Does the absence of a phase transition in SU(2) and SU(3) lattice gauge theories with Wilson action really have anything to do with continuum confinement?

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It is proven that all Abelian monopoles of SU(N) are unstable while SU(N)/Z(N) always has 2^N-2 species of stable monopoles. It is argued that the presence [absence] of a phase transition in SU(N)/Z(N) [SU(N)] lattice gauge theories for $N=2$ and 3 follows solely from the qualitative distinction between *stable* and *unstable*, and hence is a lattice artifact irrelevant to the continuum limit. The SU($N\geq 4$) transitions are briefly discussed.

It was Creutz who first pointed out that for $N=2$ (Ref. 1) and $N=3$ (Ref. 2) SU(N) lattice gauge theories with Wilson action exhibit a single phase for all accessible values of β while their SU(N)/Z(N) counterparts[3,4] show a single sharp first-order phase transition as the system is cooled from high to low temperature. The SU(2) and SU(3) models indicate a significant "activity" in the so-called crossover region, but this activity is too weak to produce a bona fide phase transition in the thermodynamic sense. This absence of a transition was interpreted as evidence for the proposal[5] that the center of the gauge group is the ultimate cause of confinement [its presence having "frustrated" the transition visible in SU(N)/Z(N), $N=2,3$] since confinement in the high-temperature phase is easy to establish using strong-coupling expansions.[6] Enthusiasm for this interpretation was only slightly dampened by the subsequent discovery that for $N=4$, 5, and 6 (Refs. 7–9, respectively) SU(N)/Z(N) has a first-order transition just as SU(N)/Z(N) does.[10] This transition for SU($N\geq 4$) was thought[7] to be an artifact of having used only the fundamental-representation Wilson action and not expected to be deconfining, however. Now while it is true that a renormalization-group-"improved" action can be constructed[11] for which these transitions are absent, a competitive "improvement" scheme[12] does not appear to have this property so the situation remained confused.

In a recent article[13] (hereafter referred to as I) and in Ref. 14 the formal continuum limit of lattice gauge theories with Wilson action[6] has been reexamined. For the *non-Abelian* case the usual Yang-Mills expression[15] must be altered to include generalized vortex and monopole degrees of freedom in addition to the familiar gluons and instantons. These results are sketched in Ref. 14 and full derivations are presented in I. For the purposes of this paper it suffices to know that a *bare* monopole configuration is parametrized by specifying (i) the boundary of an open, oriented two-surface S and (ii) a nontrivial root lattice vector of the *group* in question defined by

$$\exp\left[i\sum_K {}^*E_K T_K\right]=\mathbb{1}\ \ [\mathrm{SU}(N)/Z(N)]\ ,$$
$$\exp\left[i\sum_K {}^*e_K t_K\right]=\mathbb{1}\ \ [\mathrm{SU}(N)]\ , \tag{1}$$

where t and T denote the fundamental (N dimensional) and adjoint (N^2-1 dimensional) representations of the Lie algebra common to SU(N) and SU(N)/Z(N), respectively. The indices being summed over in (1) take values $K=j^2-1$, $j=2,3,\ldots,N$ labeling the elements of the maximal Abelian subalgebra of SU(N). We denote by $\{{}^*\mathbf{e}\}$ and $\{{}^*\mathbf{E}\}$ the periodic lattices of vectors satisfying (1) and remark that $\{{}^*\mathbf{E}\}\supset\{{}^*\mathbf{e}\}$ and $\exp(i\,{}^*\mathbf{E}\cdot t)=\exp[i(2\pi/N)k]$ for some $k=0,1,2,\ldots,N-1$ meaning that the vectors ${}^*\mathbf{E}$ organize themselves naturally into N classes labeled by k. The sublattice corresponding to $k=0$ is of course $\{{}^*\mathbf{e}\}$. These lattices are explicitly displayed for $N=2,3,4$ in I where it is furthermore shown that the vectors ${}^*\mathbf{E}$ and ${}^*\mathbf{e}$ act as generalized magnetic charges[16] replacing the Dirac value[17] 2π times an integer familiar from the U(1) case. It is the quantitative distinction between $\{{}^*\mathbf{E}\}$ and $\{{}^*\mathbf{e}\}$ taken together *with the qualitative difference in the stability properties of the associated monopoles* (to be discussed below) that provide a simple explanation of the phase-transition pattern in SU(N) and SU(N)/Z(N) Monte Carlo (MC) data.

MONOPOLE STABILITY PROPERTIES

To proceed quantitatively we introduce the background fields

$$g\hat{A}^{\mu}_{K}={}^*E_K a^{\mu}(x;S)\ \ [\mathrm{SU}(N)/Z(N)]\ ,$$
$$g\hat{A}^{\mu}_{K}={}^*e_K a^{\mu}(x;S)\ \ [\mathrm{SU}(N)] \tag{2}$$

and the matrix

$$\nu=\frac{1}{2\pi}\sum_K {}^*E_K T_K\equiv\frac{1}{2\pi}{}^*\mathbf{E}\cdot T\ , \tag{3}$$

where the potential function $a^{\mu}(x;S)$ is defined in the Appendix. The stability[18] (or fluctuation) operators in the background fields (2) (in a background-field gauge[19,13]) are given by

$$H^{\mu\nu}_{ab}(\nu)=-(D_{\hat{A}})^2_{ab}g^{\mu\nu}-4\pi i\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}(\nabla\Lambda{}^*a)^{\lambda\rho} \tag{4}$$

with $(D_{\hat{A}})^{\mu}_{ab}=\nabla^{\mu}\delta_{ab}-igT^K_{ab}\hat{A}^{\mu}_K$ as usual. In I it is shown that for arbitrary N, ${}^*\mathbf{E}={}^*\mathbf{E}(k;n_3,n_8,\ldots,n_{N^2-1})$ with k an integer obeying $0\leq k\leq N-1$ and the $N-1$ n_K's are unrestricted integers. More important for our present, restricted purposes is the spectrum of ν (denoted by $\{\nu\}$) written in terms of k and the n_K's. For general N it is

$$\{v\}=\{\mp(k+2n_3+\cdots+n_{N^2-1}),\mp(k+n_3+2n_8+\cdots+n_{N^2-1}),\ldots,\mp(k+n_3+\cdots+2n_{N^2-1}),\; \mp(n_3-n_8),\mp(n_3-n_{15}),\ldots,\mp(n_8-n_{15}),\ldots,\mp(n_{(N-1)^2-1}-n_{N^2-1}),0,0,\ldots,0\} \tag{5}$$

with a total of $N-1$ zero eigenvalues and $\frac{1}{2}(N-1)(N-2)$ of the form $\mp(n_{J^2-1}-n_{K^2-1})$, $2\le J<K\le N$. The $\mp$ means that the particular eigenvalue appears with both signs and the total number of eigenvalues (for fixed k and n_K) is easily verified to be N^2-1 as it should be. For $N=2$, $\{v\}=\{\mp(k+2n_3),0\}$, $k=0,1$.

Definition. A configuration $\hat{A}$ will be called *stable* if its corresponding stability operator has a non-negative spectrum. Otherwise it is unstable.

In a thorough analysis Brandt and Neri[20] have shown that the necessary and sufficient condition for stability in the above sense[21] is that each eigenvalue of $v({}^*\mathbf{E})$ satisfies the inequality $|v|\le 1$. The simplicity of this criterion leads to the following theorem.

Theorem. All nontrivial root lattice vectors of SU(N) [the ${}^*\mathbf{e}$'s in (1)] are unstable. The trivial vector corresponding to $n_K=0$ for all K is of course stable.

To demonstrate this recall that the vectors ${}^*\mathbf{e}(n_K)$ are obtained from the ${}^*\mathbf{E}(k;n_K)$ by setting $k=0$ so we begin by setting $k=0$ in (5). The theorem is now obvious for $N=2$ and follows for $N\ge 3$ in such a straightforward manner that we defer writing down the analytic proof to a separate publication.[22] By contrast SU(N)/Z(N) possesses nontrivial *stable* vectors ${}^*\mathbf{E}$ for *all N*. There are two of them for $N=2$ [$k=1$, $n_3=0$, and $k=1$, $n_3=-1$ in (5)] and it can be shown[22] that there exist 2^N-2 such stable vectors for a general N although we will not need their explicit form here. This existence of stable versus unstable bare monopoles already points to a *qualitative dynamical distinction* between SU(N) and SU(N)/Z(N) based theories and constitutes the crucial technical observation of this paper. For later comparison purposes a final quantity of interest turns out to be the number of root lattice vectors for which exactly two eigenvalues of v have absolute value 2, the rest obeying $|v|\le 1$. Such vectors represent in some sense a "minimal departure from stability" and their number as a function of N is recorded in Table I.

MONOPOLE (IN)STABILITY AND PHASE TRANSITIONS

The concept "stability" introduced in the previous section is a mathematical property of the configurations (2), or more precisely of their fluctuation operators $H(v)$. To make contact with physics we consider the bare monopole contribution to the continuum SU(N)/Z(N) partition function (gauge terms are suppressed) given schematically by

$$Z=\sum_{\{\partial S,{}^*\mathbf{E}\}}\exp[-\mathscr{A}^{\rm cl}(\partial S,{}^*\mathbf{E})]\int\mathscr{D}\delta A\,\exp\{-[\tfrac{1}{2}\delta A H(v)\delta A+gC_3\delta A^3+gC_3'\delta A^3(g\hat{A})+g^2C_4\delta A^4]\}\,, \tag{6}$$

where the $\sum$ is over closed loops ∂S and root lattice vectors $\{{}^*\mathbf{E}\}$ as specified in I, C_3, C_3', and C_4 are the familiar tensors characterizing cubic and quartic interaction terms in a background field $\hat{A}$ (Ref. 19), $H(v)$ is given by (4) and

$$\mathscr{A}^{\rm cl}=\frac{1}{4g^2}\int[dx][{}^*\mathbf{E}(\nabla\Lambda{}^*\alpha)^{\mu\nu}]^2 =\frac{|{}^*\mathbf{E}|^2}{2g^2}\int[dx][d_x']\,{}^*X^\mu(x)D(x-x')\,{}^*X^\mu(x')\,, \tag{7}$$

${}^*X^\mu(x)$ is the current of "true magnetic charge" having space-time support equal to ∂S (see Appendix) and $\mathscr{A}^{\rm cl}$ represents the Coulomb self-action of this current. Although (6) and (7) are derived (in I) for the continuum, we imagine now "looking back" onto the lattice (reintroduction of a cutoff so that $\mathscr{A}^{\rm cl}$ is rendered finite) and formulate our main observation as the following hypothesis.

TABLE I. The number of root lattice vectors ${}^*\mathbf{e}$ of SU(N) having $|v|=2$ for precisely two eigenvalues with the remaining eigenvalues obeying $|v|\le 1$. $\eta(N)$=(the number of unstable eigenvalues of v)/(the total number of eigenvalues of v).

N	SU(N)	$\eta(N)$
2	2	2/3
3	6	1/4
4	12	2/15
5	20	1/12
6	30	2/35
⋮	⋮	⋮
10	90	2/99
11	110	1/60

Hypothesis. The phase transition in SU(N)/Z(N) ($N=2,3$) gauge theories as the system is cooled from high to low temperature $T(=\beta^{-1}=g^2)$ through the crossover region is caused by the freezing out of *stable* monopoles. Likewise, the strong "activity" seen in SU(2) (Ref. 1) and SU(3) (Ref. 2) is also caused by the freezing out of (${}^*\mathbf{e}$-type) monopoles, but these monopoles, by the theorem proven above, are all *unstable*. Of course, that the crossover region has something to do with monopoles is, by itself, not a new idea;[23,24] what it new is the suggestion that the distinction between stable and unstable is responsible for the difference between phase transition and "activity." To see why this is so, let ${}^*\mathbf{E}$ be a stable vector so that its corresponding $H(v)$ is a strictly positive operator (by definition). To compute the contribution of such a stable monopole to the partition function the gluon functional integral can be evaluated perturbatively because the Gaussian approximation [set $g=0$ in (6)] exists. This means that the basic pointlike character of stable monopoles is only perturbatively modified by their coupling to the gluon field $\delta A^{\mu}(x)$ and it is the freezing out of these *pointlike* states that is responsible for the *sharp* transition in $SU(N)/Z(N)$ ($N=2,3$) systems. For the case of an unstable monopole, however, where $H(\nu)$ has negative eigenvalues,[25] the situation is completely different. Were we to set $g=0$ in (6), the Gaussian integral would not exist. Mathematically this means that the term $g^2C_4(\delta A)^4$ must be invoked to secure convergence of the $\int \mathscr{D}\delta A$ in (6); physically this means that a configuration parametrized by ∂S and an unstable $^*\mathbf{E}$ (or $^*\mathbf{e}$) can no longer be viewed as a pointlike object.[26] In short, interaction between unstable monopoles and gluon degrees of freedom is nonperturbative in character and leads to an intrinsically extended object. It is then plausible, all other things being equal, that the freezing out of such "fattened" configurations will have a less dramatic effect on thermodynamic quantities then the freezing out of their pointlike relatives since their effective classical action, although still infinite, is less divergent than in the pointlike case.[27] This is the proposed mechanism by which the first-order $SU(N)/Z(N)$ ($N=2,3$) transitions are softened into mere "activities" for SU(2) and SU(3). Of course taken by itself this argument, although relatively convincing, would only be circumstantial evidence for the new interpretation. There is, however, a more important reason. It is shown in I that the ground states of SU(N) and $SU(N)/Z(N)$ based gauge theories, in the absence of matter fields and in the continuum limit, are in fact the same. This fact alone already precludes any difference in structure in the crossover region which depends on the center of the gauge group from being relevant in the continuum and confirms the new interpretation in a simple and striking way. Since the difference between phase transition and "activity" measured on the lattice finds a simple explanation in the continuum concept of stability versus instability of precisely those configurations which are known to drive the crossover, and since these configurations are not in fact present in the continuum limit (in four dimensions) we conclude that the absence of a phase transition in SU(2) and SU(3) lattice gauge theories has nothing to do with continuum confinement. Rather than the Abelian monopoles (2), whose lattice counterparts play a significant role in the crossover region, it can be shown[13,14] that it is the Abelian and non-Abelian singular pure gauge configurations introduced in I that control ground-state structure in the continuum limit.

It remains to comment on the phase transition observed in $SU(N\geq 4)$ lattice gauge theories in four dimensions.[7–9] Since it is known[28] that pure $Z(N)$ gauge theories for $N\geq 4$ exhibit *three* phases, it is tempting to speculate that the phase transition showing up in $SU(N\geq 4)$ has something to do with "the second $Z(N)$ transition." From our vantage point in the continuum we can say nothing directly about this. A glance at Table I, however, suggests a more mundane possibility. If $^*\mathbf{e}$ is a root lattice vector the spectrum of whose $\nu(^*\mathbf{e})$ contains only two eigenvalues *violating* the stability criterion $|\nu|\leq 1$ (for definiteness we assume in Table I that the magnitude of these "unstable" eigenvalues also equals 2) then the ratio of the number of unstable eigendirections to the total number of eigendirections (defined to be) $\eta(\nu)=2/(N^2-1)$. In the continuum, of course, the distinction between stable and unstable has an absolute meaning. On a finite lattice, however, and in an MC experiment which must select its equilibrium configurations in a finite number of sweeps it is plausible that as N *becomes large* the distinction between stable and unstable effectively disappears $[\eta(\nu)\rightarrow 0]$ for these "minimally unstable" vectors $^*\mathbf{e}$ (whose number increases rapidly with N as given in Table I), the *stable* eigendirections dominate and what would have been only an "activity" effectively hardens into a transition. Reality (on the lattice) is likely to be a combination of the "second $Z(N)$ transition" and "activity hardening" pictures and it will be interesting to remeasure and reanalyze the $SU(N\geq 4)$ transition regions with an eye to resolving this issue. This remains to be done.

DISCUSSION AND SUMMARY

We have presented strong evidence that the answer to the question posed in the title of this paper is *no.* It was proven that for SU(N) all Abelian monopoles are unstable while for $SU(N)/Z(N)$ there always exist 2^N-2 distinct species of stable monopoles. It was then suggested that the measured difference between phase transition and "activity" in $SU(N)/Z(N)$ vs SU(N) ($N=2,3$) lattice gauge theories is caused by the difference between stable and unstable monopoles. The freezing out of stable monopoles (which retain their pointlike character) as the system is cooled through the crossover region leads to a bona fide phase transition in the thermodynamic sense, while the freezing out of unstable monopoles (which necessarily become extended objects) leads only to a well-defined "activity." For large N (≥ 4?) certain unstable monopoles can act as effectively stable ones (on the lattice) and simulate a hardening of the "activity" into a transition.

This elementary, if unexpected, explanation of one of the standard lattice gauge signals for confinement is not as catastrophic as it might appear at first glance. It only means that measurements of the average plaquette energy at the present level of accuracy yield no information about the continuum. But what about measurements of fundamental representation Wilson loops, string tensions, etc.?[28] The answer to this question is the following: since the presence of Abelian monopoles, whose effects must fade away as the true continuum is approached, always contaminates the MC data at finite β and lattice spacing, all traces of this "Abelian background" must be carefully isolated and deleted before one can claim that continuum relevant quantities have been reliably determined. Otherwise, the signal of ultimate interest is being systematically masked by these lattice artifacts. Now it is the main result of I that the ground state of SU(N) continuum gauge theory *is* anomalous (nonperturbative), and that the expected value of a fundamental representation Wilson loop *is* anticipated to show nonperimeter behavior, for reasons, however, which are completely different from those responsible for area-law behavior in the strong-coupling and crossover regions.[6] In the final analysis the anomalous ground state is caused by a subtle quantum-mechanical effect (paramagnetic instability) involving the intrinsic magnetic moments of the colored gluons.[13] Can we be sure that being in the so-called "scaling region" (usually accepted as being equivalent to the continuum limit in numerical experiments[28]) is sufficient to enable us to begin seeing such delicate effects? We venture to conjecture that one must be *deep* in the "scaling region" before the effects of the paramagnetic instability become numerically significant and stress the need for vastly improved measurements in this region before we can draw any conclusions about the continuum.

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APPENDIX

Let $\xi^\mu(\sigma_1,\sigma_2)$ be a convenient parametrization of an oriented two-surface S, and define

$$d\sigma^{\mu\nu}(\xi)=[(\partial\xi^\mu/\partial\sigma_1)(\partial\xi^\nu/\partial\sigma_2) - (\partial\xi^\mu/\partial\sigma_2)(\partial\xi^\nu/\partial\sigma_1)] , \tag{A1}$$

$$d^*\sigma^{\mu\nu}(\xi)=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}d\sigma^{\lambda\rho} , \quad \epsilon^{1234}=1 , \tag{A2}$$

$$m^{\mu\nu}(x;S)=\int_S d^*\sigma^{\mu\nu}(\xi)\delta^4[x-\xi(\sigma_1,\sigma_2)] . \tag{A3}$$

The antisymmetric tensor $m^{\mu\nu}$ can be decomposed[13] according to

$$m^{\mu\nu}(x;S)=[\nabla\Lambda a(x;S)]^{\mu\nu}+\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}[\nabla\Lambda^* a(x;\partial S)]^{\lambda\rho} , \tag{A4}$$

where $(\nabla\Lambda a)^{\mu\nu}\equiv\nabla^\mu a^\nu-\nabla^\nu a^\mu$. Calling $X^\mu\equiv\nabla^\nu m^{\mu\nu}$ and $^*X^\mu=\nabla^{\nu*}m^{\mu\nu}$ we can write (suppressing gauge terms)

$$\begin{aligned} a^\mu(x;S)&=\int[d\xi]D(x-\xi)X^\mu(\xi) , \\ {}^*a^\mu(x;\partial S)&=\int[d\xi]D(x-\xi)\,{}^*X^\mu(\xi) , \end{aligned} \tag{A5}$$

where $D(x-\xi)$ is the negative inverse Laplacian. It follows from the definition of m that $^*X^\mu=\int_{\partial S}d\xi^\mu\delta^4[x-\xi]$ so (A5) implies that $^*a^\mu=0$ (or more generally a pure gauge) when the boundary of S vanishes.

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