Ultralocal limit of the gravitational field coupled to a scalar field

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(Received 18 August 1986)

The ultralocal limit of the gravitational field and the gravitational field coupled to a scalar field are studied from both the classical and quantum standpoints. For the classical cases, use will be made of the minisuperspace formalism. In the quantum case, techniques for quantizing ultralocal self-interacting scalar fields are extended to deal with the gravitational field. It will be shown that although the singularity behavior of the gravitational field by itself is characterized by Kasner solutions at each spatial point classically, the quantum case requires the addition of a repulsive scattering potential in order to have well-defined self-adjoint field operators. For the gravitational field coupled to a massive scalar field, both the singularity behavior and when the mass terms become important in the ultralocal limit are studied classically and from the quantum standpoint. Classically, the singularity behavior of the gravitational field is modified by the scalar field so that the solutions are Kasner-like in a four-dimensional hyperspace minisuperspace which allows for expansions in all three axes in space-time. The quantum case for the singularity behavior has solutions which are confined by repulsive potential barriers to regions where only the states with the lowest "angular momentum" can evolve out of. Adding matter terms introduces solutions that show the periodic motion of a harmonic oscillator. For the quantum case, a discrete eigenvalue spectrum occurs for the scalar degree of freedom. It is therefore seen that in the ultralocal limit, the quantum and classical cases are significantly different.

I. INTRODUCTION

Investigations into the quantum structure of gravitation have focused primarily on the weak-field properties. An interesting alternative approach first suggested by Isham[1] and developed by Pilati[2–4] and Isham[5] is to study the strong-coupling limit of general relativity in the Hamiltonian formulation and to quantize the theory in this limit. In the strong-coupling limit, the spatial gradient terms drop out and the full theory is recovered by adding them in as a perturbation. This approach represents developing a perturbation theory around the strong-coupling limit which can give information complementary to that which is usually obtained in a weak-field perturbation theory.

In this paper, the classical version of the strong-coupling limit for the gravitational field will be briefly reviewed and a quantum formulation will be developed. The strong-coupling limit of the gravitational field implies ultralocality, i.e., that the dynamics at each spatial point are decoupled. The quantization of ultralocal field theories has been extensively studied by Klauder[6–10] and his techniques will be extended to deal with the gravitational field. Both the classical and quantum versions of the gravitational field coupled to a scalar field in the ultralocal limit will then be developed and elucidated.

We will use the constraint formalism[11–14] for both the classical and quantum versions of the theory. Specifically, we will eliminate all but one constraint, the super-Hamiltonian. This defines the physically allowable solutions of the equations of motion. In the quantum case, the constraint condition is implemented as an operator constraint defining the physically allowable spectrum of states.

For the gravitational field by itself, classically the ultralocal limit results in a collection of "Kasner universe" solutions.[2] Our quantum formulation introduces repulsive terms into the super-Hamiltonian, which produces a scattering phenomenon in quantum minisuperspace unlike earlier work.[2] We therefore see differences between the classical and quantum formulations of ultralocal gravity.

The effect of introducing a scalar field coupled to the gravitational field in the ultralocal limit will also be investigated. We will look at the cases where the self-interaction and matter terms of the scalar field are small and where they effect the solutions. In the former, classically it will be shown that the solutions evolve along the null cone of a four-dimensional space called "hyperspace minisuperspace" which consists of a three-dimensional minisuperspace[15] and a dimension for the degree of freedom of the scalar field. This evolution along the null cone is the four-dimensional analog of the solely gravitational solutions propagating on the null cone in minisuperspace. For the quantum solutions, repulsive barriers must be introduced that confine the solutions to specific regions in "hyperspace minisuperspace."

In the domain where the matter terms are no longer negligible, we will see that the repulsive terms added to the quantum constraint introduces scattering and we also have a discrete eigenvalue spectrum for the scalar degree of freedom.

Section II deals with the quantization of ultralocal self-interacting scalar fields in general. Sections III and IV deal with the ultralocal limits of the gravitational field and the gravitational field coupled to a scalar field, respectively.

II. ULTRALOCAL QUANTIZATION

A. Interacting scalar fields

The techniques developed by Klauder[6–10] for quantizing self-interacting scalar fields can easily be extended to deal with two or more different types of fields interacting with each other.

Consider a classical Hamiltonian of the form

$$H=\int d\mathbf{x}[\tfrac{1}{2}p_1{}^2(\mathbf{x})+\tfrac{1}{2}p_2{}^2(\mathbf{x})+\tfrac{1}{2}m_1{}^2\phi_1{}^2 +\tfrac{1}{2}m_2{}^2\phi_2{}^2+V(\rho,\phi_1,\phi_2)] \ , \tag{2.1}$$

where ρ is the coupling constant between the ϕ_1 and ϕ_2 fields. In this Hamiltonian, we have dropped the spatial gradient terms. This type of theory is characterized by the fact that distinct spatial points characterize statistically independent fields for all times. Because the dynamics at each spatial point are independent, the classical-field-theory Hamiltonian reduces to an infinite collection of two-degrees-of-freedom Hamiltonians, one at each space point. The quantum theory in a sense is also characterized by an underlying two-degrees-of-freedom system. In the ultralocal quantum theory, we can form the non-Fock representations of ϕ_1 and ϕ_2:

$$\phi_1(\mathbf{x})=\int\int d\lambda\, d\Omega\, B^\dagger(\mathbf{x},\lambda,\Omega)\lambda B(\mathbf{x},\lambda,\Omega) \ , \tag{2.2}$$

$$\phi_2(\mathbf{x})=\int\int d\lambda\, d\Omega B^\dagger(\mathbf{x},\lambda,\Omega)\Omega B(\mathbf{x},\lambda,\Omega) \ , \tag{2.3}$$

where

$$B(\mathbf{x},\lambda,\Omega)=A(\mathbf{x},\lambda,\Omega)+c(\lambda,\Omega) \ . \tag{2.4}$$

λ,Ω are defined by $-\infty<\lambda<\infty$, $-\infty<\Omega<\infty$, and $c(\lambda,\Omega)$ is a function of both λ and Ω space in order to represent the interaction between the two fields and is the two-dimensional analog of the one-dimensional model function $c(\lambda)$ The quantum form for the Hamiltonian is

$$H=\int\int\int d\mathbf{x}\, d\lambda\, d\Omega\, B^\dagger(\mathbf{x},\lambda,\Omega)h(\lambda,\Omega)B(\mathbf{x},\lambda,\Omega) \ , \tag{2.5}$$

where $h(\lambda,\Omega)$ is the analogous Hamiltonian in the λ-Ω base space. Using the commutation relations

$$[B(\mathbf{x},\lambda,\Omega),B^\dagger(\mathbf{x},\lambda,\Omega)]=\delta(\mathbf{x}-\mathbf{x}')\delta(\lambda-\lambda')\delta(\Omega-\Omega') \ , \tag{2.6}$$

we can express renormalized products of the fields in the product space by

$$\phi_1{}^n(\mathbf{x})_r=\int\int d\lambda\, d\Omega B^\dagger(\mathbf{x},\lambda,\Omega)\lambda^n B(\mathbf{x},\lambda,\Omega) \ , \tag{2.7}$$

$$\phi_2{}^m(\mathbf{x})_r=\int\int d\lambda\, d\Omega\, B^\dagger(\mathbf{x},\lambda,\Omega)\Omega^m B(\mathbf{x},\lambda,\Omega) \ , \tag{2.8}$$

$$[\phi_1{}^n(\mathbf{x})\phi_2{}^m(\mathbf{x})]_r=\int\int d\lambda\, d\Omega\, B^\dagger(\mathbf{x},\lambda,\Omega)\lambda^n\Omega^m B(\mathbf{x},\lambda,\Omega) \ . \tag{2.9}$$

The model function $c(\lambda,\Omega)$ must satisfy the conditions

$$\int\int c^2(\lambda,\Omega)d\lambda\, d\Omega=\infty \ , \tag{2.10}$$

$$\int\int \lambda^n c^2(\lambda,\Omega)d\lambda\, d\Omega<\infty \ , \tag{2.11}$$

$$\int\int \Omega^m c^2(\lambda,\Omega)d\lambda\, d\Omega<\infty \ , \tag{2.12}$$

$$\int\int \lambda^n\Omega^m c^2(\lambda,\Omega)d\lambda\, d\Omega<\infty \ . \tag{2.13}$$

These conditions can be satisfied by choosing the model function to be of the form[16]

$$c(\lambda,\Omega)=\frac{e^{-y(\lambda,\Omega)}}{|\lambda^n+\Omega^m|^\gamma} \ . \tag{2.14}$$

This form also allows us to express rotationally invariant interactions between the fields. The Hamiltonian H is given by Eq. (2.5) with h given by

$$h=-\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}-\frac{1}{2}\frac{\partial^2}{\partial\Omega^2}-\frac{1}{2}\frac{\partial^2 y}{\partial\lambda^2}-\frac{1}{2}\frac{\partial^2 y}{\partial\Omega^2}+\frac{1}{2}\left[\frac{\partial y}{\partial\lambda}\right]^2+\frac{1}{2}\left[\frac{\partial y}{\partial\Omega}\right]^2 +\frac{\gamma\left[n\lambda^{n-1}\dfrac{\partial y}{\partial\lambda}+m\Omega^{m-1}\dfrac{\partial y}{\partial\Omega}\right]}{2(\lambda^n+\Omega^m)}-\frac{\gamma[n(n-1)\lambda^{n-2}+m(m-1)\Omega^{m-2}]}{4(\lambda^n+\Omega^m)}+\frac{\gamma(\gamma/2+1)(n^2\lambda^{2n-2}+m^2\Omega^{2m-2})}{4(\lambda^n+\Omega^m)} \ . \tag{2.15}$$

We choose $n=m=2$, for which h reduces to

$$h=-\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}-\frac{1}{2}\frac{\partial^2}{\partial\Omega^2}-\frac{1}{2}\frac{\partial^2 y}{\partial\lambda^2}-\frac{1}{2}\frac{\partial^2 y}{\partial\Omega^2}+\frac{1}{2}\left[\frac{\partial y}{\partial\lambda}\right]^2+\frac{1}{2}\left[\frac{\partial y}{\partial\Omega}\right]^2+\frac{\gamma\left[\lambda\dfrac{\partial y}{\partial\lambda}+\Omega\dfrac{\partial y}{\partial\Omega}\right]}{\lambda^2+\Omega^2}+\frac{\gamma^2}{2(\lambda^2+\Omega^2)} \ . \tag{2.16}$$

The last term is the regularization term that makes the operator $-\frac{1}{2}\partial^2/\partial\lambda^2-\frac{1}{2}\partial^2/\partial\Omega^2$, well defined. The exponential in $c(\lambda,\Omega)$ in Eq. (2.14) not only gives the λ-Ω space analog of the usual potential-energy terms in the expression for $h(\lambda,\Omega)$, but also an unexpected term,

$$\frac{\gamma\left[\lambda\dfrac{\partial y}{\partial\lambda}+\Omega\dfrac{\partial y}{\partial\Omega}\right]}{\lambda^2+\Omega^2} \ . \tag{2.17}$$

If we look at the limit of this term as $\Omega\to 0$, we get (γ/λ)

$\partial y/\partial\lambda$ which is the same type of term that appeared in the single-field case. Here, however, this does not appear as a typical potential-energy term of the form $\lambda^n\Omega^m$ but represents a complex coupling between the fields because of the $1/(\lambda^2+\Omega^2)$ dependence. This is chosen to represent a potential-energy term because of its form when either Ω or $\lambda\to 0$ and because it is not required to make the kinetic-energy terms well defined.

B. Constraint formalism applied to ultralocal theories

Consider a classical super-Hamiltonian constraint that comes from a parametrized field theory in Hamiltonian form:

$$H(\mathbf{x})=\tfrac{1}{2}p_{\rm cl}{}^2(\mathbf{x})+\tfrac{1}{2}m^2\phi_{\rm cl}{}^2(\mathbf{x})+V(\phi_{\rm cl}(\mathbf{x}))=0 . \quad (2.18)$$

This super-Hamiltonian can be incorporated into a quantum theory as a constraint that defines allowable states of the system to obtain the physical subspace:

$$H(\mathbf{x})\,|\,\psi\rangle=0 . \quad (2.19)$$

In the ultralocal quantum theory, let

$$H(\mathbf{x})\int d\lambda\, B^{\dagger}(\mathbf{x},\lambda)hB(\mathbf{x},\lambda) . \quad (2.20)$$

We require that

$$H(\mathbf{x})\,|\,0\rangle=0 , \quad (2.21)$$

which implies that

$$hc(\lambda)=0 , \quad (2.22)$$

where the model function $c(\lambda)$ must satisfy

$$\int c^2(\lambda)d\lambda=\infty , \quad (2.23)$$

$$\int \lambda^n c^2(\lambda)d\lambda<\infty . \quad (2.24)$$

If the $|\,\psi\rangle$ are the coherent states defined by[17]

$$|\,\psi\rangle=\exp\left[-\tfrac{1}{2}\int\int |\,\psi(\mathbf{x},\lambda)\,|^2 d\mathbf{x}\,d\lambda\right] \times\exp\left[\int\int \psi(\mathbf{x},\lambda)A^{\dagger}(\mathbf{x},\lambda)d\mathbf{x}\,d\lambda\right]|\,0\rangle , \quad \psi(\mathbf{x},\lambda)\in L^2 , \quad (2.25)$$

then the constraint equation (2.18) is satisfied if

$$h\psi(\mathbf{x},\lambda)=0 . \quad (2.26)$$

Therefore the constraint equation can be written as a constraint on the Hilbert space spanned by ψ. This constraint formalism can easily be extended to include more than one field.

III. ULTRALOCAL GRAVITY

A. Classical case

The classical Hamiltonian formulation of gravity involves four constraint equations per space-time point. These can be written in the ultralocal limit as

$$H(\mathbf{x})=g^{-1/2}G_{ijkl}(\mathbf{x})\pi^{ij}(\mathbf{x})\pi^{kl}(\mathbf{x})=0 , \quad (3.1)$$

$$H_i(\mathbf{x})=-2g_{ik}\pi^{kj}{}_{lj}=0 , \quad (3.2)$$

where $H(\mathbf{x})$ and $H_i(\mathbf{x})$ are called the super-Hamiltonian and supermomentum, respectively.

Using a gauge condition proposed by Pilati,[2] the degrees of freedom of the system can be reduced from 6 each for the gravitational field and its conjugate momenta to 3 each. These are $\tau_+,\tau_-,\tau,\pi_+,\pi_-,\pi$, where τ,π are the intrinsic time and its associated conjugate momentum. The commutation relations for the gravitational degrees of freedom are

$$[\tau_+(\mathbf{x}),\pi_+(\mathbf{x}')]=[\tau_-(\mathbf{x}),\pi_-(\mathbf{x}')]=\delta(\mathbf{x}-\mathbf{x}') ,$$
$$[\tau(\mathbf{x}),\pi(\mathbf{x}')]=\delta(\mathbf{x}-\mathbf{x}') , \quad (3.3)$$
$$[\tau_i,\tau_i]=[\pi_i,\pi_i]=0,\ \ i=+,- .$$

It has been shown[2] that the super-Hamiltonian can then be written as

$$H(\mathbf{x})=\pi_+{}^2(\mathbf{x})+\pi_-{}^2(\mathbf{x})-\pi^2(\mathbf{x})=0 . \quad (3.4)$$

Since the ultralocal limit was employed and spatial points are dynamically decoupled, the problem of dealing with an infinite-degrees-of-freedom super-Hamiltonian has been reduced to an infinite number of three-degrees-of-freedom super-Hamiltonians. In other words, this can be looked at as independent super-Hamiltonians at each space point. From this viewpoint, use can be made of Misner's minisuperspace formalism.[15]

In a flat minisuperspace, the super-Hamiltonian can be written in the general case in the form

$$H=\pi_+{}^2+\pi_-{}^2-\pi^2+V(\tau_+,\tau_-,\tau) . \quad (3.5)$$

It has been shown that, in the case when $V(\tau_+,\tau_-,\tau)=0$, the constraint equation defines a Kasner universe,[18] an anisotropic expanding universe where distances expand along two dimensions and contract along the third. The constraint equation (3.5) with $V=0$ gives the result that the path of a Kasner universe in minisuperspace is on a "null cone." Equation (3.4) can be considered as defining an infinite collection of Kasner universes—one at each spatial point.

B. The quantum case

Following Pilati,[2] we will represent the remaining gravitational degrees of freedom and their conjugate momenta by the non-Fock expressions:

$$\tau(\mathbf{x})=\int\int\int d\beta_+d\beta_-d\Omega\, B^{\dagger}(\mathbf{x},\beta_+,\beta_-,\Omega)\Omega \times B(\mathbf{x},\beta_+,\beta_-,\Omega) , \quad (3.6)$$

$$\pi(\mathbf{x})=\int\int\int d\beta_+d\beta_-d\Omega\, B^{\dagger}(\mathbf{x},\beta_+,\beta_-,\Omega)\left[-i\frac{\partial}{\partial\Omega}\right] \times B(\mathbf{x},\beta_+,\beta_-,\Omega) , \quad (3.7)$$

$$\tau_+(\mathbf{x})=\int\int\int d\beta_+d\beta_-d\Omega\, B^{\dagger}(\mathbf{x},\beta_+,\beta_-,\Omega)\beta_+ \times B(\mathbf{x},\beta_+,\beta_-,\Omega) , \quad (3.8)$$

$$\pi_+(\mathbf{x}) = \int\int\int d\beta_+ d\beta_- d\Omega\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega)\left[-i\frac{\partial}{\partial\beta_+}\right] \times B(\mathbf{x},\beta_+,\beta_-,\Omega) , \quad (3.9)$$

$$\tau_-(\mathbf{x}) = \int\int\int d\beta_+ d\beta_- d\Omega\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega)\beta_- \times B(\mathbf{x},\beta_+,\beta_-,\Omega) , \quad (3.10)$$

$$\pi_-(\mathbf{x}) = \int\int\int d\beta_+ d\beta_- d\Omega\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega)\left[-i\frac{\partial}{\partial\beta_-}\right] \times B(\mathbf{x},\beta_+,\beta_-,\Omega) . \quad (3.11)$$

We can define the quantum commutation relations

$$[B(\mathbf{x},\beta_+,\beta_-,\Omega),B^\dagger(\mathbf{x}',\beta'_+,\beta'_-,\Omega')] = \delta(\mathbf{x}-\mathbf{x}')\delta(\beta_+-\beta'_+)\delta(\beta_--\beta'_-)\delta(\Omega-\Omega') , \quad (3.12)$$

where $B(\mathbf{x},\beta_+,\beta_-,\Omega) = A + c(\beta_+,\beta_-)$. The Hamiltonian constraint can be represented by

$$H(\mathbf{x}) = -\int\int\int d\beta_+ d\beta_- d\Omega\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega) \times\left[\frac{\partial^2}{\partial{\beta_+}^2}+\frac{\partial^2}{\partial{\beta_-}^2}-\frac{\partial^2}{\partial\Omega^2}+V(\beta_+,\beta_-)\right] \times B(\mathbf{x},\beta_+,\beta_-,\Omega) , \quad (3.13)$$

where $V(\beta_+,\beta_-)$ gives the regularization-type terms for the kinetic-energy terms. As was pointed out, V can be determined from the model function $c(\beta_+,\beta_-)$ and the condition

$$hc(\beta_+,\beta_-) = 0 . \quad (3.14)$$

In earlier work,[2] the choice $c(\beta_+,\beta_-)=1$ was made. This then gives $V(\beta_+,\beta_-)=0$. With this choice of the model function, the conditions that the model function must satisfy in order that the fields be well defined are not satisfied. The independent degrees of freedom of the gravitational field can be looked at as independent massless scalar fields. With this, the choice

$$c(\beta_+,\beta_-) = \frac{1}{|{\beta_+}^2+{\beta_-}^2|^\gamma} , \quad (3.15)$$

which leads to well-defined field operators, will be made. The quantum Hamiltonian constraint which defines the allowable spectrum of states is then given by

$$H(\mathbf{x}) = -\int\int\int d\beta_+ d\beta_- d\Omega\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega) h \times B(\mathbf{x},\beta_+,\beta_-,\Omega) , \quad (3.16)$$

with

$$h = \frac{\partial^2}{\partial{\beta_+}^2}+\frac{\partial^2}{\partial{\beta_-}^2}-\frac{\partial^2}{\partial\Omega^2}-\frac{4\gamma^2}{{\beta_+}^2+{\beta_-}^2} . \quad (3.17)$$

It is convenient to make a change of coordinates where $u=({\beta_+}^2+{\beta_-}^2)^{1/2}$ and $\phi=\tan(\beta_+/\beta_-)$. As was shown, if the states $|\psi\rangle$ satisfying $H|\psi\rangle=0$ are coherent states, then the constraint equation reduces to a base-space constraint equation $h\psi=0$. This can be solved when h is given by Eq. (3.17) to yield solutions of the form

$$\psi(\mathbf{x},u,\phi,\Omega) = e^{\pm i\gamma(\mathbf{x})\phi} e^{\pm i\beta(\mathbf{x})\Omega}[\alpha(\mathbf{x})u]^{1/2} J_n(\alpha u) , \quad (3.18)$$

where $\alpha^2(\mathbf{x}) + \beta^2(\mathbf{x}) = 0$, $\rho^2(\mathbf{x})+\gamma^2(\mathbf{x})=0$.

In the coherent-state representation, we can define the states

$$|\psi\rangle = \exp\left[-\tfrac{1}{2}\int\int |\psi(\mathbf{x},u,\phi,\Omega)|^2 d\mathbf{x}\, dR\right] \times\exp\left[\int\int \psi(\mathbf{x},u,\phi,\Omega)A^\dagger(\mathbf{x},u,\phi,\Omega)d\mathbf{x}\, dR\right]|0\rangle , \quad (3.19)$$

where $\psi(\mathbf{x},u,\phi,\Omega)$ is given in Eq. (3.18) and dR represents a generic differential depending on the coordinate system chosen for the degrees of freedom of the gravitational field. These states form the physical subspace. From Eq. (3.17), when ${\beta_+}^2+{\beta_-}^2$ is large, the repulsive term is negligible. This corresponds to a model function that is a constant. In this case we would get that the states look like "quantum Kasner states." For the high-energy limit, the degrees of freedom are decoupled, but at lower energy they are coupled through the repulsive term.

In analogy with work done on quantum cosmological models, the repulsive term can be considered as a scattering potential where, as $4\gamma^2/({\beta_+}^2+{\beta_-}^2)\to 0$, one gets the asymptotic incoming and outgoing states of the system which correspond to Kasner states. In earlier work[2] done on the quantum ultralocal limit of the gravitational field, there was no scattering potential present because of the choice of model function $c(\beta_+,\beta_-)=1$. Our solutions are independent cosmological-type solutions at each spatial point. Applying the coherent-state representation turns these into solutions of field operators. We see in our formulation a scattering phenomenon in the quantum ultralocal case not present classically.

IV. GRAVITY COUPLED TO A MASSIVE SCALAR FIELD IN THE ULTRALOCAL LIMIT

A. The classical case

We will now discuss the ultralocal limit of a massive self-interacting scalar field with the gravitational field. First, the ultralocal limit will be discussed classically, then in the next section the resulting theory will be quantized. The super-Hamiltonian and supermomentum that result from putting the theory in Hamiltonian form are given by

$$H = G_{ijkl}\pi^{ij}\pi^{kl} - g^{-1/2}R + \tfrac{1}{2}g^{-1/2}p^2 + g^{1/2}g^{ab}\phi_{,a}\phi_{,b} + \tfrac{1}{2}g^{1/2}m^2\phi^2 + g^{1/2}V(\rho,\phi) = 0 , \quad (4.1)$$

$$H_i = -2{\pi_i}^j{}_{|j} + p\phi_{,i} = 0 , \quad (4.2)$$

where $V(\rho,\phi)$ in Eq. (4.1) represents a self-interaction term for the scalar field and ρ is a coupling constant that measures the strength of the interaction. The commutation relation for the scalar field and its conjugate momentum is given by

$$[\phi(\mathbf{x}),p(\mathbf{x}')] = \delta(\mathbf{x}-\mathbf{x}') . \quad (4.3)$$

For the gravitational field, we choose the same commutation relations as were considered for the gravitational field alone in Eqs. (3.3). Because the degrees of freedom of the gravitational field are independent of the scalar field, the same gauge condition used for the gravitational field by itself can be chosen and, taking the ultralocal limit of the super-Hamiltonian, we get

$$H=\frac{e^{-3\tau/2}}{6}(\pi_+{}^2+\pi_-{}^2-\pi^2)+\frac{e^{-3\tau/2}}{2}p^2+\tfrac{1}{2}e^{3\tau/2}m^2\phi^2+e^{3\tau/2}V(\rho,\phi)=0\ . \tag{4.4}$$

By a conformal transformation and by redefining ϕ,p by the transformations

$$\phi(\mathbf{x})=\sqrt{3}\phi'(\mathbf{x})\ , \tag{4.5}$$

$$p(\mathbf{x})=\frac{1}{\sqrt{3}}p'(\mathbf{x})\ , \tag{4.6}$$

the super-Hamiltonian constraint can be written as

$$H=\pi_+{}^2+\pi_-{}^2-\pi^2+p'^2+9m^2\phi'^2e^{3\tau}+V(\rho,\phi)e^{3\tau}=0\ . \tag{4.7}$$

The rate of change of an operator with respect to the invariant supertime parameter Z is defined by

$$\dot{F}=\frac{\partial F}{\partial Z}=[F,H]\ . \tag{4.8}$$

Using this relationship, we get the following equations of motion for the field and their momenta:

$$\begin{aligned}&\dot{\tau}=-2\pi\ ,\\&\dot{\pi}=-27m^2\phi^2e^{3\tau}-3e^{3\tau}V(\rho,\phi)\ ,\\&\dot{\tau}_+=2\pi_+\ ,\\&\dot{\pi}_+=0\ ,\\&\dot{\tau}_-=2\pi_-\ ,\\&\dot{\pi}_-=0\ .\end{aligned} \tag{4.9}$$

These equations of motion describe the paths through a "hyperspace superspace" parametrized by the invariant supertime Z. This space consists of the three degrees of freedom for the gravitational field with an added dimension for the degree of freedom associated with the scalar field. Because spatial points are decoupled through the ultralocality condition, the dynamics can be described by paths through an independent "hyperspace minisuperspace" at each spatial point. The ultralocality condition here reduces the dynamics of the fields to the dynamics of independent four-degrees-of-freedom systems at each spatial points. Hyperspace minisuperspace consists of minisuperspace, the finite-dimensional analog of superspace and the one degree of freedom of the scalar field. The super-Hamiltonian in the ultralocal limit constrains the system to an allowable path in each "hyperspace minisuperspace." The constraint is reminiscent of the form for those occurring in the minisuperspace models for which our treatment of the gravitational field by itself was analogous to, but here with an added dimension for the scalar field. By choosing Pilati's gauge condition for the metric tensor discussed in the preceding section and representing the super-Hamiltonian as we have in Eq. (4.7), we have defined a hyperspace minisuperspace at each spatial point that is flat and four dimensional. This four-dimensional space is useful because we can define a Lorentzian metric that is given by

$$ds'^2=G_{AB}dg^Adg^B\ , \tag{4.10}$$

where $G_{AB}=\mathrm{diag}(-1,1,1,1)$ or

$$ds'^2=-d\tau^2+d\tau_+{}^2+d\tau_-{}^2+d\phi^2\ . \tag{4.11}$$

The $H=0$ constraint can now be written as

$$\left[\frac{ds'}{dZ}\right]^2=-36m^2\phi^2e^{3\tau}-4V(\rho,\phi)e^{3\tau}\ . \tag{4.12}$$

Since the right-hand side of Eq. (4.12) is negative, the dynamic evolution at each space point in hyperspace minisuperspace is timelike and becomes null as $V(\rho,\phi)\to 0$.

When $T\to-\infty$, which represents the singularity behavior, the matter terms in Eqs. (4.7) and (4.12) become negligible. We get that

$$H=-\pi^2+\pi_+{}^2+\pi_-{}^2+p^2\ , \tag{4.13}$$

$$\left[\frac{ds'}{dZ}\right]^2=0\ . \tag{4.14}$$

Equation (4.14) shows that the path of the system at each space point in the four-dimensional hyperspace minisuperspace is on the null cone. Also, obviously Eqs. (4.13) and (4.14) describe the motion in this space of a massless scalar field coupled to gravity. The super-Hamiltonian here is the four-dimensional analog of the three-dimensional super-Hamiltonian of Eq. (3.3) that defines a Kasner universe. We can therefore say that Eqs. (4.13) and (4.14) define a "Kasner-like" universe solution at each spatial point. We can consider this to represent the singularity behavior at each spatial point. It also represents the strong-coupling limit of the gravitational field coupled to the scalar field. In this limit the mass terms of the scalar field become negligible. It will be seen that the addition of the scalar field will affect the singularity behavior of the gravitational field.

The four-dimensional Kasner-like universe solutions that are described by Eqs. (4.13) and (4.14) have different properties than the three-dimensional case. The equations of motion we obtain from Eq. (4.13) for this case are simply

$$\dot{\tau}=-2\pi\ , \tag{4.15}$$

$$\dot{\tau}_\pm=2\pi_\pm\ , \tag{4.16}$$

$$\dot{\phi}=2p\ , \tag{4.17}$$

$$\dot{\pi}=0\ , \tag{4.18}$$

$$\dot{\pi}_\pm=0\ , \tag{4.19}$$

$$\dot{p}=0\ . \tag{4.20}$$

Equations (4.15)–(4.20) imply the momenta are independent of the supertime Z. We want to look at the line element in space-time. Following Liang[19] and Francisco and Pilati,[20] we choose a synchronous reference frame where $g_{0i}=0$, $g_{00}=-1$, and choose $dZ=dt$. If we rescale the time parameter so that[20] $dt=(1/6\pi)d(\ln t)$, then we get

$$\dot{\tau}_{\pm}=\frac{d\tau_{\pm}}{(1/6\pi\tau)d\tau}=2\pi_{\pm}\ ; \tag{4.21}$$

therefore

$$\dot{\tau}_{\pm}=\frac{1}{3}\frac{\pi_{\pm}}{\pi\tau} \tag{4.22}$$

and

$$\tau_{\pm}=\frac{1}{3}\frac{\pi_{\pm}}{\pi}\ln\tau\ . \tag{4.23}$$

We also get

$$\dot{\tau}=-\frac{1}{3\tau} \tag{4.24}$$

and

$$\tau=-\tfrac{1}{3}\ln\tau\ . \tag{4.25}$$

For the scalar field, we have

$$\dot{\phi}=\frac{1}{3\pi}\frac{p}{\tau}\ , \tag{4.26}$$

and therefore

$$\phi=\frac{1}{3\pi}p\ln\tau\ . \tag{4.27}$$

We can redefine the gravitational dynamical degrees of freedom by

$$\beta_1=\tau_++\sqrt{3}\tau_-,\quad \beta_2=\tau_+-\sqrt{3}\tau_-,\quad \beta_3=-2\tau_+\ , \tag{4.28}$$

where $\sum_i\beta_i=0$ and $g=\det g_{ij}=e^{-6\tau}$. In this parametrization, the metric tensor g_{ij} can be written as

$$g_{ij}=\begin{bmatrix} e^{2(-\tau+\beta_1)} & & \\ & e^{2(-\tau+\beta_2)} & \\ & & e^{2(-\tau+\beta_3)} \end{bmatrix}. \tag{4.29}$$

If we set[20]

$$\begin{aligned} p_1&=\frac{1}{3}\left[\frac{\pi_++\sqrt{3}\pi_-}{\pi}\right]+\frac{1}{3}\ ,\\ p_2&=\frac{1}{3}\left[\frac{\pi_+-\sqrt{3}\pi_-}{\pi}\right]+\frac{1}{3}\ ,\\ p_3&=\frac{-2}{3}\frac{\pi_+}{\pi}+\frac{1}{3}\ , \end{aligned} \tag{4.30}$$

then we can write Eq. (4.29) as

$$g_{ij}=\begin{bmatrix} \tau^{2p_1} & & \\ & \tau^{2p_2} & \\ & & \tau^{2p_3} \end{bmatrix}, \tag{4.31}$$

$$\sum_{i=1}^{3}p_i=1\ . \tag{4.32}$$

If the scalar field is ignored for the moment, it is easy to obtain the condition

$$\sum_{i=1}^{3}p_i^{\,2}=1\ . \tag{4.33}$$

Equations (4.32) and (4.33) define a Kasner universe. The line element in space-time for the Kasner universe is given by

$$ds^2=-d\tau^2+\tau^{2p_1}dx^2+\tau^{2p_2}dy^2+\tau^{2p_3}dz^2\ . \tag{4.34}$$

Equations (4.32) and (4.33) require that one of the p's be nonpositive.[18] For example,

$$-\tfrac{1}{3}\le p_1\le 0\ . \tag{4.35}$$

This is why, for the Kasner universe, two of the axes are expanding and one is contracting.

Adding the scalar field in the limit as $\tau\to-\infty$ results in a four-dimensional "Kasner universe." The same parametrization as in (4.28) can be used in this case and p_i can be defined by Eq. (4.30) because the degrees of freedom of the gravitational field are independent of the scalar field in this constraint formalism. While Eq. (4.32) still holds, Eq. (4.33) is no longer true. Because of our super-Hamiltonian (4.13), we get $\pi_+^{\,2}+\pi_-^{\,2}=\pi^2-p^2$ instead of $\pi_+^{\,2}+\pi_-^{\,2}=\pi^2$. This gives

$$\sum_{i=1}^{3}p_i^{\,2}=1-\tfrac{2}{3}p^2/\pi^2 \tag{4.36}$$

instead of Eq. (4.33). Choosing $2/3p^2=q^2$, we get

$$\sum_{i=1}^{3}p_i^{\,2}=1-q^2\ . \tag{4.37}$$

Equations (4.32) and (4.37) are the same equations for the p_i's as obtained by Belinskii and Khalatnikov by different means[21] when they investigated the effect of a massless scalar field on a cosmological singularity.

In Eq. (4.37), q can vary in the range

$$-(\tfrac{2}{3})^{1/2}\le q\le(\tfrac{2}{3})^{1/2}\ . \tag{4.38}$$

The added degree of freedom from the introduction of the scalar field now allows p_1,p_2,p_3 to all be positive, which means that in this case at each space point instead of having the typical "Kasner universe" solution with two expanding and one contracting axes, here we have allowable solution where all three axes are expanding. When $q>\frac{1}{2}$ or $q<-\frac{1}{2}$, this is the only solution. This type of solution will be called "quasi-Kasner." Belinskii and Khalatnikov further show that these results hold when the p_i's and q are spatially dependent as in our case.

The solutions in this case have some interesting properties. Depending on the values for q, the solutions are either the Kasner universe solution or the quasi-Kasner universe solution. It has been shown[21] that even if the initial solution is a Kasner universe, adding the $g\,^3R$ term (which is small near the singularity) to the super-Hamiltonian introduces scattering which will evolve into a quasi-Kasner universe with all the p_i's positive. For the case of the gravitational field by itself, the $g\,^3R$ term scatters the system from one Kasner universe solution to another. Therefore, even classically, new features appear when adding the scalar field in this limit.

When τ gets larger, we can start seeing the effect of the potential terms of the scalar field. When $\tau \to 0$, with the self-interaction term $V(\rho,\phi)=0$, the super-Hamiltonian of Eq. (4.7) becomes

$$H(\mathbf{x})=\pi^2(\mathbf{x})+\pi_+{}^2(\mathbf{x})+\pi_-{}^2(\mathbf{x})+p^2(\mathbf{x})+9m^2\phi^2(\mathbf{x})=0 \ . \tag{4.39}$$

The equations of motion are

$$\begin{aligned}
&\dot\tau(\mathbf{x})=-2\pi(\mathbf{x}) \ , \\
&\dot\pi(\mathbf{x})=0 \ , \\
&\dot\tau_\pm(\mathbf{x})=2\pi_\pm(\mathbf{x}) \ , \\
&\dot\pi_\pm(\mathbf{x})=0 \ , \\
&\dot\phi(\mathbf{x})=2p(\mathbf{x}) \ , \\
&\dot p(\mathbf{x})=-18m^2\phi(\mathbf{x}) \ , \\
&\dot H(\mathbf{x})=0 \ .
\end{aligned} \tag{4.40}$$

From Eqs. (4.40), $\pi,\pi_\pm$ are constants of motion (independent of the supertime Z); therefore, we can write

$$\pi_+(\mathbf{x})=\alpha(\mathbf{x}), \quad \pi_-(\mathbf{x})=\beta(\mathbf{x}), \quad \pi(\mathbf{x})=\gamma(\mathbf{x}) \ ,$$

so we can write the super-Hamiltonian as

$$H=-\gamma^2(\mathbf{x})+\alpha^2(\mathbf{x})+\beta^2(\mathbf{x})+p^2(\mathbf{x})+9m^2\phi^2(\mathbf{x})=0 \ , \tag{4.42}$$

where $-\gamma^2(\mathbf{x})+\alpha^2(\mathbf{x})+\beta^2(\mathbf{x})+p^2(\mathbf{x})\le 0$. Using the equations of motion for ϕ,p, we get

$$\tfrac{1}{2}\dot\phi^2(\mathbf{x})+18m^2\phi^2(\mathbf{x})=2[\gamma^2(\mathbf{x})-\alpha^2(\mathbf{x})-\beta^2(\mathbf{x})] \ , \tag{4.43}$$

where $2[\gamma^2(\mathbf{x})-\alpha^2(\mathbf{x})-\beta^2(\mathbf{x})]\ge 0$. We see that the classical constraint equation (4.39) defines a harmonic-oscillator-type solution for the scalar degree of freedom with positive energy given by the constants on the right-hand side of the equation. The solution of Eq. (4.40) is

$$\phi(\mathbf{x})=\frac{1}{3m}[\gamma^2(\mathbf{x})-\alpha^2(\mathbf{x})-\beta^2(\mathbf{x})]^{1/2}\sin(36m^2Z) \ , \tag{4.44}$$

where we have chosen $\phi=0$ at $Z=0$. The solution describes a system in the scalar field degree of freedom at each spatial point scattering off the walls of the potential executing simple harmonic motion. Since

$$\begin{aligned}
&\tau(\mathbf{x})=-2\gamma(\mathbf{x})Z \ , \\
&\tau_+(\mathbf{x})=2\alpha(\mathbf{x})Z \ , \\
&\tau_-(\mathbf{x})=2\beta(\mathbf{x})Z \ ,
\end{aligned} \tag{4.45}$$

we can write the solution in Eq. (4.44) as

$$\phi(\mathbf{x})=\frac{1}{6mZ}(\tau^2-\tau_+{}^2-\tau_-{}^2)\sin(36m^2Z) \ . \tag{4.46}$$

Equation (4.12) can be written as

$$\left[\frac{ds'}{dZ}\right]^2=4[\gamma^2(\mathbf{x})-\alpha^2(\mathbf{x})-\beta^2(\mathbf{x})]\sin^2(36m^2Z) \ . \tag{4.47}$$

We can get an idea of the effect of the super-Hamiltonian when $\tau>0$ in the general case by looking at Fig. 1, where we can see the general behavior of the poten-

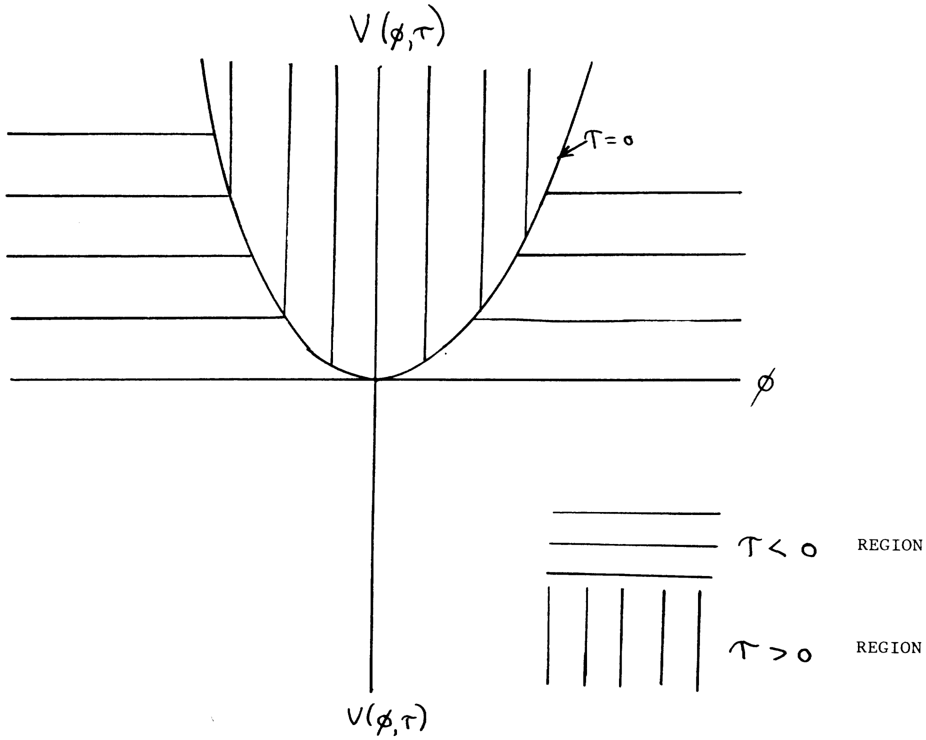

FIG. 1. Intrinsic time regions.

tial term as a function of ϕ and π. The scalar field degree of freedom in this case consists of harmonic-oscillator-type solutions for a potential that is contracting. The super- Hamiltonian in this formalism defines the allowable paths in hyperspace minisuperspace. It is a constraint on the possible values of the degrees of freedom of the system. In the case of the constraint that we are dealing with, $\rho \to 0$ so $V(\rho,\phi)$ can be neglected, the trajectory for the scalar field degree of freedom parametrized by the supertime Z is confined to a parabolic potential that is closing with increasing τ. In a sense, the description of what occurs in this formalism is that there is "scattering" off the walls of this potential. Our description of what occurs for the classical field theory in the ultralocal limit is that the solution consists of these solutions at each spatial point.

B. Quantum case

The super-Hamiltonian constraint in Eq. (4.7) is imposed in the quantum formulation by turning the constraint into an operator constraint on the state vectors $|\psi\rangle$. This defines the allowable spectrum of states. As in the discussion of the gravitational field by itself, use will be made of the coherent-state representations to study the ultralocal quantum theory. We will proceed to construct a Hilbert space and a representation of the canonical commutation relations. The Hilbert space is a Fock space defined in terms of creation and annihilation operators:

$$A(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda), \quad A^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) \tag{4.48}$$

satisfying

$$[A(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda),A^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)] = \delta(\mathbf{x}-\mathbf{x}')\delta(\beta_+-\beta'_+)\delta(\Omega-\Omega')\delta(\lambda-\lambda') \tag{4.49}$$

and

$$A\,|\,0\rangle = 0 \,. \tag{4.50}$$

We are interested in the "translated Fock" representations. For the case of the gravitational field coupled to a scalar field, these are defined by

$$B(\mathbf{x},\lambda,\beta_+,\beta_-,\Omega,\lambda) = A(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) + c(\beta_+,\beta_-,\Omega,\lambda) \,. \tag{4.51}$$

For the gravitational field coupled to a scalar field, the super-Hamiltonian is intrinsic time dependent, unlike that for the gravitational field by itself. We expect the ground state to depend on the intrinsic time; therefore, in this case, the model function is chosen to be a function of the intrinsic time.

The gravitational degrees of freedom are given by the variables $\tau_+,\tau_-,\pi_+,\pi_-,\pi,\tau$, while the scalar degrees of freedom are given by ϕ,p. The ultralocal representations of these operators is given by the expressions

$$\begin{aligned}
\tau_+(\mathbf{x}) &= \int\int\int\int d\beta_+ d\beta_- d\Omega\, d\lambda\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)\beta_+ B(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) \,,\\
\pi_+(\mathbf{x}) &= -i\int\int\int\int d\beta_+ d\beta_- d\Omega\, d\lambda\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)\frac{\partial}{\partial\beta_+} B(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) \,,\\
\tau_-(\mathbf{x}) &= \int\int\int\int d\beta_+ d\beta_- d\Omega\, d\lambda\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)\beta_- B(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) \,,\\
\pi_-(\mathbf{x}) &= -i\int\int\int\int d\beta_+ d\beta_- d\Omega\, d\lambda\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)\frac{\partial}{\partial\beta_-} B(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) \,,\\
\tau(\mathbf{x}) &= \int\int\int\int d\beta_+ d\beta_- d\Omega\, d\lambda\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)\Omega B(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) \,,\\
\pi(\mathbf{x}) &= -i\int\int\int\int d\beta_+ d\beta_- d\Omega\, d\lambda\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)\frac{\partial}{\partial\Omega} B(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) \,,\\
\phi(\mathbf{x}) &= \int\int\int\int d\beta_+ d\beta_- d\Omega\, d\lambda\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)\lambda B(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) \,,\\
p(\mathbf{x}) &= -i\int\int\int\int d\beta_+ d\beta_- d\Omega\, d\lambda\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)\frac{\partial}{\partial\lambda} B(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) \,.
\end{aligned} \tag{4.52}$$

Equation (4.7) for the classical super-Hamiltonian can be written in the quantum case as

$$H(\mathbf{x}) = -\int\int\int d\beta_+ d\beta_- d\Omega\, d\lambda\, B^\dagger(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)h \times B(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda) \,, \tag{4.53}$$

with

$$h = \frac{\partial^2}{\partial\beta_+{}^2} + \frac{\partial^2}{\partial\beta_-{}^2} + \frac{\partial^2}{\partial\lambda^2} - \frac{\partial^2}{\partial\Omega^2} - \bar{V}(\beta_+,\beta_-,\Omega,\lambda) \,,$$

where $\bar{V}(\beta_+,\beta_-,\Omega,\lambda)$ contains the interaction terms and the regularization terms. In order to link the dynamics with the field representation

$$-\left[\frac{\partial^2}{\partial\beta_+{}^2} + \frac{\partial^2}{\partial\beta_-{}^2} + \frac{\partial^2}{\partial\lambda^2} - \frac{\partial^2}{\partial\Omega^2} - \bar{V}(\beta_+,\beta_-,\Omega,\lambda)\right] c(\beta_+,\beta_-,\Omega,\lambda) = 0 \,. \tag{4.54}$$

Based on the form of the classical super-Hamiltonian, it is reasonable to assume

$$\bar{V}(\beta_+,\beta_-,\Omega,\lambda)=9m^2\lambda^2e^{3\Omega}+V(\rho,\lambda,\Omega)+v(\beta_+,\beta_-,\Omega,\lambda)\ , \tag{4.55}$$

where the self-interaction term $V(\rho,\lambda,\Omega)$ has been separated from the regularization terms given by $v(\beta_+,\beta_-,\Omega,\lambda)$. Equation (4.54) can then be written as

$$\left[\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}+\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\Omega^2}-9m^2\lambda^2e^{3\Omega}-V(\rho,\lambda,\Omega)-v(\beta_+,\beta_-,\Omega,\lambda)\right]c=0\ . \tag{4.56}$$

We must solve for the model function $c(\beta_+,\beta_-,\Omega,\lambda)$ that satisfies this equation. The model function $c(\beta_+,\beta_-,\Omega,\lambda)$ that yields the pseudofree theory when both the self-interaction terms and the interaction between the scalar field and the gravitational field are shut off by taking $\lambda\to 0$ needs to be found. The interaction terms would then be added by including the $y(\lambda,\Omega)$ term in the exponential portion of the model function, as was shown in Sec. II. For the pseudofree case, when $\lambda\to 0$, Eq. (4.56) becomes

$$\left[\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}+\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\Omega^2}+v(\beta_+,\beta_-,\Omega,\lambda)\right]\times c(\beta_+,\beta_-,\Omega,\lambda)=0\ . \tag{4.57}$$

We wish to maintain the "Lorentz invariance" of the quantum version of the classical super-Hamiltonian with the interaction of the scalar field and the gravitational field shut off. If we choose

$$c(\beta_+,\beta_-,\Omega,\lambda)=\frac{1}{|\beta_+{}^2+\beta_-{}^2+\lambda^2-\Omega^2|^{\gamma}}\ , \tag{4.58}$$

then

$$h=\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}+\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\Omega^2}-\frac{4\gamma(\gamma-1)}{\beta_+{}^2+\beta_-{}^2+\lambda^2-\Omega^2}\ . \tag{4.59}$$

With this choice, h is still Lorentz invariant, as it is in the classical case. The last term is a regularization term for the kinetic energy. To add the interactions, we consider a model function of the form

$$c(\beta_+,\beta_-,\Omega,\lambda)=\frac{e^{-y(\lambda,\Omega)}}{|\beta_+{}^2+\beta_-{}^2+\lambda^2-\Omega^2|^{\gamma}}\ . \tag{4.60}$$

Using the relationship

$$h=\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}+\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\Omega^2}-\frac{1}{c}\left[\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}+\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\Omega^2}\right]c\ , \tag{4.61}$$

we get h in its most general form:

$$h=\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}+\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\Omega^2}-\frac{4\gamma(\gamma-1)}{\beta_+{}^2+\beta_-{}^2+\lambda^2-\Omega^2}-\frac{4\gamma\left[\lambda\frac{\partial y}{\partial\lambda}+\Omega\frac{\partial y}{\partial\Omega}\right]}{\lambda^2+\beta_+{}^2+\beta_-{}^2-\Omega^2}+\frac{\partial^2 y}{\partial\lambda^2}-\left[\frac{\partial y}{\partial\lambda}\right]^2-\frac{\partial^2 y}{\partial\Omega^2}+\left[\frac{\partial y}{\partial\Omega}\right]^2\ . \tag{4.62}$$

The potential-energy terms break the symmetry in the problem. As in the classical case, both the $\Omega\to 0$ and $\Omega\to-\infty$ limits will be investigated. First, from Eq. (4.55), as $\Omega\to 0$,

$$\bar{V}(\beta_+,\beta_-,\Omega,\lambda)=9m^2\lambda^2+V(\rho,\lambda)+v(\beta_+,\beta_-,\Omega,\lambda)\ . \tag{4.63}$$

From Eqs. (4.62) and (4.63), we can conclude, as $\Omega\to 0$, that the form of h is

$$h=\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}+\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\Omega^2}-\frac{4\gamma(\gamma-1)}{\beta_+{}^2+\beta_-{}^2+\lambda^2}-\frac{12m\lambda^2\gamma}{\lambda^2+\beta_+{}^2+\beta_-{}^2}-9m^2\lambda^2-V(\rho,\lambda)+3m\ . \tag{4.64}$$

Equation (4.53) with h given above is the quantum analog of the classical constraint (4.39). We see from these equations for h that although classically the scalar field is coupled only to the intrinsic time in the ultralocal limit, in the quantum case, it is also coupled to the other dynamical degrees of freedom of the gravitational field. Even when there is no scalar field present, we still get an interaction between the degrees of freedom of the gravitational field that regularizes the kinetic energy terms. The justification for this is that the states are prepared for the interaction, the effects of which can never fully be shut off. In general relativity, the interactions can never be fully shut off.

As was done in Sec. III for the gravitational field by itself, a constraint of the form $H(x)|\psi\rangle=0$ can be implemented by finding the states ψ that satisfy $h\psi=0$ and then using these states to define a coherent-state representation which satisfies the constraint condition defined by $H(x)$. Let us look at the physical subspace of states that satisfy the equation

$$h\psi(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)=0\ , \tag{4.65}$$

where h is defined by Eq. (4.64). We will look at two approximate cases for this equation, when $\beta_+{}^2+\beta_-{}^2\gg\lambda^2$ and when $\lambda^2\gg\beta_+{}^2+\beta_-{}^2$.

We will first look at the case where $\beta_+{}^2+\beta_-{}^2\gg\lambda^2$. Using a binomial expansion and keeping the highest-order terms, Eq. (4.65) becomes

$$\left[\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}+\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\Omega^2}-\frac{4\gamma(\gamma-1)}{\beta_+{}^2+\beta_-{}^2}\right.$$
$$\left.-9m^2\lambda^2-V(\rho,\lambda)+3m\right]\psi=0\ . \quad (4.66)$$

As in the classical case, the self-interaction term $V(\rho,\lambda)$ will be shut off, so we are looking at the pseudofree limit. The solutions to this constraint equation are

$$\psi(\mathbf{x},u,\phi,\Omega,\lambda)=e^{\pm i\epsilon\Omega}[2^{-j/2}\pi^{-1/4}(j)^{-1/2}e^{-9m^2\lambda^2/2}H_j(3m\lambda)]e^{i[n^2-4\gamma(\gamma-1)-1/4]^{1/2}\phi}[\alpha(x)u]^{1/2}J_n(\alpha(x)u)\ , \quad (4.67)$$

where j must be an integer, but n need not be and

$$\alpha^2(x)+\epsilon^2(x)+\delta^2(x)=0\ . \quad (4.68)$$

The constraint equation (4.66) is now represented by the constraint on the allowable constants (constants in $\beta_+,\beta_-,\Omega,\lambda$ space) given in Eq. (4.68) with the physical subspace given by Eq. (4.67). Equations (4.67) and (4.68) represent the solutions of the constraint equation near $\Omega\approx 0$. Let us compare this to the classical solutions. In the classical case near $\tau\to 0$, in the ϕ degree of freedom, the system is confined to a parabolic potential, while in the gravitational degrees of freedom the system is free. In the quantum case, we have a repulsive potential well in the gravitational degrees of freedom as in the case of the gravitational field alone with the system scattering off the well. The degree of freedom associated with the scalar field in the classical case exhibited simple harmonic motion. Here, of course, this degree of freedom is quantized and has bound states with discrete eigenvalues.

For the case where $\lambda^2 \gg \beta_+{}^2+\beta_-{}^2$, the constraint equation (4.65) becomes

$$\left[\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}+\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\Omega^2}-\frac{4\gamma(\gamma-1)}{\lambda^2}\right.$$
$$\left.-9m^2\lambda^2-U(\rho,\lambda)\right]\psi=0\ , \quad (4.69)$$

where we will again take the limit $\rho\to 0$, so $V(\rho,\lambda)=0$. The solutions to this are

$$\psi(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)=B_n(\lambda)e^{i[\gamma(x)\beta_+ +\delta(x)\beta_- -\rho(x)\Omega]}\ , \quad (4.70)$$

where

$$B_n(\lambda)=(12m)^{1/4}\left[\frac{\Gamma(n+1)}{\Gamma(a+n+1)}\right]^{1/2}(3m\lambda^2)^{(2a+1)/4}$$
$$\times\exp\left[\frac{-3m\lambda^2}{2}\right]L_n{}^a(3m\lambda^2)\ , \quad (4.71)$$

$$\alpha_n{}^2=6m(2n+a+\gamma+\tfrac{1}{2})\ , \quad (4.72)$$

where $a=\frac{1}{2}(16\gamma^2-15)^{1/2}$ and $L_n{}^a$ are Laguerre polynomials.[22]

Here the scalar degree of freedom has a a discrete eigenvalue spectrum with eigenfunctions localized in the positive and negative λ half-plane because of the infinite repulsive well at $\lambda=0$. The scalar field degree of freedom consists of bound states that have a discrete eigenvalue spectrum.

The constraint equation on the allowable eigenvalue spectrum for the degrees of freedom of the system at each spatial point is given by

$$\gamma^2(x)+\delta^2(x)-\rho^2(x)=\alpha_n{}^2(x)\ . \quad (4.73)$$

We can also look at the case for the super-Hamiltonian when $\Omega\to-\infty$ as we did classically in Eq. (4.13). We want the potential terms to drop out here also as they did in the classical case. Therefore we want $y(\lambda,\Omega)=0$ when $\Omega\to-\infty$. In this case we get, for h,

$$h=\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}+\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\Omega^2}-\frac{4\gamma(\gamma-1)}{\beta_+{}^2+\beta_-{}^2+\lambda^2-\Omega^2}\ . \quad (4.74)$$

This represents shutting off the interaction between the intrinsic time and the scalar field. It also represents the quantum constraint in the pseudofree limit for a massless self-interacting scalar field coupled to the gravitational field. This constraint in the $\beta_+,\beta_-,\Omega,\lambda$ space is "Lorentz invariant." The last term represents a singular potential well.

The constraint equation $h\psi(\mathbf{x},\beta_+,\beta_-,\Omega,\lambda)=0$ can be solved by making use of the four-dimensional Laplacian.[23,24] Because of the form of the repulsive well, we can divide "hyperspace minisuperspace" into two regions. First, when $\beta_+{}^2+\beta_-{}^2+\lambda^2>\Omega^2$, Eq. (4.74) becomes

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{3}{\rho}\frac{\partial}{\partial\rho}-\frac{L^2}{\rho^2}-\frac{4\gamma(\gamma-1)}{\rho^2}\right]A(\rho,\alpha,\theta,\phi)=0\ , \quad (4.75)$$

where $\rho^2=\beta_+{}^2+\beta_-{}^2+\lambda^2-\Omega^2\geq 0$, and L^2 represents the magnitude of an "angular momentum operator." When $\Omega^2>\beta_+{}^2+\beta_-{}^2+\lambda^2$, with $\rho^2=\Omega^2-\beta_+{}^2+\beta_-{}^2+\lambda^2\geq 0$, Eq. (4.75) still is valid. The solution to this equation are of the form

$$A\rho^{-1+[1+4\gamma(\gamma-1)+4J(J-1)]^{1/2}}B^J_{M\mu}(\alpha,\theta,\phi)\ , \quad (4.76)$$

where $B^J_{M\mu}(\alpha,\theta,\phi)$ is the four-dimensional analog of the three-dimensional spherical harmonics that satisfy

$$L^2B^J_{M\mu}(\alpha,\theta,\phi)=4J(J+1)B^J_{M\mu}(\alpha,\theta,\phi)\ . \quad (4.77)$$

In order to have well-defined solutions as $\rho\to 0$, it is re-

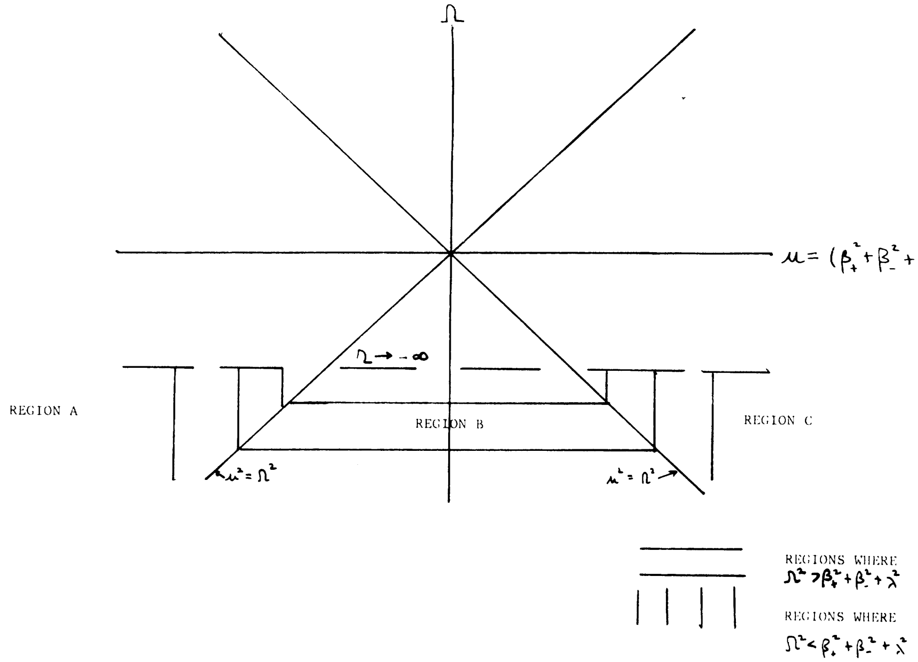

FIG. 2. Hyperspace minisuperspace near singularity.

quired that

$$J(J+1) \geq \gamma(1-\gamma) . \tag{4.78}$$

For both cases that we have discussed, when $J(J+1) > \gamma(1-\gamma)$ and $\rho \to 0$, $\psi \to 0$. Only when $J(J+1) = \gamma(1-\gamma)$ is $\psi \neq 0$ at the $\beta_+{}^2 + \beta_-{}^2 + \lambda^2 = \Omega^2$ barrier. Looking at Fig. 2 we see that the condition $\beta_+{}^2 + \beta_-{}^2 + \lambda^2 = \Omega^2$ represents a singular potential that extends infinitely far out from the origin and for $\Omega \to -\infty$ divides the space into three regions. Regions A and C are the regions where $\beta_+{}^2 + \beta_-{}^2 + \lambda^2 > \Omega^2$, while for region B, $\Omega^2 > \beta_+{}^2 + \beta_-{}^2 + \lambda^2$. For $J(J+1) > \gamma(1-\gamma)$, the physical subspaces defined by the constraint equation for these regions are disconnected. In other words, solutions cannot cross the asymptotes defined by the potential barrier and are constrained to evolve in their respective region. We expect that this produces scattering from the asymptotes. Only when $J(J+1) = \gamma(1-\gamma)$, which would give the lowest allowable "angular momentum states," are the regions connected.

V. CONCLUSION

We have looked at the ultralocal limit of the super-Hamiltonian in both the classical and quantum cases for the gravitational field coupled to the scalar field. We also developed the quantum version of the gravitational field by itself.

In the classical cases the ultralocal limit allowed us to consider that the dynamics at each spatial point are decoupled and we were able to treat the fields at each spatial point as finite degree of freedom systems. We then made use of Misner's minisuperspace formalism[18] to determine the effect of the super-Hamiltonian on "hyperspace minisuperspace," i.e., minisuperspace with an added degree of freedom for the scalar field. With the addition of the scalar field, the super-Hamiltonian becomes a constraint equation for a four-degrees-of-freedom system in the ultralocal limit. We looked at the constraint when $\tau \to -\infty$, which represents the singularity behavior of gravity coupled to a scalar field. In this limit, which also represents the strong-coupling limit of the theory, we found that the solutions were "quasi-Kasner"; i.e., they evolve along the null cone of the four-dimensional "hyperspace minisuperspace." For the gravitational field by itself it had been found that the solutions also evolve along the null cone, but in a three-dimensional minisuperspace. We found that the quasi-Kasner solutions in four-dimensional space have different properties from the Kasner solutions in three-dimensional space. The quasi-Kasner solution has the property that because of the added degree of freedom of the scalar field all three axes can expand. This is the same result that was obtained by Belinskii and Khalitnikov[21] by entirely different means. Our final solution for this limit was that of a collection of quasi-Kasner solutions, one at each spatial point.

When $\tau = 0$ we found that the matter terms of the scalar field have an effect on the allowable solutions of the super-Hamiltonian. Specifically, for the case where we had a massive scalar field that had no other self-interaction terms, the solutions in the scalar degree of freedom are those of a harmonic oscillator. We surmised that at larger τ, i.e., $\tau > 0$, the solutions are the same, but with this harmonic motion in the scalar degree of freedom becoming stronger.

We also developed quantum versions of these cases. We generalized Klauder's work[6–10] on the quantization of self-interacting ultralocal fields and reexamined Pilati's results[2] for the ultralocal quantization of the gravitational field by itself. Our results differ from this earlier work in that in order to have well-defined field operators we require an infinite repulsive well at the origin of the three-dimensional space underlying the ultralocal field. This produces solutions to the super-Hamiltonian which can be described as a scattering phenomenon where, far from the potential well, the solutions are Kasner-like. Our asymptotic states far from the scattering potential are the same as Pilati's.

Quantizing the ultralocal limit for the gravitational field coupled to a scalar field, we looked at the cases where $\Omega=0$, and where $\Omega \rightarrow -\infty$, which represents the strong-coupling limit. We found that near $\Omega \simeq 0$ we still had scattering from a repulsive well, but also solutions that have a discrete eigenvalue spectrum in the scalar degree of freedom where, in the classical case, we had a harmonic-oscillator-type solution.

Looking at the solutions to the Hamiltonian constraint when $\Omega \rightarrow -\infty$, we found that the repulsive well we introduced resulted in solutions that are constrained to regions defined by ${\beta_+}^2+{\beta_-}^2+\lambda^2>\Omega^2$ or $\Omega^2>{\beta_+}^2+{\beta_-}^2+\lambda^2$, and with the solutions unable to evolve through the potential barriers defined by ${\beta_+}^2+{\beta_-}^2+\lambda^2=\Omega^2$, except for the lowest "angular momentum state."

We have looked at the physical subspace defined by the quantum super-Hamiltonian in the ultralocal limit near the singularity and when matter effects become important. We would like to look at the physical subspace for the ultralocal limit in the general case for the interaction of the intrinsic time and the self-interacting scalar field. In order to do this, we must solve the equation for $y(\lambda,\Omega)$ appearing in the model function that gives this interaction in the λ-Ω space. This equation can be solved numerically.

An important aspect of these ultralocal theories is that they could serve as a starting point for a perturbation expansion which could give information that is complementary to that which is usually obtained in a weak-field perturbation theory. To develop a perturbation expansion around an ultralocal limit requires adding in the gradient terms. The method of doing this is currently unknown. However, Francisco and Pilati[20] have looked at this problem for the gravitational field by itself when the term $g\,{}^3R$ begins to become important in the super-Hamiltonian. They develop an approximation to the quantum evolution in this case that makes use of the results of the mixmaster universe[15] model of Misner for a homogeneous universe, which may be useful.

ACKNOWLEDGMENTS

The author would like to thank Dr. John Klauder and Dr. Ronald Mallett for many helpful discussions.

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