Higher-dimensional black holes in compactified space-times

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The Majumdar-Papapetrou solutions of the Einstein-Maxwell equations are generalized to asymptotically flat $(N+1)$-dimensional space-times. These new solutions are then used to construct black-hole solutions in which the extra dimensions are compactified on a torus. Conjectures about similar constructions for the vacuum Einstein equations are also discussed.

I. INTRODUCTION

Recently, a study of black-hole solutions to Einstein's equations in higher-dimensional space-time was made in Ref. 1. All of the solutions discussed there are in asymptotically flat Minkowski space-times, that is, solutions in which spatial infinity has the topology of S^{N-1}. A realistic model based on a higher-dimensional theory must compactify the extra dimensions beyond those of the $(3+1)$-dimensional space-time observed at low energies. Therefore the relevance of these solutions to the observed world may seem somewhat remote. In Ref. 1 it was conjectured that these solutions may be useful in determining the mechanism by which the extra dimensions are compactified. It was also pointed out that the higher-dimensional solutions would approximately describe the short-range geometry of a black hole placed at a specific point on the compact manifold. This paper provides examples of exact solutions describing such a black hole with the extra dimensions compactified on a torus. These examples are made using solutions of the Einstein-Maxwell equations which generalize the Majumdar-Papapetrou metrics to higher-dimensional space-times.

The paper is organized as follows. In Sec. II the Majumdar-Papapetrou metrics in $3+1$ dimensions are considered. (These will be referred to as MP metrics below.) Analogues of the MP metrics are found for $(N+1)$-dimensional space-times, and the properties of these new solutions are discussed. In Sec. III the above solutions are used to produce an exact solution of the Einstein-Maxwell solutions on a compactified space-time. In Sec. IV the extension of the above construction to the case of the vacuum Einstein equations is discussed. A toy model is produced describing a black hole in a compactified four-dimensional space-time. Section V concludes with a discussion of the possible physical implications of these solutions.

The conventions of this paper are those established in Ref. 1. In particular, the metric for $(N+1)$-dimensional flat space-time is $\eta_{\mu\nu}=\text{diag}(-1,+1,+1,\ldots)$. The Riemann tensor is defined by $(\nabla_\mu\nabla_\beta-\nabla_\beta\nabla_\mu)u^\rho=R^\rho{}_{\nu\mu\beta}u^\nu$, and the Ricci tensor and scalar are $R_{\mu\nu}=R^\alpha{}_{\mu\alpha\nu}$ and $R=g^{\mu\nu}R_{\mu\nu}$. The speed of light is set $c=1$, but Newton's constant G is explicitly retained in formulas. The Einstein-Maxwell action is

$$\int\left[\frac{1}{16\pi G}R-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right]\sqrt{-g}\,d^{N+1}x\ . \tag{1.1}$$

For an isolated gravitating system, the mass may be determined by considering the leading perturbations of the metric from flat space in the asymptotic region far from the system, $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ (Refs. 1 and 2). If the metric is chosen to satisfy the harmonic gauge condition $(h^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}h^\alpha{}_\alpha)_{,\nu}=0$, these leading perturbations take the form

$$\begin{aligned}
h_{00}&\approx\frac{16\pi G}{(N-1)A_{N-1}}\frac{M}{r^{N-2}}\ ,\\
h_{ij}&\approx\frac{16\pi G}{(N-1)(N-2)A_{N-1}}\frac{M}{r^{N-2}}\delta_{ij}\ ,\\
h_{0i}&=O(1/r^{N-1})\ .
\end{aligned}\tag{1.2}$$

Here A_N denotes the area of a unit N-sphere, and r is the radial coordinate defined as in flat space: $r=(x^{1^2}+x^{2^2}+\cdots)^{1/2}$. The charge of an isolated system is defined as

$$Q=\tfrac{1}{2}\int F^{\mu\nu}dS_{\mu\nu}\ , \tag{1.3}$$

where the integral is over any $(N-1)$-dimensional surface enclosing the system. Thus asymptotically the electric field around a point charge is

$$F^{0k}=\frac{Q}{A_{N-1}}\frac{x^k}{r^N}\ . \tag{1.4}$$

One may note that the present definition (1.3) differs from that given in Ref. 1 by a factor of A_{N-1}, which then makes its appearance in (1.4). This particular definition is chosen here on the rationale that while solutions must depend on the dimension of space-time, definitions should not. It is merely a matter of convenience, and one can simply replace Q by $A_{N-1}Q$ in all the formulas given here in order that they agree with the previous definition.

II. GENERALIZED MP METRICS

In $3+1$ dimensions it is well known that in the nonrelativistic limit a system of massive charged particles will be in static equilibrium if the charge-to-mass ratio is chosen to balance the gravitational attraction and the

Coulombic repulsion. Majumdar and Papapetrou[3] independently found solutions of the source-free Einstein-Maxwell equations corresponding to this balanced situation. Hartle and Hawking[4] considered the maximal analytic extension of these metrics. They found that the solution for a system of point charges may be interpreted as the metric for a corresponding system of extreme charged black holes, while solutions corresponding to extended sources contained naked singularities. In this section their discussion[4] is extended to space-times with $N+1$ dimensions.

First, the balance condition in higher-dimensional space-times will be derived for the conventions which were introduced above. The Newtonian gravitational force is found as usual by considering the geodesic equations in the nonrelativistic approximation. Using (1.2) this procedure yields

$$F_{\text{Newton}} = -\frac{8\pi G}{A_{N-1}}\frac{N-2}{N-1}\frac{M_1 M_2}{r^{N-1}} , \qquad (2.1)$$

where the overall minus sign indicates that the force is attractive. To determine the Coulomb force between a pair of point charges, one may insert their electric fields as given by (1.4) into the Maxwell energy density derived from the action (1.1). Integrating this energy density over all space yields two divergent self-energy terms, while the cross term in the expression yields the Coulombic interaction energy. Differentiating this term yields

$$F_{\text{Coulomb}} = \frac{1}{A_{N-1}}\frac{Q_1 Q_2}{r^{N-1}} , \qquad (2.2)$$

where the overall sign here indicates a repulsive force between two charges of the same sign. Now these forces will be balanced for any configuration of pointlike massive charges if all of the charges have the same sign, and the charge-to-mass ratio is

$$\frac{|Q|}{M} = \left[8\pi G\frac{N-2}{N-1}\right]^{1/2} . \qquad (2.3)$$

Just as in $3+1$ dimensions, this balance condition is also precisely the condition for a static charged black hole to have a degenerate horizon.[1]

Now solutions of the source-free Einstein-Maxwell equations corresponding to the above case are desired. Consider the following metric ansatz:

$$ds^2 = -U^{-2}(x^j)dt^2 + U^{2\lambda}(x^j)\delta_{ij}dx^i dx^j , \qquad (2.4)$$

where $\lambda=(N-2)^{-1}$. The x^j are Cartesian coordinates on a flat N-dimensional space, which will be called the background space as in Ref. 4. For the Maxwell form choose

$$A = \pm K U^{-1}(x^j)dt , \qquad (2.5)$$

where $K=[(1/8\pi G)(N-1)/(N-2)]^{1/2}$. Asymptotically U will be normalized to 1 so that the line element reduces to flat space there, but as a consequence the Maxwell potential tends to a pure gauge $\pm K\,dt$ instead of vanishing. The remarkable feature of this ansatz is that solving the source-free Einstein-Maxwell equations simply requires that $U(x^j)$ satisfy Laplace's equation for the flat background space:

$$\nabla^2 U = \frac{\partial^2 U}{\partial(x^1)^2} + \frac{\partial^2 U}{\partial(x^2)^2} + \cdots = 0 . \qquad (2.6)$$

These solutions then generalize the MP metrics to higher-dimensional space-times.

In three dimensions determining solutions of Laplace's equation is a well-studied problem,[5] and the results are trivially extended to N dimensions. Begin by considering monopole sources, in which case U takes the form

$$U(x^j) = 1 + \sum_a \frac{\mu_a}{r_a^{\,N-2}} ,$$
$$r_a = [(x^1-x_a^1)^2 + (x^2-x_a^2)^2 + \cdots]^{1/2} . \qquad (2.7)$$

Considering widely separated sources one may take a sphere enclosing each source individually, and using (1.2) and (1.3) the mass and charge corresponding to each source is found to be

$$M_a = \frac{(N-1)A_{N-1}}{8\pi G}\mu_a ,$$
$$Q_a = \mp\left[\frac{(N-1)(N-2)}{8\pi G}\right]^{1/2} A_{N-1}\mu_a . \qquad (2.8)$$

Therefore the charge-to-mass ratio for these solutions is indeed the desired equilibrium ratio given in (2.3).

As expected with only one source the metric reduces to the analogue of an extreme Reissner-Nordström black hole. This fact can be made more evident by first choosing standard spherical coordinates on the background space, and then transforming to a new radial coordinate with $\tilde{r}^{N-2} = r^{N-2} + \mu$. The resulting metric is just that presented in Ref. 1:

$$ds^2 = -\left[1-\frac{\mu}{\tilde{r}^{N-2}}\right]^2 dt^2 + \left[1-\frac{\mu}{\tilde{r}^{N-2}}\right]^{-2} + d\tilde{r}^2\tilde{r}^2 d\Omega^{N-1} , \qquad (2.9)$$

where $d\Omega^{N-1}$ is the line element on a unit $(N-1)$-sphere. The vector potential here is related to that given in Ref. 1 by a simple gauge transformation, $A \to A \pm K\,dt$. If μ (and hence the mass) is negative the metric singularity at $\tilde{r}^{N-2}=0$ is in fact a naked curvature singularity. Therefore in the following analysis only positive μ will be considered. As in $3+1$ dimensions[6] one is easily able to extend the coordinate patch covered by $(t,\tilde{r})$. One constructs the null coordinates

$$dv_{\pm} = dt \pm \left[1-\frac{\mu}{\tilde{r}^{N-2}}\right]^{-1} d\tilde{r} . \qquad (2.10)$$

Radial lines of constant $v_+(v_-)$ are infalling (outgoing) lightlike geodesics. Now given in terms of $(v_+,\tilde{r})$, the metric is singularity-free on the future horizon, while the $(v_-,\tilde{r})$ coordinates provide a regular extension across the past horizon. The interior regions contain timelike singularities at $\tilde{r}=0$, and each may be extended to two different exterior regions. The resulting global topology is essentially the same as for the extreme Reissner-Nordström black hole in $3+1$ dimensions,[6] and is illustrated by the Penrose diagram in Fig. 1.

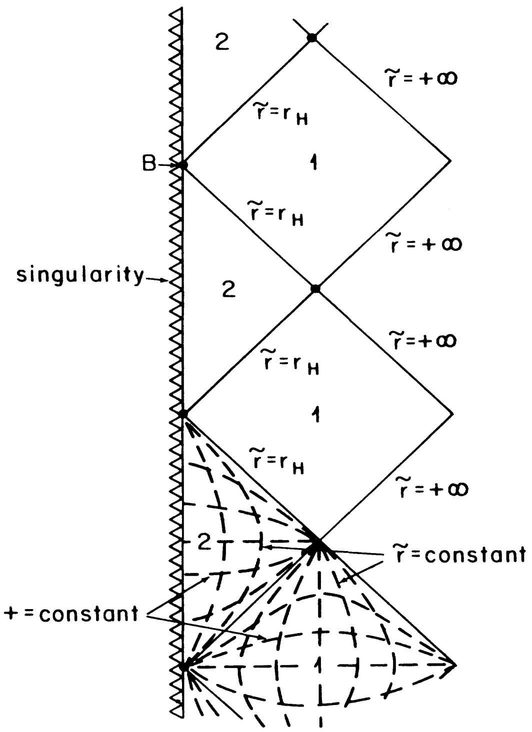

FIG. 1. The Penrose diagram for the analogue of an extreme Reissner-Nordström black hole. In $N+1$ dimensions each point on the diagram represents an $(N-1)$-sphere. There is a single degenerate horizon at $\tilde{r}=r_H=\mu^{1/(N-2)}$ and the singularity forms a timelike boundary of the interior regions labeled 2. The point labeled B on the horizon is an infinite proper distance from any point a finite coordinate distance away in the adjacent exterior region.

Next consider the case of two monopole sources in which U has the form

$$U=1+\frac{\mu_1}{r_1^{N-2}}+\frac{\mu_2}{r_2^{N-2}} . \tag{2.11}$$

Examining the components of the Maxwell field strength and the Riemann tensor in an orthonormal basis, one finds that they remain finite as $r_1 \to 0$ and in fact they are independent of μ_2 there. This indicates the singularity at $r_1=0$ is only a coordinate singularity. To construct explicitly a coordinate system which is nonsingular there, begin by transforming to polar coordinates about $r_1=0$. If μ_1 and μ_2 are separated by a in the background space and θ is the angle measured from the $\mu_1-\mu_2$ axis (see Fig. 2), then r_2 is given by

$$r_2{}^2=r_1{}^2+a^2-2ar_1\cos\theta . \tag{2.12}$$

Introducing a new radial coordinate given by $r_1=R^\lambda$, U takes the form

$$\begin{aligned} U&=1+\frac{\mu_1}{R}+\frac{\mu_2}{r_2(R,\theta)^{N-2}} , \\ &\underset{R\to 0}{\sim} 1+\frac{\mu_1}{R}+\frac{\mu_2}{a^{N-2}}+O(R^\lambda) . \end{aligned} \tag{2.13}$$

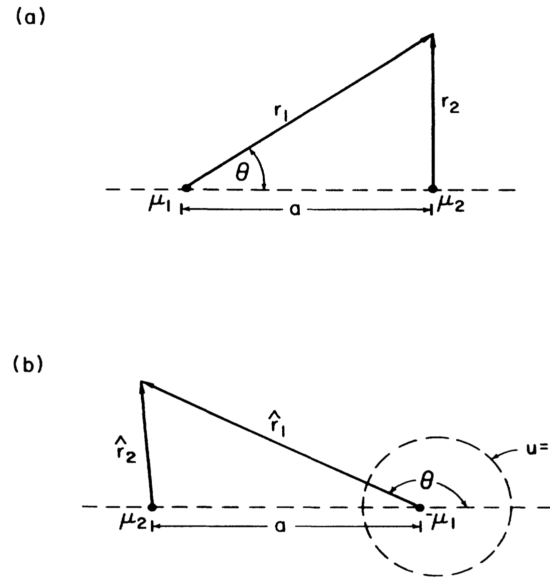

FIG. 2. The exterior background space for the MP solution with two black holes is shown in (a). The two monopole sources μ_1 and μ_2 are at the points $r_1=0$ and $r_2=0$, respectively. In the space-time these points are actually the horizons of the two black holes. Passing through $r_1=0$ one enters a region where the background space is as shown in (b). The surface $U=0$ in this space is actually a point in the space-time and the location of the curvature singularity.

Now construct coordinates

$$dv_\pm=dt\pm[V(R,\theta)dR+W(R,\theta)d\theta] , \tag{2.14}$$

where

$$\begin{aligned} V(R,\theta)&=\lambda R^{\lambda-1}U^{1+\lambda} \\ &=\lambda R^{\lambda-1}\left[1+\frac{\mu_1}{R}+\frac{\mu_2}{r_2(R,\theta)^{N-2}}\right]^{1+\lambda} . \end{aligned} \tag{2.15}$$

For (2.14) to be a valid coordinate transformation, the term added to dt must be an exact differential which requires $\partial W/\partial R=\partial V/\partial\theta$. Hence W is defined as

$$\begin{aligned} W&=\int \frac{\partial V}{\partial\theta}dR \\ &=\int \lambda(1+\lambda)R^{\lambda-1}U^\lambda\frac{\partial U}{\partial\theta}dR \\ &=-2\lambda(1+\lambda)(N-2)a\sin\theta\int\frac{R^{2\lambda-1}U^\lambda}{r_2(R,\theta)^N}dR , \end{aligned} \tag{2.16}$$

where the integration constant is chosen such that $W(R=0,\theta)=0$. One finds that near $R=0$

$$W\approx-\frac{2(1+\lambda)(N-2)}{a^{N-1}\mu_1{}^\lambda}R^\lambda\sin\theta+O(R^{2\lambda}) . \tag{2.17}$$

With these coordinates the metric becomes

$$ds^2 = -U^{-2}dv_\pm{}^2 \pm 2U^{-2}dv_\pm(V\,dR + W\,d\theta) - 2\frac{VW}{U^2}dR\,d\theta - \left[\frac{W}{U}\right]^2 d\theta^2 + U^{2\lambda}R^{2\lambda}(d\theta^2 + \sin^2\theta\,d\Omega^{N-2})$$

$$\underset{R\to 0}{\sim} \pm 2\lambda\mu_1{}^{\lambda-1}dv_\pm dR + \mu_1{}^{2\lambda}(d\theta^2 + \sin^2\theta\,d\Omega^{N-2}) + O(R^\lambda) \tag{2.18}$$

which is nonsingular at $R=0$. Therefore we may extend the manifold to negative values of R with

$$U = 1 - \frac{\mu_1}{|R|} + \frac{\mu_2}{r_2(|R|,\theta)^{N-2}},$$
$$V = \lambda |R|^{\lambda-1}|U|^{1+\lambda}, \tag{2.19}$$
$$W = \int \frac{\partial V}{\partial\theta}dR,$$

where

$$r_2{}^2 = |R|^{2\lambda} + a^2 + 2a|R|^\lambda\cos\theta. \tag{2.20}$$

This last step requires some extra comment. There is a problem arising from the fractional exponent $\lambda=(N-2)^{-1}$ occurring in various places. In fact no problem occurs for odd values of N. If R is negative, one has $r_1 = R^\lambda = -|R|^\lambda$ and (2.20) follows immediately from (2.11) by naively continuing r_1 to negative values. The problem occurs with even N, in which case $r_1{}^{N-2}=R$ is only valid for positive R. The same situation occurs for the single monopole solution discussed above. For even N the transformation $\tilde{r}^{N-2}=r^{N-2}+\mu$ is only valid outside of the horizon $\tilde{r}^{N-2}>\mu$. Constructing a sensible isotropic radial coordinate $\hat{r}$ inside the horizon requires $\tilde{r}^{N-2}=\mu-\hat{r}^{N-2}$. For even N in the metric (2.18) one encounters expressions with $R^{M\lambda}$ with even M, and for negative R one defines these expressions as $(R^M)^\lambda$. The exceptions to M being even are in r_2 and V, and one may check that (2.19) and (2.20) provide the correct interior metric for even, as well as odd N, by considering the continuity of the components of the electric field across the horizon in an orthonormal frame.

To facilitate the discussion one may introduce a set of background space coordinates in the interior region $R<0$ with

$$\hat{r}_1 = |R|^\lambda,\quad \hat{r}_2{}^2 = \hat{r}_1{}^2 + a^2 + 2a\hat{r}_1\cos\theta. \tag{2.21}$$

Now the metric and the vector potential have the form given in (2.4) and (2.5) with

$$U = 1 - \frac{\mu_1}{\hat{r}_1{}^{N-2}} + \frac{\mu_2}{\hat{r}_2{}^{N-2}}. \tag{2.22}$$

Here U has the functional form of a potential for a source $-\mu_1$ at $\hat{r}_1=0$, and another monopole with μ_2 at a distance a along the $\theta=\pi$ axis (see Fig. 2). Near $\hat{r}_1=0$, U has large negative values, and near $\hat{r}_2=0$ it has large positive values. Therefore there is an intermediate surface on which $U=0$ and the metric becomes singular. One sees that this metric singularity is actually a true singularity from the Maxwell field invariant

$$F_{\mu\nu}F^{\mu\nu} = -2K^2\left[\frac{\nabla_i U}{U^{1+\lambda}}\right]^2 \tag{2.23}$$

which diverges as $U\to 0$. In (2.22) U is normalized to 1 at asymptotic infinity. This value is then the dividing point for equipotential surfaces which enclose the point $\hat{r}_1=0$ and those which enclose $\hat{r}_2=0$. Therefore the singular surface $U=0$ encloses the point $\hat{r}_1=0$. It should be stressed that the background coordinates were introduced to simplify the examination of the interior region. The actual geometry of this region bears no resemblance to that of the background space illustrated in Fig. 2(b). For example, $\hat{r}_1=0$ and $t=$const does not label a point but a surface with a finite volume $A_{N-1}\mu_1{}^{(N-1)\lambda}$, while the surface with $t=$const and $U=0$ is actually a point with zero volume.

The transformation between v_+ and v_-

$$v_+ = v_- + 2\int V(R,\theta)dR \tag{2.24}$$

is divergent at $R=0$. (Note that this divergence is independent of θ, and hence is not determined by $\int W\,d\theta$.) Therefore the (v_+,R) and (v_-,R) coordinate patches only partially overlap, and the coordinate transformations (2.14) give inequivalent extensions of the exterior region. With v_+ the metric is regular on the future horizon, while it is singularity-free on the past horizon when expressed in terms of the coordinate v_-. Each interior region has a timelike singularity at $U=0$ and (2.24) provides two inequivalent extensions of these regions to different exterior regions. Hence ignoring the presence of the second source at $r_2=0$, the topology of the black hole at $r_1=0$ is analogous to that of the single monopole solution illustrated in Fig. 1. Examining $r_2=0$ one also finds a second black hole with a new set of extensions. In general one need not identify any of the regions found in extending the original coordinate patch, and one is lead to an infinite array of coordinate patches illustrated in Fig. 3. Of course one may choose to identify any equivalent regions to reduce this infinite array (see Ref. 4 for an especially simplifying choice).

For the general case of an arbitrary number of monopole sources (2.7), the above results will also hold. The metric (2.4) is singular at $r_a=0$, but a suitable choice of coordinates will reveal that the geometry is actually regular. Each source will correspond to a black hole, and as the number of sources increases the global topology of the maximally extended manifold will become more and more complex.

There are many solutions to Laplace's equation (2.6) other than the simple monopole solutions discussed above.

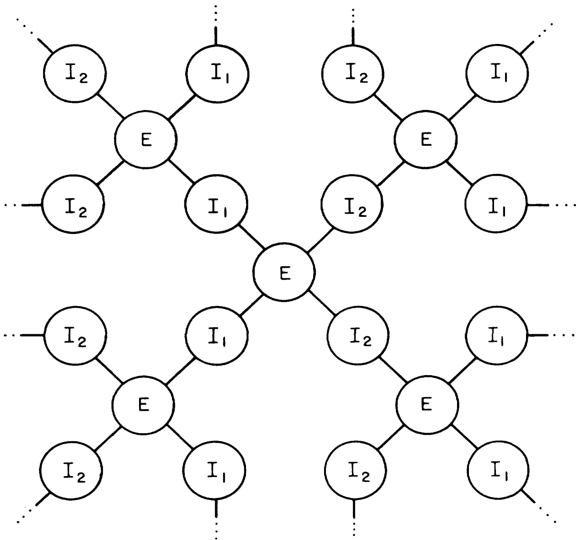

FIG. 3. The most general extension of the MP metric with two monopole sources. There are three types of regions: E is the exterior region, I_1 is the region inside the horizon at $r_1=0$, and I_2 is the interior region for the horizon at $r_2=0$.

In 3 + 1 dimensions, Hartle and Hawking[4] proved that for any of these alternative solutions the corresponding MP spaces contain naked singularities. The trivial extension of their analysis to $N+1$ dimensions is given here, and the same results are still found to hold.

Consider a solution U of (2.6) which approaches unity in an asymptotic region of the background space. Hence (2.4) provides an asymptotically flat MP metric which may be extended inward until U either diverges or becomes zero. In the latter case along some curve approaching $U=0$, the field invariant $F^2=-2K^2(\nabla_i U/U^{1+\lambda})^2$ diverges indicating the presence of a naked singularity. If U diverges it may do so either at a point or along some extended region in the N-dimensional background space. If there exists an equipotential surface which contains the point where U diverges (as would occur for a dipole source, for instance), then a naked singularity occurs by the following argument. Approaching the point along the specified surface, U is constant but $\nabla_i U$ must diverge, and therefore F^2 diverges at this point. Therefore to avoid a naked singularity, U must diverge when approached in any direction from a region of finite U. This may be expressed by saying that $|U|$ must be bounded below at the singularity. In the case of a point divergence one may extend a theorem from potential theory in three dimension[5] to show that U must then have the form

$$U(x^j)=\frac{\mu}{r^{N-1}}+f(x^j)\ , \qquad (2.25)$$

where μ is a constant, r is the background distance from the singularity, and $f(x^j)$ is a regular function at $r=0$. This is the case of monopole sources considered above.

The case of extended singularities is more interesting. Close to the singularity one may assume that U approaches the form for a "flat" source. Then it is easy to show for an M-dimensional source in an N-dimensional space, U diverges no faster than

$$U(x^j)\sim\frac{f(\hat{x}^k)}{\hat{r}^{N-M-2}}\ ,\quad N-M>2\ , \qquad (2.26)$$

where $\hat{x}^k$ are coordinates tangent to the singular surface and $\hat{r}$ is the orthogonal distance away from the surface in the background space. The behavior given in (2.26) leads to a divergence in F^2. In the special case of $N-M=2$, U has a logarithmic divergence which again leads to an infinity in F^2. (Of course for $N-M=1$, no divergence occurs.)

This result was somewhat unexpected. In 3 + 1 dimensions the result that extended sources lead to singularities is ensured by a theorem[7] requiring any spacelike cross section of a stationary horizon to have spherical topology. This property seems to continue in higher dimensions here, since the only nonsingular solutions have horizons whose topology is that of an $(N-1)$-sphere. It was noted in Ref. 1 that the proof of the theorem mentioned above cannot be applied in more than 3 + 1 dimensions. In fact simple examples can be constructed of objects with extended horizons by adding extra flat dimensions to a vacuum black-hole metric. Examining those solutions in the asymptotic regions, one finds that $h_{ab}=0$ for a or b being among the extra dimensions. A nonrelativistic source is inappropriate for such gravitational fields. One requires extra stress components $T_{ab}=-\delta_{ab}T_{00}/(N-M-1)$ parallel to the surface $(a,b>N-M)$ for an M-dimensional source in $N+1$ dimensions. This situation is similar to a cosmic-string solution[8] which is equivalent to an extra flat dimension added to a $(2+1)$-dimensional metric. Considering the case of a nonrelativistic source, the exact solution for the exterior of a static homogeneous dustlike M-dimensional source in an $(N+1)$-dimensional space-time (with $N>3$ and $N-M>2$) is

$$ds^2=-\left[\frac{1-\frac{\mu}{r^K}}{1+\frac{\mu}{r^K}}\right]^{2F}dt^2+\left[1-\frac{\mu}{r^K}\right]^{2G}\left[1+\frac{\mu}{r^K}\right]^{2H}(dr^2+r^2d\Omega^{N-M-1})+\left[\frac{1-\frac{\mu}{r^K}}{1+\frac{\mu}{r^K}}\right]^{-2F/(N-2)}\delta_{ab}dy^a dy^b\ , \qquad (2.27)$$

where y^a are the M tangential coordinates and

$$F=\left[\frac{N-M-1}{N-M-2}\frac{N-2}{N-1}\right]^{1/2},$$
$$G=\frac{1}{N-M-2}-\frac{F}{N-2}, \qquad (2.28)$$
$$H=\frac{1}{N-M-2}+\frac{F}{N-2}, \quad K=N-M-2 .$$

These solutions for $M>0$ are all singular if continued to $r^K=\mu$. The MP metrics for extended sources are similarly singular since they are exterior solutions for charged dustlike sources with no extra stresses.[9] It is possible to form nontrivial extended solutions with matter fields by considering the Riessner-Nordström solution with equal electric and magnetic charges. This solution has $F_{\mu\nu}F^{\mu\nu}=0$ and therefore simply adding extra flat directions produces a valid solution for the higher-dimensional Einstein-Maxwell equations containing an extended horizon with nontrivial "hair."

As a final comment on these solutions, recall that in 3 + 1 dimensions, the Maxwell field strength and its dual are both two-forms. Therefore a duality rotation may be applied to the MP solutions leading to a static solution with massive sources carrying both electric and magnetic charges. As discussed in Ref. 1 in higher $(N+1)$-dimensional spaces the dual to the Maxwell field is an $(N-1)$-form field. If one is willing to consider such a field, duality rotations may be performed to the generalized MP solutions to yield static solutions with massive sources with both Maxwell electric charges, and generalized $(N+1)$-dimensional magnetic charges. Such $(N-1)$-form fields could also be used to extend the above construction for solutions containing extended horizons with a nontrivial flux of matter fields.

COMPACTIFIED SOLUTIONS

Now the construction of compactified solutions from the higher-dimensional MP metrics is discussed. First the simplest example with a black hole in five dimensions will be considered. The construction is relatively simple. Consider an open five-dimensional MP metric with an infinite line of identical monopole sources μ equally spaced with separation a. Then U becomes

$$U=1+\sum_{n=-\infty}^{+\infty}\frac{\mu}{r_n{}^2},$$
$$r_n{}^2=x^2+y^2+z^2+(w+na)^2 . \qquad (3.1)$$

The resulting metric (2.4) and Maxwell vector potential (2.5) are periodic in the coordinate w with a period a. Therefore one may simply choose to identify the points with w and $w+a$. Spatial infinity for the new space-time then has the compactified topology $S^2\times S^1$. Therefore the new solution may be described as a charged five-dimensional black hole in a Kaluza-Klein background $M^4\times S^1$. One may regard this construction as finding the MP solution for a background space $R^3\times[-a/2,+a/2]$ with periodic boundary conditions by placing image sources in the intervals $[na-a/2,na+a/2]$. The expression (3.1) for U may be summed to the closed form

$$U=1+\frac{\pi\mu}{a\rho}\frac{\sinh 2\pi\frac{\rho}{a}}{\cosh 2\pi\frac{\rho}{a}-\cos 2\pi\frac{w}{a}}$$
$$=1+\frac{\pi\mu}{a\rho}\frac{\sinh\pi\frac{\rho}{a}\cosh\pi\frac{\rho}{a}}{\sin^2\pi\frac{w}{a}+\sinh^2\pi\frac{\rho}{a}}, \qquad (3.2)$$

where $\rho^2=x^2+y^2+z^2$. These expressions are still rather unwieldy, but explicitly display the periodic dependence on w. For $\rho,w\ll a$, the latter expression in (3.2) is easily seen to simplify to

$$U=1+\frac{\mu}{\rho^2+w^2}+\frac{\pi^2}{3}\frac{\mu}{a^2}+O\left[\frac{\rho^2}{a^2},\frac{w^2}{a^2}\right]. \qquad (3.3)$$

This expression is the single monopole solution for five-dimensional asymptotically flat Minkowski space plus a constant which is essentially the potential at $r_0=0$ due to all of the image sources, and terms which vanish as $r_0\to 0$. This reduction explicitly demonstrates that the short-range geometry is described by the solutions constructed in noncompactified space-times. One can check that the surface gravity, the area of the horizon, and the topology are not affected by the external potential. The topology of the region $r_0<a$ is similar to that of an extreme Reissner-Nordström black hole as illustrated in Fig. 1. The topology of the asymptotic exterior region differs though since w is periodic. Define two quantities

$$M_5=\frac{3\pi}{4G}\mu, \quad |Q_5|=\left[\frac{3\pi}{G}\right]^{1/2}\pi\mu, \qquad (3.4)$$

which are the mass and the magnitude of the charge for the single source in (3.3) if it occurred in an open five-dimensional space.

Standard Kaluza-Klein theory[10] begins with the Einstein action in five dimensions and expands about a background space-time of $M^4\times S^1$. The massless modes observed in four dimensions are those which are independent of w and include the graviton arising from the four-dimensional metric, a U(1) gauge field from extra off-diagonal components of the metric, and a scalar from the last diagonal component of the metric. It should be stressed that the action (1.1) considered here contains an elementary Maxwell field. Therefore the gauge field in the present solution is totally unrelated to the Kaluza-Klein gauge field. The higher-dimensional Einstein-Maxwell action may seem less aesthetic than the pure Einstein action usually considered in Kaluza-Klein theories, but it appears that elementary gauge fields will be necessary to produce a realistic grand unified theory.[11] In the solutions considered here the Kaluza-Klein scalar or the dilaton does play a nontrivial role since $\exp(2\phi)=g_{ww}=U^{2\lambda}$. The action (1.1), when compactified on $M^4\times S^1$ and restricted to massless fields as is appropriate for the asymptotic region of the black-hole solution, becomes

$$\int e^{\phi}\left[\frac{1}{16\pi G_4}R^{(4)}-\frac{1}{4}F^{(4)\mu\nu}F^{(4)}_{\mu\nu}\right]\left[-g^{(4)}\right]^{1/2}d^4x\ , \quad (3.5)$$

where $G_4=G/a$ is the effective four-dimensional Newton's constant and the gauge field has also been rescaled $A^{(4)}=\sqrt{a}\,A$. $R^{(4)}$ is the Ricci scalar calculated with the four-dimensional metric taken as the components of $g^{\mu\nu}$ with $\mu,\nu=0,1,2,3$. In (3.5), terms involving both the Kaluza-Klein gauge field and the fifth component of the Maxwell potential have been neglected. The unusual couplings of the dilaton in the reduced action lead to some strange properties of Kaluza-Klein solutions.[12] It is often convenient to redefine the four-dimensional metric by a conformal transformation involving the dilaton which removes the dilaton coupling to the Ricci scalar and diagonalizes the kinetic terms of the graviton and the dilaton. This transformation is not performed here.

The first expression in (3.2) is useful to examine the asymptotic limit $\rho\rightarrow\infty$:

$$U\rightarrow 1+\frac{\pi\mu}{a\rho}+\frac{2\pi\mu}{a\rho}\exp\left[-2\pi\frac{\rho}{a}\right]\cos 2\pi\frac{w}{a}+\cdots\ . \quad (3.6)$$

Examining the asymptotic properties of the metric and Maxwell field one finds that

$$M_{\text{grav}}=\frac{\pi\mu}{aG_4}\ ,\quad |Q|=\left[\frac{3\pi}{G_4}\right]^{1/2}\frac{\pi\mu}{a}\ . \quad (3.7)$$

M_{grav} is the mass determined by comparing g_{00} with (1.2), and may be called the gravitational mass because it appears in the Newtonian force law derived by examining nonrelativistic geodesics. One may also consider computing the inertial mass of the solution using a gravitational stress-energy pseudotensor[2] of the five-dimensional metric to find $M_{\text{inert}}=3\pi\mu/4aG_4$. That these two masses do not coincide is not a violation of the equivalence principle, but rather a result of the long-range dilaton field and its unusual coupling in (3.5) (Ref. 12). Note though that the inertial mass coincides with M_5 given in (3.4) and that $|Q|=|Q_5|/\sqrt{a}$.

Considering $|Q|/M_{\text{grav}}$ for (3.7) one finds that it differs from the balance ratio (2.3) appropriate for $N=3$ by a factor of $\sqrt{3}/2$. This may seem curious since it is expected that any number of such objects will remain in static equilibrium. A solution appropriate to describe such a situation could be found by simply extending the above construction to include several parallel lines of sources with equal spacings in the five-dimensional background space-time. The disagreement with (2.3) is simply a result of the force laws (2.1) and (2.2) for point sources in 3 + 1 dimensions being inappropriate. One should instead calculate the force per unit length for parallel line sources in 4 + 1 dimensions. In that case one finds that Coulomb's law is actually unmodified in this particular case but the effective Newton's law is replaced by

$$F_{\text{Newton}}=-\frac{4}{3}\frac{G_4M_1M_2}{r}\ . \quad (3.8)$$

This describes the effective long-range interaction between two point masses in $M^4\times S^1$, but breaks down when $r\approx a$. The factor 4/3 in (3.8) leads to the discrepancy between (3.7) and (2.3). This factor also gives an explanation of why M_{grav} is larger than M_{inert} by exactly the same factor. Of course such factors would simply be absorbed into the definitions of mass and charge of an observer in the low-energy Kaluza-Klein world, and no discrepancies would be apparent to him. One may also note at this point that an uncharged body such as a four-dimensional Schwarzschild black hole is expected to attract the objects described by the new solutions above. This expectation arises from considering the behavior of the corresponding objects in an uncompactified five-dimensional space.

In (3.6) the next-to-leading contribution is explicitly shown. From this term one can see that the higher modes of the metric, dilaton, and gauge fields are exponentially suppressed as expected for the interpretation of these modes as particles with mass $2\pi\hbar/a$ in the usual low-energy expansion. Typically in examining Kaluza-Klein theory such modes are ignored since the masses are typically of the order of the Planck mass and are hence deemed irrelevant for a discussion of low-energy field theory. Despite the potential involving such massive modes, the mass of the black-hole solution is neither quantized nor necessarily large. The mass (3.7) is determined by the free parameter μ which may be made arbitrarily small. This freedom reflects the fact that one is investigating a classical solution. For a realistic theory, the charges of the matter fields will be quantized and hence the charge of a black hole resulting from the gravitational collapse of such matter will also be quantized. Therefore a "physical" black hole may be expected to have a charge and mass on the order of the Planck scale. Perhaps equally interesting is that the parameter μ can be made arbitrarily large without producing singularities or affecting the asymptotic scale of the compactified dimension.

One may also note that the interpretation of the higher modes as massive particles is only applicable in an $M^4\times S^1$ background or asymptotically in the black-hole space-times. In the solutions given here the size of the compact dimension grows as one approaches the black hole. In fact for $t=$const and $\rho=0$, the proper distance from $w=0$ to $w=a$ is infinite. This divergence is related to the well-known fact that for the extreme Reissner-Nordström solution, the point labeled B in Fig. 1 is an infinite proper distance from points in the exterior adjacent region (which holds for arbitrary space-time dimensions).

An essential difference between the solution considered here and previous nontrivial Kaluza-Klein metrics[12–14] is that it is only independent of w asyptotically. Alternatively one may say the $\partial/\partial w$ is only an asymptotic Killing vector. Presumably this lack of symmetry could be detected in scattering particles off of such an object. The lack of w symmetry would result in the excitation of nontrivial modes about the compact direction in the scattering process which would be observed as the production of particles with Kaluza-Klein charge. This is not a very practical experiment though because energy conservation would require that for low-energy scattering these excited modes be bound states, and so these particles would simply appear to be absorbed by the black hole.

Since asymptotically the higher modes are exponentially suppressed, one is left with a certain amount of nonuniqueness. In 3 + 1 dimensions asymptotically flat source-free static solutions are characterized by the asymptotic invariants such as mass and charge. In an asymptotically $M^4 \times S^1$ spaces, solutions are characterized by the mass, and the vector, and scalar charges[14] (of both the Kaluza-Klein and Maxwell fields in this case). If one now also allows the w invariance to be broken new solutions must be considered which only differ from those above by exponentially suppressed factors. Therefore the surface integrals are insufficient to characterize solutions in asymptotically $M^4 \times S^1$ spaces. For example, the solutions considered here would be indistinguishable at large radii from a solution with exact w symmetry produced by compactifying the five-dimensional MP solution for an appropriate line source. In this example though, the latter metric will be singular at $\rho = 0$.

The construction described at the beginning of the section may be extended to higher dimensional spaces. Begin with a background space R^N and place identical monopoles on an $(N-3)$-dimensional lattice to produce the potential

$$U = 1 + \sum_{n_i=-\infty}^{+\infty} \frac{\mu}{r(n_i)^{N-2}} , \qquad r(n_i)^2 = x^2 + y^2 + z^2 + |\mathbf{w} + n_i \mathbf{e}_i|^2 , \tag{3.9}$$

where $\mathbf{w}$ is the position vector in the $(N-3)$-dimensional background subspace and $\mathbf{e}_i$ are $N-3$ basis vectors for the lattice. Identifying points up to lattice vectors $n_i \mathbf{e}_i$ then yields a solution in a compactified background $M^4 \times T^{N-3}$. This procedure may be generalized in the obvious manner if one wishes to place more than one source in the basic unit cell of the lattice. The effective four-dimensional Newton's constant is $G_4 = G/V$ and the Maxwell potential is rescaled as $A^{(4)} = \sqrt{V}\, A$ where in these expressions V is the asymptotic volume of the compactified torus.

One can determine the leading long-range behavior of the potential (3.9) by replacing the discrete sum by an integral:

$$U \approx 1 + \frac{\mu}{V} \int \frac{d\mathbf{w}}{(\rho^2 + |\mathbf{w}|^2)^{(N-2)/2}} = 1 + \frac{\mu}{V} \frac{\pi^{(N-2)/2}}{\Gamma((N-2)/2)} \frac{1}{\rho} , \tag{3.10}$$

where $\rho^2 = x^2 + y^2 + z^2$. Deriving the mass and charge of the compactified black hole from this asymptotic formula one finds

$$\frac{|Q|}{M} = (4\pi G_4)^{1/2} \left[\frac{1}{2} \frac{N-1}{N-2} \right]^{1/2} = \left[8\pi G_M \frac{M-2}{M-1} \right]^{1/2} \left[\frac{M-2}{M-1} \frac{N-1}{N-2} \right]^{1/2} . \tag{3.11}$$

The latter expression above is for the case where an $(N+1)$-dimensional theory is reduced to background with $M+1$ noncompact dimensions. Hence in all of the compactified cases $|Q|/M$ is less than the balance ratio given in (2.3) and yet these object will remain in static equilibrium. This problem is again resolved by the fact that for the units adopted here the effective Newton's law (2.1) and Coulomb's law (2.2) are modified by overall numerical factors in the compactified spaces.

At short range $|\mathbf{w}| \ll |\mathbf{e}_i|$ these black-hole solutions will appear very similar to the solution for a single source in N dimensions. Quantities such as the surface gravity and the area of the horizon are unchanged in the compactified space. Near the horizon the topology is also similar to that of the analogue of the extreme Reissner-Nordström black hole as illustrated in Fig. 1, but of course the asymptotic region is more complicated because some of the directions are compact.

IV. VACUUM SOLUTIONS

If one wishes to consider the effects of black holes in quantum gravity, it may be more appropriate to consider vacuum solutions for Einstein's equations. Extending the construction of the previous section would call for solving for an infinite set of black holes on a regular lattice in $N+1$ dimensions. Such a proposition at first sight may seem unreasonable since without the Maxwell field there is nothing to balance the gravitational attraction of the black holes, and so it seem unlikely that nonsingular vacuum solutions would exist. Therefore first a toy model will be considered in which a four-dimensional space is compactified down to three dimensions. Then it will be argued that the existence of this solution suggests that there should also be nonsingular solutions in higher dimensions.

Begin by considering static axisymmetric solutions of Einstein's equations in 3 + 1 dimensions. In this case the metric may be reduced to the form[15]

$$ds^2 = -e^{2U} dt^2 + e^{2(k-U)}(dr^2 + dz^2) + e^{-2U} r^2 d\phi^2 . \tag{4.1}$$

For this form of the metric, solving Einstein's equations requires that U be an axisymmetric solution of Laplace's equation:[15]

$$\nabla^2 U(r,z) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right] U = 0 . \tag{4.2}$$

Given a solution for U, k may be solved by quadratures. The Schwarzschild solution results in choosing the Newtonian potential for a rod of length $2GM$ and mass M (Ref. 16). By considering the sum of potentials for a number of nonoverlapping rods, one finds solutions describing several of black holes on a common axis. These solutions are given by[16]

$$U = \sum_{i=1}^{l} U_i , \quad k = \sum_{i=1}^{l} \sum_{j=1}^{l} k_{ij} , \tag{4.3}$$

where

$$U_i=\tfrac{1}{2}\ln\frac{\rho_i+\widetilde{\rho}_i-2GM_i}{\rho_i+\widetilde{\rho}_i+2GM_i}\ ,\quad k_{ij}=k_{ji}=\frac{1}{4}\ln\frac{\rho_i\widetilde{\rho}_j+(z-z_i-GM_i)(z-z_j+GM_j)+r^2}{\rho_i\rho_j+(z-z_i-GM_i)(z-z_j-GM_j)+r^2}$$
$$+\frac{1}{4}\ln\frac{\widetilde{\rho}_i\rho_j+(z-z_i+GM_i)(z-z_j-GM_j)+r^2}{\widetilde{\rho}_i\widetilde{\rho}_j+(z-z_i+GM_i)(z-z_j+GM_j)+r^2}\ , \tag{4.4}$$

with

$$\rho^i=[r^2+(z-z_i-GM_i)^2]^{1/2},\quad \widetilde{\rho}_i=[r^2+(z-z_i+GM_i)^2]^{1/2}\ . \tag{4.5}$$

In these solutions the constants of integration have been chosen to give an asymptotically flat space and regular horizons. The metric is singular on the axis at the rods but if the mass of the rod is positive this is actually only a coordinate singularity indicating the presence of a horizon. This singularity can then be removed by an appropriate coordinate transformation.[16]

The solution though does have coordinate singularities on axis in between the rods.[17] On the axis one has

$$k_{ij}\,|_{r=0}=\begin{cases}0 & \text{if } z_i\le z_j \text{ and } z<z_i+GM_i \text{ or } z>z_j+GM_j\ ,\\ k_{ij}^0=\tfrac{1}{2}\ln\left|\dfrac{(z_i-z_j-GM_i-GM_j)(z_i-z_j+GM_i+GM_j)}{(z_i-z_j+GM_i-GM_j)(z_i-z_j-GM_i+GM_j)}\right| & \text{if } z_i<z_j \text{ and } z_i+GM_i<z<z_j-GM_j\ .\end{cases} \tag{4.6}$$

The k_{ij}^0 are directly proportional to the Newtonian force between the ith and jth rods. Therefore on the axis between two rods, k is proportional to the total Newtonian force between all of the rods below the given point and those above that point. Moving along the axis as a rod is crossed, k changes by the total force on that rod. To avoid a conical singularity on the axis one must identify the angular coordinate ϕ modulo $2\pi\exp(k)\,|_{r=0}$. Therefore if ϕ is to have period 2π so that the asymptotic regions are nonsingular, then the regions between the rods will all have conical singularities.[17] These conical singularities can in some sense be interpreted as struts needed to balance the gravitational attraction between the black holes.[17]

Now consider the above solution but instead of a finite number of black holes consider the potential for an infinite set of rods with identical masses $M_n=M$ centered at $z_n=na$. This solution suffers from a number of defects but these problems arise only because of the low dimension of the space being considered. Begin by considering the asymptotic behavior of the potential U. For large r, one sees that the sum in (4.3) diverges, but in replacing the sum by an integral, one is able to extract the asymptotic potential of an infinite line source with linear density M/a:

$$\begin{aligned}U\,|_{r\to\infty}&\approx-\sum_{n=-\infty}^{+\infty}\frac{GM}{(r^2+n^2a^2)^{1/2}}\\&\approx-\frac{2GM}{a}\int_0^{+\infty}\frac{dl}{(r^2+l^2)^{1/2}}\\&\approx\frac{2GM}{a}\ln|r|-D\ ,\end{aligned} \tag{4.7}$$

where D is a divergent constant. Even ignoring D, the space-time will not be asymptotically flat. This fact is well known for the exterior solutions of an infinite dust cylinder in 3 + 1 dimensions.[18] As mentioned above, the constants of integration in (4.3) were chosen to produce an asymptotically flat space but since the space in question is not asymptotically flat in any event there is no reason not to make a new choice. Redefine

$$U=\sum_{n=-\infty}^{+\infty}U_n-\sum_{n=1}^{+\infty}\ln\frac{1-\dfrac{GM}{na}}{1+\dfrac{GM}{na}}\ , \tag{4.8}$$

where U_n is defined as in (4.4). The new potential is (4.8) is finite asymptotically and allows the space-time to be continued through the horizon. Note that in higher dimensions similar problems would not occur. The potential for an extended source in higher dimensions vanishes asymptotically, and so the metric is asymptotically flat as is demonstrated explicitly by (2.27).

In the present solution there is no problem of k on the axis changing as a rod is crossed. This is because the total Newtonian force on each rod is exactly zero since with an infinite number of rods there is reflection symmetry about the center of each of the rods. In evaluating $k\,|_{r=0}$ as given by (4.3) and (4.4), one determines the total force of half the rods on the other half of the rods, and this quantity diverges. This problem is not fatal though since the divergent quantity is only a constant and may be eliminated again by a new choice of the constant of integration. Redefine

$$\begin{aligned}k=&\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}k_{nm}\\&-\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\ln\left[1-\frac{4G^2M^2}{(n+m+1)^2a^2}\right]\ ,\end{aligned} \tag{4.9}$$

where (4.4) still defines k_{nm}. With this definition k vanishes on the axis between rods and so ϕ may simply be identified with period 2π. Therefore the problem of conical singularities has been solved. With (4.9), k is also nonsingular asymptotically and on the horizons. Note that the sum of forces considered above would not diverge in a higher number of dimensions.

Now identifying points with z and $z+a$, the solution becomes the metric for a four-dimensional black hole in a compactified space-time. The exterior manifold may be extended across past and future horizons, and the interior regions each have a spacelike singularity. Hence in the vicinity of the horizon the geometry is similar to that of a Schwarzschild black hole. Asymptotically the geometry is more complicated because z is periodic and the metric does not approach flat space. This latter property is simply due to the low dimension of the space-time. The existence of this solution in $3+1$ dimensions which is free of naked singularities suggests that similar compactified vacuum solutions in higher dimensions should exist. The key point is essentially that in an infinite array of masses the gravitational forces will balance. Of course in the Newtonian problem this is not a stable equilibrium and a slight perturbation upsetting the symmetry of the array would result in the collapse of the system. In the compactified case the symmetry of the image sources is imposed by the periodic boundary conditions. Unfortunately the four-dimensional solution has no obvious extension to higher dimensions. The essential problem is that in the above case surfaces of constant r are cylindrical with no intrinsic curvature, but in higher dimensions the corresponding Einstein equations are complicated by intrinsic curvature of such surfaces.

This section will be concluded with some conjectures about the properties of higher-dimensional vacuum black-hole solutions in compactified space-times. The basic conjecture is that asymptotically the metric for a black hole compactified in a torus will appear like the dustlike M-dimensional source in $N+1=M+4$ dimensions given in (2.27) with y^a now being periodic coordinates on an M-dimensional torus. (In such a solution δ_{ab} may be replaced by some other constant g_{ab} to describe a more general torus.) This form occurs because asymptotically the higher modes about the compact dimensions will be exponentially suppressed as seen in (3.6). Note the four-dimensional solution discussed above is singular if $GM>a/2$ so that the rods touch and one has an infinite line source. Therefore it is likely that in the nonsingular solutions of interest G_4M will be smaller than the dimensions of the compact manifold. (This situation differs from the compact MP solutions where the mass parameter was totally arbitrary.) The leading asymptotic perturbations of the compactified black-hole metric will be in the diagonal components

$$\begin{aligned} h_{00}&=4G_4\frac{N-2}{N-1}\frac{M}{r}\ ,\\ h_{ij}&=\frac{4G_4}{N-1}\frac{M}{r}\delta_{ij}\ ,\\ h_{ab}&=\frac{4G_4}{N-1}\frac{M}{r}\delta_{ab}\ , \end{aligned} \tag{4.10}$$

where $G_4=G/V$ and V is the asymptotic volume of the compact manifold. The properties of such a solution can be compared with the exact black-hole solution constructed as the direct product of a four-dimensional Schwarzschild black hole with a constant torus (or any Ricci-flat manifold for that matter).

The effective gravitational mass from h_{00} in (4.10) is $M_{\text{grav}}=2M(N-2)/(N-1)$. As discussed in Sec. III this result is greater than M as a result of the modification to the effective Newtonian force law due the image sources. The asymptotic metric is also sufficient to calculate the inertial mass with an stress-energy pseudotensor,[2] and the result is $M_{\text{inert}}=M$. For the Schwarzschild solution, one has simply $M_{\text{grav}}=M_{\text{inert}}=M$. As discussed in Sec. II though, the effective source for the asymptotic fields in this solution would be nonrelativistic with stresses along the compact dimensions of the order of the mass density. Therefore the effective Newtonian force law calculated for the first case would be inapplicable here.

Assuming that the properties of the horizon are not affected in the compactified solution, one may use the surface gravity and "area" determined for an $(N+1)$-dimensional Schwarzschild-type black hole with mass M[1]:

$$\begin{aligned} \kappa_c&=\frac{N-2}{2}\left[\frac{16\pi G_4}{(N-1)A_{N-1}}VM\right]^{-1/(N-2)}\ ,\\ \mathcal{A}_c&=A_{N-1}\left[\frac{16\pi G_4}{(N-1)A_{N-1}}VM\right]^{(N-1)/(N-2)}\ . \end{aligned} \tag{4.11}$$

(Note that the "area" of the horizon actually has the dimensions of length^{N-1}.) These quantities may be compared to the Schwarzschild case where the usual four-dimensional results apply:

$$\kappa_s=\frac{1}{4G_4M}\ ,\quad \mathcal{A}_s=16\pi G_4{}^2VM^2\ . \tag{4.12}$$

One may note the fractional power of M appearing in (4.11) as well as the dependence on V. A black hole appears as a heat bath with a temperature proportional to the surface gravity $T=\hbar\kappa/(2\pi k)$ (Ref. 19) (where $\hbar$ is Planck's constant and k is Boltzmann's constant). The luminosity for a radiating body with temperature T in $N+1$ dimensions is

$$L=c\sigma_N\mathcal{A}T^{N+1}\ , \tag{4.13}$$

where $\mathcal{A}$ is the body's surface "area," σ_N is the Stefan-Boltzmann constant appropriate for $N+1$ dimensions, and c is a constant to take account of the number of species of particles that are being radiated. The luminosity gives the rate at which the mass of the black hole decreases. For a compactified black hole with an initial mass M, one finds that the lifetime of the black hole is proportional to $(G_4M)^{N/(N-2)}$. In the Schwarzschild case if G_4M is initially much greater than the typical dimension of the compact space, modes with nontrivial momenta about the compact dimensions will not be excited for most of the black hole's lifetime. In this case the standard four-dimensional result applies and one finds that the lifetime is proportional to $(G_4M)^3$. If G_4M is less than the compact dimensions, the temperature is high enough to excite modes about these directions and the black hole evaporates much faster with a lifetime proportional to $(G_4M)^N$. However if the compactification scale is of the order of the Planck mass, the black hole will not contain enough energy to excite these modes and the evaporation will continue as in four dimensions. The compactified black hole is able to excite such modes because the proper volume of the compact space increases in the vicinity of the black hole. Energy conservation though will dictate that the excited modes must be bound states. For a massless field such modes have stress components along the spatial momentum vector, and hence along the compact dimensions, comparable to the energy density. Perhaps then the fate of this black hole is that it transforms to the Schwarzschild case after all.

Following the prescription for Euclidean quantum gravity,[20] one may Euclideanize these spaces and identify the Euclidean time coordinate with a period $\beta = 2\pi/\kappa$ in order that the resulting spaces be nonsingular. One may then compare the Euclidean action of the two cases above[20]

$$I = -\frac{1}{8\pi G}\int (K - K^0)\sqrt{g}\, d^N x \ , \tag{4.14}$$

where the integral is made over a surface of constant r in the limit $r \rightarrow \infty$. K is the trace of second fundamental form on this boundary and K^0 is that of the same metric embedded in flat space. The usual Einstein volume term has been dropped from (4.14) since vacuum solutions are being considered and hence the Ricci scalar vanishes everywhere. The result for the compactified black hole is $I_c = \beta_c M/(N-1)$, while for the Schwarzschild case the standard result holds: $I_s = \beta_s$ M/2. If one compares both cases with a fixed period or temperature, then $\beta_s = \beta_c$ requires $(G_4 M_s)^{N-2} \approx G_4 M_c V$. For a nonsingular solution one requires $G_4 M_c < V^{1/(N-3)}$, and therefore it follows that $M_c < M_s$ in the cases of interest. This fact and the factor of $1/(N-1)$ in I_c reduces the action of these compactified black holes over the Schwarzschild case. This result may lead one to conclude that these "modes" are much more easily excited in a thermal bath of gravitons at high temperatures than the Schwarzschild black holes. This conclusion overlooks the nonzero dilaton charge of the compactified black holes. They could only be created if there are other species of particles which can cancel this scalar charge, so that the thermal bath remains neutral. Of course in these estimates the contributions of quantum loop effects have been ignored. Another black-hole solution which might consider in the above comparisons would be the direct product of a compactified black hole with another constant torus, which would have stresses around the constant torus. Its properties are essentially those of the compactified black-hole solution in the direct product.

A weakness with the conjectured form of the compactified vacuum solution is relating the asymptotic and short-range forms of the metric. Implicit in the above discussion is that if the mass parameter of the short-range form is M then the mass density in the long-range potential is M/V where V is the asymptotic volume of the compact space. This relation was true in the four-dimensional solution above and for the MP solutions of Sec. III. The proper volume of the compact manifold is by no means constant in the vicinity of the black hole and so it is not clear that this is the correct choice. Another problem may occur with the global topology. In the four-dimensional case, there was a potential conical singularity between the black holes. This would have results in redefining the period of the angle about the axis, but the defect angle was infinite and it was argued that this problem should be removed by redefining a constant of integration. In higher dimensions it was argued that the corresponding quantities would be finite but no attempt has been made to consider such problems with the conjectured metric above. Indeed it is not clear what the corresponding effect in a higher-dimensional space-time would be.

V. DISCUSSION

In this paper some exact solutions of the Einstein-Maxwell equations in compactified space-times have been constructed. The solutions given here display explicitly the property that the geometry in the vicinity of the horizon is like that of the corresponding solution in an asymptotically flat Minkowski space-time. It is therefore likely that similar results hold for vacuum solutions of Einstein's equations in compactified space-times. This is especially of interest for the case of spinning black holes in higher-dimensional spaces which can evade the usual restriction $J \leq G_4 M^2$ obeyed by spinning black holes in 3 + 1 dimensions.

The examination of a four-dimensional vacuum solution for black holes on a common axis suggested that similar nonsingular vacuum solutions of black holes in compactified spaces exist in higher dimensions. The essential point is that the gravitational forces in an infinite array of image masses will cancel. However it appears that in the vacuum case the mass must be less than the size of compact dimensions for the solutions to be free of naked singularities.

Compact spaces other than a torus are of interest. In the conjectured vacuum solution one may simply replace $\delta_{ab} dy^a dy^b$ by the line element on any compact Ricci-flat space, since the compact metric would only enter Einstein's equations only through its Ricci tensor or scalar. The conjectured properties of such a compactified black hole would then follow through unchanged. For the MP solutions choosing a similar compact manifold may be possible by choosing a more complicated system of image charges. Compact Ricci-flat manifolds are of interest in higher-dimensional supergravity[21] but recently they have been the focus of models compactifying superstring theories.[22] Vacuum compactified black-hole solutions would provide suitable solutions of the first-order string equations of motion[23] with which one may investigate effects of higher-order string effects.[24] The compactified MP solutions are not suitable string backgrounds because the dilaton field equation would not be satisfied by a constant dilaton field.

Ultimately one must recognize the restrictions of this work. The structure of solutions to Einstein's equations is being investigated on the compactification scale. In phenomenological models the compactification scale is usually taken to be on the order of the Planck scale although some estimates may make it a few orders of magnitude larger. This choice is made to relate Newton's constant to coupling constants for the Kaluza-Klein gauge fields.[25,26] At this length scale one should expect that corrections to low-energy Einstein or Einstein-Maxwell actions due to quantum loop[27] or perhaps stringy[23] effects will begin to play an important role. If one is considering higher-dimensional theories with elementary gauge fields, one need not push the compactification scale all the way down to the Planck scale. Fairly successful compactification scenarios have been devised for string theories in which the compactification scale is larger than the Planck scale.[22] In any event the solutions and results presented here may be useful in investigating the general properties of quantum gravity in higher-dimensional space-times.

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