S_f[g^-_{\mu\nu},\hat\phi^-+\phi^-]+\text{tadpoles})\}\ . \tag{2.35}$$

The field equations become

$$\frac{\partial S_g}{\partial g^{\mu\nu}}[g_{\mu\nu}]=-J_{\mu\nu}-\frac{\partial\Gamma_f}{\partial g^{+\mu\nu}}[g_{\mu\nu},\hat\phi,g_{\mu\nu},\hat\phi]\ . \tag{2.36}$$

In (2.36), $g^+_{\mu\nu}$ and $g^-_{\mu\nu}$ must be identified only after the variation has been taken. Until that point, $g^-_{\mu\nu}$ is to be considered as just another matter field. This ensures that $\partial\Gamma_f/\partial g^+_{\mu\nu}$ is covariantly conserved, since $\Gamma_f-S_g[g^-_{\mu\nu}]$ is invariant under coordinate changes.

From discussions above, it is clear that $2(-g)^{1/2}\partial\Gamma_f/\partial g^{+\mu\nu}$ corresponds to the expectation value of the stress tensor of the matter fields, taken with respect to the state which evolves from the in vacuum under the influence of combined gravitational and matter fields $g_{\mu\nu}$ and $\hat\phi$.

In cases where there is a preferred notion of "time," such as in static spaces or conformally static spaces like the Robertson-Walker universes, it may be more convenient to forsake a fully covariant formulation and use a preferred coordinate system with these special properties to express the effective action. We will do this for the quantum cosmological problems to be studied.

For treating problems of statistical field theory in curved space, the closed-time-path functional formalism can be generalized by considering initial states as described by a density matrix.[11–14,19] We will consider problems of this nature in our later work.[20]

## III. ANISOTROPY DISSIPATION: REAL AND CAUSAL EFFECTIVE ACTION AND FIELD EQUATIONS

In this section we will consider the dynamics of a radiation-filled Bianchi type-I universe under the influence of a massless conformal quantum scalar field. We will derive the effective field equations governing the evolution of the metric with back reaction from particle production, using the semiclassical closed-time-path (in-in) effective action. We will assume that the reader is familiar with the conventional in-out approach to this problem as discussed in Ref. 5, whose notation we will adhere to here.

The Bianchi type-I metric has the form

$$ds^2=a^2(\eta)(d\eta^2-e^{2\beta_{ij}(\eta)}dx^idx^j)\ , \tag{3.1}$$

where $a$ is the scale factor and the anisotropy matrix $\beta_{ij}(\eta)$ is symmetric and traceless. $\eta=\int dt/a$ is the conformal time (a prime denotes $d/d\eta$). The gravitational action is

$$S_g=\int d^4x(-g)^{1/2}(\Lambda_B+\kappa_BR+\alpha_{1B}R_{\mu\nu\rho\sigma}$$
$$+\alpha_{2B}R_{\mu\nu}R^{\mu\nu}+\alpha_{3B}R^2)\ , \tag{3.2}$$

where $\kappa=(16\pi G)^{-1}$ and we have added the generic $R^2$ terms in (3.2) to cancel the anticipated ultraviolet divergences. In conjunction with dimensional regularization we will use an $n$-dimensional form of the action (the form of $R^2,R_{\mu\nu}R^{\mu\nu},R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ can be found in Appendix B of Ref. 5). The scalar field action is given by

$$S_f=\tfrac{1}{2}\int d^nx(-g)^{1/2}\left[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi-\frac{n-2}{4(n-1)}R\Phi^2\right]\ . \tag{3.3}$$

For a free field theory the action is one-loop exact. Only the gravitational coupling constants need be renormalized. Their counterterms[4–6,21] are given by

$$\delta\Lambda=0,\ \ \delta\kappa=0,\ \ \delta\alpha_1=-\delta\alpha_2=\frac{1}{180(4\pi)^2\epsilon},\ \ \delta\alpha_3=0\ , \tag{3.4a}$$

where $\epsilon=n-4$. The bare quantities $\alpha_{1B},\alpha_{2B}$ take the form

$$\alpha_{1,2B}=\mu^{\epsilon}\{\alpha_{1,2}\pm[180(4\pi)^2\epsilon]^{-1}\}\ , \tag{3.4b}$$

where $\mu$ is a renormalization constant with units of mass. For simplicity one assumes that one can choose a renormalization point at which the $\mu$-independent $\Lambda$ and the $\alpha_i$ are zero. We will assume that the anisotropy is small and consider perturbation expansion of the effective action and field equations up to quadratic order in $\beta$ ($\beta^2=\beta_{ij}\beta^{ij}$ measures the shear).

To zeroth order in $\beta$ our problem reduces to that of a spatially flat radiation-filled Robertson-Walker (RW) universe with a conformal quantum field. In this case the integral over fields does not contribute to the equations for the conformal factor. The terms containing the poles in $\alpha_{1,2B}$ are total $\eta$ derivatives and may be discarded. The gravitational action of the RW universe with one-loop quantum correction is given by[4]

$$\Gamma[g^+]=\int_{-\infty}^{\infty}d\eta\left\{-6\kappa a'^2+[180(4\pi)^2]^{-1}\left[\left[\frac{a'}{a}\right]^4-3\left[\frac{a''}{a}\right]^2\right]\right\}. \tag{3.5}$$

The variation of $\Gamma$ with respect to $a$ yields Einstein's equation with a quantum source given by the trace anomaly of the scalar field. This equation being of higher order, it admits many solutions, amongst which are the conformally complete ones.[5] For $\beta\neq 0$ as a result of particle production the conformal vacuum is no longer stable, $|0_-\rangle\neq|0_+\rangle$. We must then distinguish two anisotropy fields, $\beta_{ij}^-(\eta)$ and $\beta_{ij}^+(\eta)$. However, only the equations which follow from the variation of $\Gamma$ with respect to $\beta_{ij}^+$ need be considered [cf. the discussion following Eq. (2.19)]. Now terms like $-S_g[\beta_{ij}^-]$ which yields no contribution when varied with respect to $\beta_{ij}^+$ may be dropped. In the same way, we will omit in what follows terms which do not contribute to the variation with respect to $\beta_{ij}^+$. The gravitational action, expanded to second order in $\beta$ has the form[5]

$$S_g=\int d\eta\,\mathrm{Tr}(\kappa a^2\beta'^2+[180(4\pi)^2]^{-1}\{3\epsilon^{-1}\beta''^2+3\ln(\mu a)\beta''^2-[(a''/a)\beta'^2+(a'/a)^2\beta'^2-\beta''^2]\})+O(\epsilon)\ . \tag{3.6}$$

The scalar field action can be written as $S_f=S_f^{(0)}+S_f^{(1)}+S_f^{(2)}+O(\beta^3)$, where $S_f^{(0)}$ is given by (3.3) with the isotropic form of $g^{\mu\nu}$, and

$$S_f^{(1)}=-\int d^nx a^{n-2}\beta^{ij}\partial_i\Phi\partial_j\Phi\ , \tag{3.7}$$

$$S_f^{(2)}=-\int d^nx a^{n-2}\left[\frac{(n-2)}{8(n-1)}\beta'_{ij}\beta'^{ij}\Phi^2+\beta^{ik}\beta_k{}^j\partial_i\Phi\partial_j\Phi\right]. \tag{3.8}$$

The scalar field contribution is, from (2.35),

$$\begin{aligned}\Gamma_f[\beta^+,\beta^-]&=-i\ln\int D\phi^+D\phi^-\exp\{i(S_f[\beta^+,\phi^+]-S_f[\beta^-,\phi^-])\}\\&=-i\ln\int D\phi^+D\phi^-\exp\{i(S^{(0)}[\phi^+]-S^{(0)}[\phi^-])\}\\&\quad\times\{1+iS^{(1)}[\beta^+,\phi^+]-iS^{(1)}[\beta^-,\phi^-]+iS^{(2)}[\beta^+,\phi^+]-iS^{(2)}[\beta^-,\phi^-]\\&\quad-\tfrac{1}{2}(S^{(1)}[\beta^+,\phi^+])^2-\tfrac{1}{2}(S^{(1)}[\beta^-,\phi^-])^2+S^{(1)}[\beta^+,\phi^+]S^{(1)}[\beta^-,\phi^-]+O(\beta^3)\}\ .\end{aligned} \tag{3.9}$$

To compute the functional integrals with respect to the isotropic action and take its logarithm is equivalent to computing each term as a sum of Feynman graphs and retaining only the connected ones. The two-point functions in the RW space are given by

$$\langle\phi^+(x)\phi^+(x')\rangle=i[a(\eta)a(\eta')]^{1-n/2}\Delta_F(x-x')\ , \tag{3.10a}$$

$$\langle\phi^-(x)\phi^-(x')\rangle=-i[a(\eta)a(\eta')]^{1-n/2}\Delta_D(x-x')\ , \tag{3.10b}$$

$$\langle\phi^+(x)\phi^-(x')\rangle=-i[a(\eta)a(\eta')]^{1-n/2}\Delta^+(x-x')\ , \tag{3.10c}$$

where $\Delta_{F,D}$ and $\Delta^+$ are given by Eqs. (2.25a)–(2.25c) with $m^2=0$. The terms in the in-in effective action which contribute to the variation with respect to $\beta^+$ are

$$\Gamma_f[\beta^+,\beta^-]=\frac{i}{2}\langle(S^{(1)}[\beta^+,\phi^+])^2\rangle-i\langle S^{(1)}[\beta^+,\phi^+]S^{(1)}[\beta^-,\phi^-]\rangle\ . \tag{3.11}$$

One can now begin to notice the difference from the conventional approach, i.e., the second term in (3.11) is new. We now use (3.7), Wick's theorem, Eqs. (3.10) and (2.25), to get

$$\frac{i}{2}\langle (S^{(1)}[\beta^+,\phi^+])^2\rangle = -i\int\int d^nx\, d^nx'\beta^+_{ij}(\eta)\beta^+_{kl}(\eta')\int\frac{d^np}{(2\pi)^n}e^{ip(x-x')}\times\int\frac{d^nq}{(2\pi)^n}q^iq^j(p-q)^k(p-q)^l(q^2+i\varepsilon)^{-1}[(p-q)^2+i\varepsilon]^{-1}\,, \tag{3.12a}$$

$$-i\langle S^{(1)}[\beta^+,\phi^+]S^{(1)}[\beta^-,\phi^-]\rangle = 2i\int\int d^nx\, d^nx'\beta^+_{ij}(\eta)\beta^-_{kl}(\eta')\times\int\frac{d^np}{(2\pi)^n}e^{ip(x-x')}\int\frac{d^nq}{(2\pi)^n}q^iq^j(p-q)^k(p-q)^l[2\pi i\delta(q^2)\theta(q^0)]\times\{2\pi i\delta[(p-q)^2]\theta(p^0-q^0)\}\,. \tag{3.12b}$$

Since the anisotropy depends only on $\eta$ we may integrate over the space variables to get (setting $V=1$)

$$\frac{i}{2}\langle (S^{(1)}[\beta^+,\phi^+])^2\rangle = \int\int d\eta\, d\eta'\beta^{+ij}(\eta)\beta^{+kl}(h')\int\frac{d\omega}{2\pi}e^{i\omega(\eta-\eta')}\times\left[(-i)\int\frac{d^nq}{(2\pi)^n}q_iq_jq_kq_l(q^2+i\varepsilon)^{-1}[(\omega-q^0)^2-\mathbf{q}^2+i\varepsilon]^{-1}\right]\,, \tag{3.13a}$$

$$-i\langle S^{(1)}[\beta^+,\phi^+]S^{(1)}[\beta^-,\phi^-]\rangle = 2\int d\eta\, d\eta'\beta^{+ij}(\eta)\beta^{-kl}(\eta')\int\frac{d\omega}{2\pi}e^{i\omega(\eta-\eta')}\times\left[i\int\frac{d^nq}{(2\pi)^n}q_iq_jq_kq_l[2\pi i\delta(q^2)\theta(q^0)]\times\{2\pi i\delta[(\omega-q^0)^2-\mathbf{q}^2]\theta(\omega-q^0)\}\right]\,. \tag{3.13b}$$

Observe that if the argument of both $\delta$ functions must vanish in Eq. (3.13b), then $\omega=+2q^0$. Since $q^0$ must be positive, we may write

$$-i\langle S^{(1)}[\beta^+,\phi^+]S^{(1)}[\beta^-,\phi^-]\rangle = 2\int\int d\eta\, d\eta'\beta^+_{ij}(\eta)\beta^-_{kl}(\eta')\int\frac{d\omega}{2\pi}e^{i\omega(\eta-\eta')}\theta(\omega)\times\left[i\int\frac{d^nq}{(2\pi)^n}q^iq^jq^kq^l[2\pi i\delta(q^2)\theta(q^0)]\times\{2\pi i\delta[(\omega-q^0)^2-\mathbf{q}^2]\}\right]\,. \tag{3.14}$$

One could compute (3.14) by force, but it is perhaps more suggestive to note that the $q$ integral in Eq. (3.14) (including the $i$ factor) is simply twice the imaginary part of the $p$ integral in Eq. (3.13a), computed according to the Cutkowsky rules. Now the " $+$ $+$ graph" in Eq. (3.13a) is also the scalar field contribution to the in-out effective action (see Ref. 5). It can be written as

$$\frac{i}{2}\langle (S^{(1)}[\beta^+,\phi^+])^2\rangle = \int\int d\eta\, d\eta'\beta^+_{ij}(\eta)\beta^{+ij}(\eta')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega(\eta-\eta')}\left[\frac{(-1)\omega^4}{4(4\pi)^2(n^2-1)}\left[\epsilon^{-1}+\tfrac{1}{2}\ln\frac{-(\omega^2+i\varepsilon)}{4\pi\mu^2}\right]\right]\,, \tag{3.15}$$

from which it follows that

$$-i\langle S^{(1)}[\beta^+,\phi^+]S^{(1)}[\beta^-,\phi^-]\rangle = 2\int\int d\eta\, d\eta'\beta^+_{ij}(\eta)\beta^{-ij}(\eta)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega(\eta-\eta')}\left[\frac{(-1)\omega^4}{4(4\pi)^2(n^2-1)}i\pi\theta(\omega)\right]\,. \tag{3.16}$$

Observe that only the " $+$ $+$ " term is divergent; therefore, the pole term is the same as in the in-out approach. It is convenient to introduce the notation

$$R_n(\eta) = \int\frac{d\omega}{2\pi}e^{i\omega\eta}\frac{\omega^n}{2}\left[\ln\left[\frac{-(\omega^2+i\varepsilon)}{4\pi\mu^2}\right]\right]\,, \tag{3.17a}$$

$$I_n(\eta) = \int\frac{d\omega}{2\pi}e^{i\omega\eta}\omega^n\pi\theta(\omega)\,. \tag{3.17b}$$

Comparing (3.6) and (3.15) we observe that the pole parts cancel, leaving a finite residue. Finally the equation for $\beta_{ij}$ is (we redefine $\mu$ to absorb inessential constants)

$$\frac{\partial}{\partial\beta_{ij}^{+}}(S_g+\Gamma_f)\,|_{\beta^{+}=\beta^{-}=\beta}=-2\kappa\frac{d}{d\eta}(a^2\beta_{ij}')+\frac{1}{30(4\pi)^2}\frac{d^2}{d\eta^2}[\beta_{ij}''\ln(\mu a)]+\frac{1}{90(4\pi)^2}\frac{d}{d\eta}\left\{\left[\left[\frac{a'}{a}\right]^2+\frac{a''}{a}\right]\beta_{ij}'\right\}$$
$$-\frac{1}{30(4\pi)^2}\int_{-\infty}^{\infty}d\eta'\beta_{ij}(\eta')R_4(\eta-\eta')-\frac{1}{30(4\pi)^2}\int_{-\infty}^{\infty}d\eta'\beta_{ij}(\eta')iI_4(\eta-\eta')=-J_{ij}(\eta)\ . \tag{3.18}$$

Here we have introduced an external source $J_{ij}$ in order to be able to switch on the anisotropy in the far past.

The difference between (3.18) and the equations deduced from the conventional approach[5,6] lies exclusively in the term containing the $I$ kernel. When this term is omitted, the resulting equation is neither real nor causal. This becomes obvious if we write

$$\frac{1}{2}\ln\left[\frac{-(\omega^2+i\varepsilon)}{4\pi\mu^2}\right]=\ln\frac{|\omega|}{4\pi\mu^2}-i\frac{\pi}{2}\ ,$$
$$R_4(\eta)=\left[\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^4(\cos\omega\eta)\ln\frac{|\omega|}{4\pi\mu}\right]-i\frac{\pi}{2}\frac{d^4}{d\eta^4}\delta(\eta)\ . \tag{3.19}$$

(Here and hereafter we choose the cut of the logarithm along the negative real axis.) The nonlocal part of (3.18) is given by the kernel

$$K_n(\eta)=(R_n+iI_n)(\eta)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega\eta}\omega^n\left[\ln\frac{|\omega|}{4\pi\mu}+\frac{i\pi}{2}\mathrm{sgn}(\omega)\right]$$
$$=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega\eta}\omega^n\left[\ln\frac{i(\omega-i\varepsilon)}{4\pi\mu}\right]\ . \tag{3.20}$$

To obtain the second identity in (3.18), observe that

$$\frac{\partial}{\partial\omega}\left[\ln|\omega|+i\frac{\pi}{2}\mathrm{sgn}\omega\right]=\omega^{-1}+i\pi\delta(\omega)=(\omega-i\varepsilon)^{-1}\ .$$

It is clear that for even $n$, $K(\eta)$ is real, because the real part of its Fourier transform is even and its imaginary part is odd. It is causal, because all the singularities of the integral lie in the upper complex half plane. We observe that a causal and nonlocal equation could not have been derived from an action functional depending solely on $\beta^{+}$ as in the in-out approach. In fact one can eliminate the variable $\omega$ completely. Consider the sequence of functions

$$F_n(\eta)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{(\sigma+i\omega)\eta}(\sigma+i\omega)^n\ln(\sigma+i\omega)\ , \tag{3.21}$$

then

$$F_n(\eta)=\begin{cases}n!\eta^{-1-n}\theta(\eta)\ \ (n\geq 0)\ ,\\ \dfrac{(-1)^n}{(-n)!}\eta^{-n-1}[\psi(-n+1)-\ln\eta]\theta(\eta)\ \ (n<0)\ ,\end{cases} \tag{3.22}$$

Here, $\psi(z)=(d/dz)\ln\Gamma(z)$. In principle one should use the formula for $n=4$, but the resulting $\eta'$ integral would not be well defined (this implies incorrectly applying the definition of the Fourier transform of a distribution). We will obtain a well-defined expression if we write

$$\int_{-\infty}^{\infty}d\eta'\beta_{ij}(\eta')K_4(\eta-\eta')=\int_{-\infty}^{\infty}d\eta'\beta_{ij}(\eta')\left[-\frac{\partial^5}{\partial\eta'^5}F_{-1}(\eta-\eta')\right]$$
$$=\int_{-\infty}^{\eta}d\eta'\left[\frac{\partial^5}{\partial\eta'^5}\beta_{ij}(\eta')\right][\ln(\eta-\eta')-\psi(2)]\ . \tag{3.23}$$

The term containing $\psi(2)$ is really a local term and can be absorbed in a redefinition of the unit of mass $\mu$ in Eq. (3.18). In order to find the actual evolution of $\beta_{ij}$ it is convenient[5–7] to integrate both sides of (3.18) once with respect to $\eta$. The resulting equation is

$$-2Ka^2\beta'_{ij}+\frac{1}{30(4\pi)^2}\frac{d}{d\eta}\{[\ln(\mu a)]\beta''_{ij}\}+\frac{1}{90(4\pi)^2}\left[\left[\frac{a'}{a}\right]^2+\left[\frac{a''}{a}\right]\right]\beta'_{ij}$$
$$-\frac{1}{30(4\pi)^2}\int_{-\infty}^{\eta}d\eta'\left[\frac{d^4}{d\eta'^4}\beta_{ij}\right]\ln(\eta-\eta')=-\int_{-\infty}^{\eta}d\eta' J_{ij}(\eta')=-c_{ij}(\eta') \ . \quad (3.24)$$

If the source $J_{ij}$ operates only in the distant past, we may take $c_{ij}$ to be a constant for all finite values of $\eta$. Although Eq. (3.24) is different in form from those obtained from the in-out approach,[5–7] its solution appears to be similar to the real part of the solution from the latter, as given in Refs. 5, 6, and 7. In particular, Eq. (3.24) admits the conformally complete solution with $\beta'$ going to a constant as $\eta\to-\infty$, while approaching the classical behavior $\beta'_{ij}\sim c_{ij}\eta^{-2}$ as $\eta\to+\infty$. This is the case because the behavior of $\beta_{ij}$ is dominated by the local terms in both limits.

Since our primary aim is to establish the in-in formalism and to find out how much qualitatively new physical information it contains, we will not pursue the details of the solutions. To this end it is convenient to leave the nonlocal part of Eq. (3.18) written in the frequency domain as in Eq. (3.20). The in-in effective action being real and causal (in time domain) yields equations whose imaginary terms (in frequency domain) can be unambiguously identified with dissipative processes. The kernel $K_4$ acquires (in frequency domain) an imaginary part because (and only because) the conformal vacuum is unstable in the presence of anisotropy. It is in this sense that the dissipative nature of particle creation can be made more precise in the semiclassical context.[22]

We may obtain a quantitative check on the relationship between the imaginary part of $K_4$ and particle creation if we consider the energy density of the fields, $a^4T_0^0$. This can be found from the conservation law

$$(a^4T_0^0)'=-a'\frac{\partial}{\partial a}\Gamma_f-\beta'_{ij}\frac{\partial}{\partial\beta^+_{ij}}\Gamma_f \ .$$

The contribution of the nonlocal terms in (3.16) to the energy density is

$$\frac{1}{30(4\pi)^2}\int_{-\infty}^{\eta}d\eta'\beta'_{ij}(\eta')\int_{-\infty}^{\infty}d\eta''\beta^{ij}(\eta'')K_4(\eta'-\eta'') \ . \quad (3.25)$$

As $\eta\to\infty$ we may write this in terms of $\beta_{ij}(\omega)=\int d\eta\, e^{-i\omega\eta}\beta_{ij}(\eta)$, i.e.,

$$\frac{1}{30(4\pi)^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^4\left[\ln\frac{|\omega|}{4\pi\mu}+\frac{i\pi}{2}\mathrm{sgn}(\omega)\right][\mathrm{Tr}\beta^*(\omega)\beta(\omega)](-i\omega) \ . \quad (3.26)$$

We see that in the limit $\eta\to\infty$, the contribution of the real part cancels off, while the imaginary part gives a positive definite contribution

$$\rho_p=\frac{1}{120(4\pi)^2}\int_0^{\infty}d\omega(2\omega)[\omega^4\mathrm{Tr}\beta^*(\omega)\beta(\omega)] \ . \quad (3.27)$$

The spectrum of particle pairs created by a given anisotropy history $\beta_{ij}(\eta)$ is[6]

$$P(\omega)=\frac{1}{30(\pi)^2}[\omega^4\mathrm{Tr}\beta^*(2\omega)\beta(2\omega)] \ . \quad (3.28)$$

This shows that the imaginary part $\rho_p$ responsible for particle production indeed gives the total energy of particles created in the whole history

$$\rho_p=\int_0^{\infty}d\omega(2\omega)P(\omega) \ . \quad (3.29)$$

Although Eq. (3.27) for the energy of created particles has the same form as in the in-out formulation, the actual numerical result may differ. As the in-in equation for $\beta_{ij}$ is different from the in-out equation, the evolution of $\beta_{ij}$ will be different, and so will the numerical value of $\rho_p$. The difference could arise from the specification of the vacuum: we must give not only the state of the matter field but also that of the gravitational field. The gravitational out vacuum need not be the same as the gravitational in vacuum, as the background is affected by the quantum matter fields present. Therefore, it is not clear that a particle count from comparing the in and out vacua should agree with that of comparing the in vacuum with itself. The fact that the numbers we get are close enough (because the large time behavior of $\beta_{ij}$ is the same in both formulations) shows that the approximation of having neglected the quantum nature of the gravitational field is consistent within the given accuracy.

To convince ourselves that the particles in (3.28) are really the created ones, it is enough to see that as $\eta\to+\infty$ all the vacuum-polarization terms fade away and $a^4T_0^0$ reduces to $\rho_p$+const. In other words, the newly created particles weigh on the same footing as any classical radiation originally present. Moreover, the $K_4$ is the only nonlocal kernel leading to a real causal equation with the right amount of particle production. This is so because reality and Eq. (3.25) determine the imaginary part of $K_4$, and causality implies a Kramers-Kronig relation which then fixes the real part up to local terms.

We conclude that the in-in formulation of the anisotropy dissipation problem yields quantitatively similar results as the usual in-out formulation, with the additional advantage that (a) the field equations are real and causal, so that the solutions can be identified with dynamics of physical fields and geometry, (b) the formalism provides an unambiguous description of the dissipative nature of cosmological particle-production processes.

## IV. INTERACTING FIELDS: VACUUM EXPECTATION VALUE OF THE STRESS-ENERGY TENSOR

In this section we present a calculation of the in-in effective action for self-interacting quantum fields. This example will serve to illustrate higher loop calculations, renormalization of the in-in effective action, and calculation of vacuum expectation value of the stress-energy tensor. Consider a nonconformal $\lambda\phi^4$ theory on an isotropic, spatially flat Robertson-Walker background. The metric is that of Eq. (3.1) with $\beta=0$. The gravitational action is given by (3.2). The scalar field action in $n$ dimensions is

$$S_f=\int d^nx a^n(\eta)\left\{\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left[m^2+\left[\xi+\frac{1}{12}\frac{(n-4)}{(n-1)}\right]R\right]\Phi^2-\frac{\lambda}{4!}\Phi^4\right\}, \tag{4.1}$$

where $R$ is the scalar curvature and $\xi=0,\frac{1}{6}$ denote minimal and conformal coupling, respectively. We assume that in the infinite past $m^2$, $\xi-\frac{1}{6}$, and $\lambda$ are adiabatically switched off. This defines the in vacuum to be the conformal vacuum and also defines a particular model, which has been explicitly constructed in Refs. 7 and 16. We will make a perturbative expansion of the in-in effective action in powers of these parameters, specifically to first order in $\lambda$ and to second order in $M^2=m^2+(\xi-\frac{1}{6})R$. Therefore we will consider graphs with no more than two loops. We will need to know the two-loop counter terms in $S_g^{\rm ct}$, and the one-loop counterterms in $S_f^{\rm ct}$ (we do not include a background scalar field). These have been computed in Ref. 21. To this order there is neither wave function nor coupling-constant renormalization. As $\lambda$ and $M$ are adiabatically switched on, conformal invariance is broken and the vacuum becomes unstable. We must now consider the full CTP effective action as a function of two variables $a^+$, $a^-$:

$$\Gamma[a^+,a^-]=S_g[a^+]-S_g[a^-]-i\ln\int D\phi^+D\phi^-\exp\{i[(S_f+S_f^{\rm ct})(\phi^+,a^+)-(S_f+S_f^{\rm ct})(\phi^-,a^-)]\}\ . \tag{4.2}$$

Introduce a change of variables $\phi^\pm=(a^\pm)^{1-n/2}\chi^\pm$. The Jacobian of it is simply the path integral in the free conformally invariant case, which we know is a constant. After this transformation, we get

$$\Gamma[a^+,a^-]=S_g[a^+]-S_g[a^-]-i\ln\int D\chi^+D\chi^-\exp\{i[(S_f+S_f^{\rm ct})(\chi^+,a^+)-(S_f+S_f^{\rm ct})(\chi^-,a^-)]\}\ , \tag{4.3}$$

where

$$S_f(\chi,a)=\int d^n\chi\left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\chi\partial_\nu\chi-\frac{a^2}{2}M^2\chi^2-\frac{\lambda a^{-\epsilon}}{4!}\chi^4\right], \tag{4.4}$$

$$S_f^{\rm ct}(\chi,a)=\int d^4x\left[\frac{1}{2}a^2\frac{\lambda M^2}{(4\pi)^2\epsilon}\chi^2+O(\lambda^2)\right]. \tag{4.5}$$

Here again, $\epsilon=n-4$ and $M^2=m^2+(\xi-\frac{1}{6})R$, where $R$ is the $n$-dimensional scalar curvature. We have effectively reduced our problem to that of a flat-spacetime theory with position-dependent (when $n\neq4$) interactions. Expanding up to the desired orders we find (keeping only terms which contain $a^+$)

$$\Gamma=S_g+\Delta\Gamma\ ,$$

$$\begin{aligned}
\Delta\Gamma=\frac{i}{2}\Bigg[\frac{1}{4}\int d^4xd^4x'\Bigg[&(a^2M^2)^+(x)(a^2M^2)^+(x')\langle\chi^{+2}(x)\chi^{+2}(x')\rangle_c\\
&-\tfrac{1}{2}(a^2M^2)^+(x)(a^2M^2)^+(x')\langle\chi^{+2}(x)\chi^{+2}(x')\rangle_c\\
&+\tfrac{1}{2}(a^2M^2)^+(x)\frac{\lambda}{(4\pi)^2\epsilon}(a^2M^2)^+(x')\langle\chi^{+2}(x)\chi^{+2}(x')\rangle_c\\
&-(a^2M^2)(x)\frac{\lambda}{(4\pi)^2\epsilon}(a^2M^2)^-(x')\langle\chi^{+2}(x)\chi^{-2}(x')\rangle_c\Bigg]\\
-\frac{\lambda}{6}\int d^4x\,d^4x'd^4x''\Bigg[&\frac{-3}{4\times4!}(a^2M^2)^+(x)(a^2M^2)^+(x')(a^{-\epsilon})^+(x'')\langle\chi^{+2}(x)\chi^{+2}(x')\chi^{+4}(x'')\rangle_c\\
&+\frac{3}{4\times4!}(a^2M^2)^+(x)(a^2M^2)^+(x')(a^{-\epsilon})^-(x'')\langle\chi^{+2}(x)\chi^{+2}(x')\chi^{-4}(x'')\rangle_c\\
&+\frac{6}{4\times4!}(a^2M^2)^+(x)(a^2M^2)^-(x')(a^{-\epsilon})^+(x')\langle\chi^{+2}(x)\chi^{-2}(x')\chi^{+4}(x'')\rangle_c\\
&-\frac{6}{4\times4!}(a^2M^2)^+(x)(a^2M^2)^-(x')(a^{-\epsilon})^-(x'')\langle\chi^{+2}(x)\chi^{-2}(x')\chi^{-4}(x'')\rangle_c\Bigg]\Bigg]\ .
\end{aligned} \tag{4.6}$$

We observe that the graph $\langle\chi^{-2}(x)\chi^{-2}(x')\chi^{+4}(x'')\rangle_c$ does not contribute because the corresponding integral is finite.

Introducing the Feynman rules [Eq. (2.25)] and the kernels $R_0, I_0$ from Eqs. (3.17), we get

$$\begin{aligned}\Delta\Gamma=&\frac{-1}{2(4\pi)^2}\int\int d\eta\, d\eta'(a^2M^2)^+(\eta)\left[(a^2M^2)^+(\eta')\left[\frac{\delta(\eta-\eta')}{\epsilon}+R_0(\eta-\eta')\right]+2(a^2M^2)^-(\eta')iI_0(\eta-\eta')\right]\\&+\frac{\lambda}{(4\pi)^2\epsilon}\int\int d\eta\, d\eta'(a^2M^2)^+(\eta)\left[(a^2M^2)^+(\eta')\left[\frac{\delta(\eta-\eta')}{\epsilon}+R_0(\eta-\eta')\right]+2(a^2M^2)^-(\eta')iI_0(\eta-\eta')\right]\\&-\frac{\lambda}{2(4\pi)^4}\int\int\int d\eta\, d\eta'\, d\eta''(a^2M^2)^+(\eta)\left[(a^2M^2)^+(\eta')(a^{-\epsilon})^+(\eta'')\right.\\&\qquad\times\left[\frac{\delta(\eta-\eta'')}{\epsilon}+R_0(\eta-\eta'')\right]\left[\frac{\delta(\eta'-\eta'')}{\epsilon}+R_0(\eta'-\eta'')\right]\\&\qquad+(a^2M^2)^+(\eta')I_0(\eta-\eta'')I_0(\eta'-\eta'')\\&\qquad+2i\,(a^2M^2)^-(\eta')(a^{-\epsilon})^+(\eta'')\left[\frac{\delta(\eta-\eta'')}{\epsilon}+R_0(\eta-\eta'')\right]I_0(\eta''-\eta')\\&\qquad\left.+2i\,(a^2M^2)^-(\eta')(a^{-\epsilon})^-(\eta'')\left[\frac{\delta(\eta'-\eta'')}{\epsilon}+R_0^*(\eta'-\eta'')\right]I_0(\eta'-\eta'')\right].\end{aligned}$$

Finally, after a straightforward calculation we obtain

$$\begin{aligned}\Delta\Gamma=&\frac{-1}{2(4\pi)^2\epsilon}\int d\eta\, a^{+4}M^{+4}\\&-\frac{1}{2(4\pi)^2}\int\int d\eta\, d\eta'(a^2M^2)^+(\eta)[(a^2M^2)^+(\eta')R_0(\eta-\eta')+2(a^2M^2)^-(\eta')iI_0(\eta-\eta')]\\&+\frac{\lambda}{2(4\pi)^4}\int d\eta\, a^{+4}M^{+4}[\epsilon^{-2}+\epsilon^{-1}\ln a^+-\tfrac{1}{2}(\ln a^+)^2]\\&+\frac{\lambda}{(4\pi)^4}\int\int d\eta\, d\eta'(a^2M^2)^+(\eta')[(a^2M^2)^+(\eta')\ln a^+(\eta)R_0(\eta-\eta')+(a^2M^2)^-(\eta')\ln a^+(\eta)iI_0(\eta-\eta')\\&\qquad+(a^2M^2)^-(\eta')\ln a^-(\eta')iI_0(\eta-\eta')]\\&-\frac{\lambda}{2(4\pi)^4}\int\int d\eta\, d\eta'(a^2M^2)^+(\eta)\{(a^2M^2)^+(\eta')\int d\eta''[R_0(\eta-\eta'')R_0(\eta''-\eta')+I_0(\eta-\eta'')I_0^*(\eta''-\eta')]\\&\qquad+2(a^2M^2)^-(\eta')\int d\eta''[R(\eta-\eta'')iI_0(\eta''-\eta')+R_0^*(\eta''-\eta)iI_0(\eta-\eta'')]\}\,.\end{aligned}\tag{4.7}$$

Up to first order in $\lambda$ and second order in $M^2$ we have

$$S_g=\int d\eta\, a^n\left[\kappa R+\frac{1}{2(4\pi)^2\epsilon}M^4-\frac{\lambda}{2(4\pi)^4\epsilon^2}M^4+\frac{1}{180(4\pi)^2\epsilon}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-R_{\mu\nu}R^{\mu\nu})\right],$$

where as before we are assuming $\Lambda=\alpha_1=\alpha_2=\alpha_3=0, \mu=1$. The full effective action is therefore $\Gamma=S_E+\Gamma_f$, where $S_E$ is Einstein's action, and

$$\Gamma_f=\frac{1}{180(4\pi)^2}\int d\eta\left[\left[\frac{a}{a}\right]^4-3\left[\frac{a''}{a}\right]^2+90a^4M^4\left[\ln a-\frac{\lambda}{2}(\ln a)^2\right]\right]+\text{nonlocal terms}\,,\tag{4.8}$$

the nonlocal terms being the same as in Eq. (4.7).

We see that overlapping divergences have canceled out, as expected. To better appreciate the content of Eq. (4.8), let us compute the trace of the energy-momentum tensor

$$T=-a^{-3}\frac{\partial}{\partial a^+}\Gamma_f[a^+,a^-]\,|_{a^+=a^-=a}\,.$$

The variation is to be taken with $a^-$ held constant. We get, after identifying $a^+$ and $a^-$,

$$
\begin{aligned}
T(\eta)=\text{conformal anomaly}&-\frac{1}{2(4\pi)^2a^2}\int d\eta'\left[\frac{\partial}{\partial a(\eta)}(a^4M^4\ln a)(\eta')\right]\\
&+\frac{\lambda}{4(4\pi)^4a^2}\int d\eta'\left[\frac{\partial}{\partial a(\eta)}[a^4M^4(\ln a)^2](\eta')\right]\\
&+\frac{1}{(4\pi)^2a^3}\int d\eta'\left[\frac{\partial}{\partial a(\eta)}(a^2M^2)(\eta')\right]\int d\eta''(a^2M^2)(\eta'')K_0(\eta'-\eta'')\\
&-\frac{\lambda}{(4\pi)^4a^3}\int d\eta'\left[\frac{\partial}{\partial a(\eta)}(a^2M^2)(\eta')\right]\int d\eta''(a^2M^2)(\eta'')[\ln a(\eta)a(\eta')]K_0(\eta'-\eta'')\\
&-\frac{\lambda}{(4\pi)^4a^3}\int d\eta'\left[\frac{\partial}{\partial a(\eta)}\ln a(\eta')\right](a^2M^2)(\eta')\int d\eta''(a^2M^2)(\eta'')K_0(\eta'-\eta'')\\
&+\frac{\lambda}{(4\pi)^4a^3}\int d\eta'\left[\frac{\partial}{\partial a(\eta)}(a^2M^2)(\eta')\right]\int d\eta''(a^2M^2)(\eta'')\int d\eta'''K_0(\eta'-\eta''')K_0(\eta'''-\eta'')\ . \qquad (4.9)
\end{aligned}
$$

$K_0$ is the kernel defined in Eq. (3.18). We observe that $T$ is real and depends causally on the evolution of the conformal factor.

In the massless, free field case ($m=\lambda=0$), the nonlocal term in (4.9) reduces to the one computed by Davies and Unruh[16] (the local terms need not be identical since they depend upon the renormalization prescription). This reconfirms that (4.9) represents the in-in expectation value in the state which reduces to the conformal vacuum in the conformal limit. The contribution of the nonlocal terms to the energy is, in the limit $\eta\to\infty$,

$$
\begin{aligned}
\Delta(a^4T_0^0)&=\int_{-\infty}^{\infty}d\eta\, a'a^3T(\eta)\\
&=\frac{1}{(4\pi)^2}\int d\eta\left[\frac{d}{d\eta}(a^2M^2)(\eta)\right]\int d\eta'(a^2M^2)(\eta')K_0(\eta-\eta')\\
&+\frac{\lambda}{(4\pi)^4}\int d\eta\left[\frac{d}{d\eta}(a^2M^2)(\eta)\right]\int d\eta'(a^2M^2)(\eta')\int d\eta''K_0(\eta-\eta'')K_0(\eta''-\eta')\\
&-\frac{\lambda}{(4\pi)^4}\int d\eta\left[\frac{d}{d\eta}(a^2M^2)(\eta)\right]\int d\eta'(a^2M^2\ln a)(\eta')K_0(\eta-\eta')\\
&-\frac{\lambda}{(4\pi)^4}\int d\eta\left[\frac{d}{d\eta}(a^2M^2\ln a)(\eta)\right]\int d\eta'(a^2M^2)(\eta')K_0(\eta-\eta')\ .
\end{aligned}
$$

Introducing the Fourier transforms of $K_0$, $a^2M^2$, and $a^2M^2\ln a$, we obtain

$$
\Delta(a^4T_0^0)=\frac{1}{4(4\pi)^2}\int_0^{\infty}d\omega(2\omega)\left[\left[1+\frac{\lambda}{(4\pi)^2}\ln\frac{|\omega|}{4\pi\mu}\right]|a^2M^2(\omega)|^2-2\frac{\lambda}{(4\pi)^2}\mathrm{Re}\{(a^2M^2\ln a)(\omega)[(a^2M^2)(\omega)]^*\}\right]\ .
$$

(As usual the dependence on the renormalization scale $\mu$ is only apparent, since the full effective action is $\mu$ independent: a change in $\mu$ would be compensated by a change in the renormalized gravitational constants and $M^2$.) We find, as discussed in Sec. III, that particle production depends only on the imaginary parts of the kernels in the frequency domain. This also reaffirms that self-interaction enhances particle production, and that the correction appears already to first order in the coupling constants.[17,21]

In summary, we have computed the in-in effective action and the expectation values of the energy-momentum tensor. Renormalization requires only counterterms for the in-out effective action. The equations for the background metric are real, causal, and nonlocal, the nonlocal terms being related to particle-production processes.

## V. REMARKS

In this paper we have given an expository introduction to the in-in or closed-time-path functional formulation of quantum field theory in curved spacetime and applied it to two concrete model calculations in quantum cosmology. These models chosen for their simplicity and generality serve to illustrate the characteristics of this method in contrast to other ones in use.

As discussed in the Introduction the main advantages of the in-in formulation are essentially twofold. (a) Its ability to produce real and causal equations of motion offers more physically interpretable results. (b) Its capability of dealing with different two-point functions (the Feynman, causal, and correlation functions) on the same footing offers the possibility to include statistical mechanical descriptions of quantum fields in a natural way.

From the technical viewpoint the closed-time-path method does not involve difficulties much beyond those of the in-out method. All the major techniques of quantum field theory such as Wick's theorem, renormalization-group theory, background-field methods, etc., can be adapted to this formalism. The closed-time-path effective action contains the usual effective action as a particular case.

Because of the above reasoning, the closed-time-path functional formalism has the potential to become an important tool for discussing quantum and statistical processes in curved spacetimes, especially for dynamical and nonequilibrium systems like that occurring in the early Universe and in black holes. Problems like particle production and back reaction, finite-temperature field theory, transport theory, dissipative processes, dynamical critical phenomena are all amenable to treatments by this formalism.

*Note added in proof.* After this work had been accepted for publication, we received a paper by R. Jordan [University of Texas—Austin report (unpublished)] on a related problem.

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