Origin of external sources for classical Yang-Mills fields

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An effort to derive the classical Yang-Mills equation with an external color four-current density $j^{a\mu}(x)$ from a more fundamental theory is presented. It is shown that in the case of an arbitrary static external color charge, i.e., when $j^{a\mu}=\delta_0^\mu \rho^a(\mathbf{x})$, the classical Yang-Mills equation follows from the closed set of classical Yang-Mills and Dirac equations in the limit $m \to \infty$, where m is the mass of the Dirac particle. In the case of the classical Yang-Mills equation with an external color current j^{ai}, $i=1,2,3$, no such derivation is found.

The classical non-Abelian gauge theory with a fixed external color four-current density $j^{a\mu}(x)$ can be regarded as a model on which we can study the effects of the presence of color charged matter. The basic equation of the theory is the nonhomogeneous Yang-Mills equation

$$D_\lambda^a(A)F^{\lambda\mu}(A) \equiv \partial_\lambda F^{a\lambda\mu} - f_{abc}A_\lambda^b F^{c\lambda\mu} = -j^{a\mu} , \qquad (1)$$

where

$$F_{\lambda\mu}^a(A) = \partial_\lambda A_\mu^a - \partial_\mu A_\lambda^a - f_{abc}A_\mu^b A_\lambda^c . \qquad (2)$$

It has been investigated since 1976. Rather exciting discoveries have been made: e.g., instabilities,[1] total screening solutions,[2] and bifurcating solutions.[3] Numerous subsequent investigations[4] have significantly enlarged the body of knowledge about Eq. (1) and its solutions. The emerging picture is abounding in fascinating phenomena which reflect the nonlinear character of Eq. (1).

Unfortunately, the value of these results is diminished by the lack of their interpretation within the framework of underlying physical theory: i.e., quantum chromodynamics. Another serious objection is that the physical origin of the external color four-current $j^{a\mu}$ is not clear even on the level of unquantized theory. This is due to the fact that $j^{a\mu}$ has been introduced as a rather formal mathematical object with no apparent relation to quark fields.

In this paper we would like to consider the latter objection. Our considerations are restricted to unquantized Yang-Mills and quark fields which obey the fundamental set of classical Yang-Mills and Dirac equations. If the non-Abelian gauge theory is in a confining phase the direct physical relevance of such considerations will be rather limited. Nevertheless we believe that they can provide valuable information about mathematical properties of non-Abelian gauge theories in the presence of color charged matter. Moreover, there exists also a theoretical possibility of a nonconfining phase which may even be realized as a quark-gluon plasma. In this phase the classical Yang-Mills and quark fields have much more direct physical applications.

On this unquantized level we make two attempts to obtain Eq. (1). The first one is an effort to reduce the fundamental set of Yang-Mills and Dirac equations to Eq. (1). The idea is that in the large-m limit, where m is the mass of the Dirac particle, the Dirac equation becomes unimportant. This approach is partially successful. It yields Eq. (1) with static external color charge density only: i.e.,

$$j^{a\mu} = \delta_0^\mu \rho^a(\mathbf{x}) . \qquad (3)$$

The second approach is more phenomenological and it is not successful. Here we assume that the external four-current $j^{a\mu}$ is produced by a macroscopic experimental setup, just like the ordinary electric four-current j^μ considered in classical electrodynamics. We find out that Eq. (1) does not follow from a Yang-Mills equation for the total non-Abelian gauge field present in such experimental circumstances, essentially because of the nonlinearity of the Yang-Mills equation. On the whole, our considerations provide a physical motivation for introducing the external color charge density (3), while the doubts about the physical origin of the external color current j^{ai}, $i=1,2,3$ persist.

Now, let us present the two approaches in more detail. The starting point for the first approach is the fundamental set of coupled classical Yang-Mills and Dirac equations

$$\partial_\lambda G^{a\lambda\mu} - f_{abc}B_\lambda^b G^{c\lambda\mu} = -\bar{\psi}\gamma^\mu T^a \psi , \qquad (4)$$

$$\gamma^\mu(i\partial_\mu - \hat{B}_\mu)\psi - m\psi = 0 . \qquad (5)$$

We choose the Yang-Mills self-coupling constant $g=1$ and we set $c=\hbar=1$. The space-time metric is $(1,-1,-1,-1)$. We use the Dirac representation for the γ^μ matrices

$$\gamma^0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} .$$

T^a are Hermitian generators of the gauge group SU(n) with the corresponding structure constants f_{abc}, $\hat{B}_\mu = B_\mu^a T^a$, $G_{\lambda\mu}^a$ is the field-strength tensor (2) corresponding to the potential $B^{a\mu}$, i.e., $G_{\lambda\mu}^a = F_{\lambda\mu}^a(B^{c\nu})$. The fact that we consider only one Dirac field is not essential. The considerations presented below can easily be generalized to any number of Dirac fields. Then, the color four-current density present on the right-hand side (RHS) of Eq. (4) will have the form of the sum of color four-currents of all Dirac fields.

For finite momenta of the Dirac particle the large-m limit is equivalent to a nonrelativistic limit. Therefore we may apply a well-known reasoning[5] which leads to the nonrelativistic approximation for Dirac equation (5) known as the Pauli equation. We find that in the large-m limit the Dirac equation (5) has the approximate solution

$$\psi=\exp(-imx^0)\begin{bmatrix}\varphi\\ \frac{1}{2m}\sigma_i\pi^i\varphi\end{bmatrix}, \tag{6}$$

where

$$\pi^i=-i\frac{\partial}{\partial x^i}-\hat{B}^i$$

and the 2-component spinor φ obeys the Pauli equation

$$(i\partial_0-\hat{B}_0)\varphi=\frac{1}{2m}\pi^2\varphi+\frac{1}{4m}\epsilon_{iks}\sigma_s G^a_{ks}T^a\varphi . \tag{7}$$

The corresponding approximate form of the Dirac four-current density is

$$\bar{\psi}\gamma^0T^a\psi=\varphi^\dagger T^a\varphi+O(m^{-2}) , \tag{8}$$

$$\bar{\psi}\gamma^iT^a\psi=O(m^{-1}) , \tag{9}$$

where $O(m^{-n})$, $n=1,2$, denotes terms which vanish like m^{-n} when $m\rightarrow\infty$. Thus, we find that in the limit $m\rightarrow\infty$ the Dirac four-current density is

$$\bar{\psi}\gamma^\mu T^a\psi=\delta^\mu_0\varphi^\dagger T^a\varphi , \tag{10}$$

and that the spinor φ obeys the simple equation

$$(i\partial_0-\hat{B}_0)\varphi=0 . \tag{11}$$

A general solution of Eq. (11) has the form

$$\varphi(\mathbf{x},x^0)=V_B(\mathbf{x},x_0)\chi(\mathbf{x}) , \tag{12}$$

where

$$V_B(\mathbf{x},x_0)=T\exp\left[-i\int_{t_0}^{x_0}dt\,\hat{B}_0(\mathbf{x},t)\right] . \tag{13}$$

Here $T\exp$ denotes the time-ordered exponential, t_0 is a fixed instant of time, and the spinor $\chi(\mathbf{x})$ is arbitrary.

Formula (12) can be regarded as a gauge transformation to a temporal gauge. The corresponding formula for the transformation of the gauge potentials is

$$\hat{B}_\mu=V_B\hat{A}_\mu V_B{}^{-1}+i(\partial_\mu V_B)V_B{}^{-1} , \tag{14}$$

where $\hat{A}_\mu$ is a new gauge potential. It is easy to see that

$$\hat{A}_0=0 . \tag{15}$$

Substituting (14) and (10) into Eq. (4) we obtain Eq. (1) in the temporal gauge (15) with the static external color charge density (3), where

$$\rho^a(\mathbf{x})=\chi^\dagger(\mathbf{x})T^a\chi(\mathbf{x}) . \tag{16}$$

Because of (3), Eq. (15), a solubility condition for Yang-Mills equation (1),

$$\partial_\mu j^{a\mu}-f_{abc}A^b_\mu j^{c\mu}=0 , \tag{17}$$

is automatically satisfied in our case. Thus, we do not find any restrictions on $\rho^a(\mathbf{x})$.

The fact that we have obtained $\hat{A}_0=0$ is not essential. It is easy to check that the solubility condition (17) implies that in Eq. (1) one can always pass to the gauge (15) is $j^{ai}=0$ for $i=1,2,3$.

Thus, we have shown the relation between Eq. (1) and the set of Eqs. (4) and (5) in the limit $m\rightarrow\infty$. The relation holds only when $j^{a\mu}$ has the form (3), i.e., when the external color current density vanishes.

Nonzero Dirac color current j^{ai} appears in the next approximation to the set of Eqs. (4) and (5), when also the terms of the order m^{-1} are taken into account in addition to the already considered terms of the order m^0, see (8) and (9). Then Eq. (11) has to be replaced by the Pauli equation (7). In this case we are not able to derive an external current approximation because it is not possible to find a general solution of the Pauli equation (7) for generic $\hat{B}_\mu$. Therefore, in the order m^{-1}, one has to consider the rather nontrivial set of Yang-Mills and Pauli equations. Let us remark here that a similar set of equations, consisting of the Yang-Mills equation and an equation obtained from the Pauli equation (7) by dropping out the spin term $\epsilon_{kis}\sigma_s G^a_{ks}T^a$, has already been considered.[6]

In the second approach we try to justify the presence of the external color four-current $j^{a\mu}$ in Eq. (1) by referring to a macroscopic experimental setup in a laboratory, in an analogy to macroscopic external electric currents. For instance, let us consider the case when an external electric four-current j^μ is produced by a mechanical arm moving an electrically charged particle along a fixed line with a fixed velocity, in a vacuum. The particle is kept on its route by forces which are essentially of electromagnetic nature. Here we mean, for instance, the forces which prevent the particle from escaping from the arm. These forces are due to very intense, short-range microscopic electromagnetic fields $F^{\mu\nu}_{\text{micr}}$. The microscopic field $F^{\mu\nu}_{\text{micr}}$ has, of course, its sources j^μ_{micr}—nuclei and electrons of materials used to build the arm. The microscopic electromagnetic field, together with a long-range field $F^{\mu\nu}$, which we usually ascribe to the prepared external current j^μ, form the total electromagnetic field $F^{\mu\nu}_{\text{tot}}$ which characterizes the experimental setup. This total electromagnetic field and the sources of both kinds of electromagnetic fields form a solution of an extremely complicated, nonlinear set of equations consisting of a Maxwell equation and equations of motion for the sources. The Maxwell part of this set of equations reads

$$\partial_\mu F^{\mu\nu}_{\text{tot}}=-j^\nu_{\text{tot}} , \tag{18}$$

where

$$j^\nu_{\text{tot}}=j^\nu+j^\nu_{\text{micr}} \tag{19}$$

is the total electric four-current.

As one can see from textbooks on electrodynamics, $F^{\mu\nu}_{\text{micr}}$ and j^μ_{micr} usually are not included into considerations, in spite of the fact that they are crucial for the existence of the external current j^μ. The fact that this is not an error is due to the linearity of Maxwell equations.

Namely, $F^{\mu\nu}_{\rm micr}$ and $j^{\mu}_{\rm micr}$ are assumed to be related also by a Maxwell-type equation

$$\partial_\mu F^{\mu\nu}_{\rm micr} = -j^{\nu}_{\rm micr} \ . \qquad (20)$$

This is an assumption because, in fact, only $F^{\mu\nu}_{\rm tot}$ and $j^{\mu}_{\rm tot}$ exist; indeed, a probing charge feels $F^{\mu\nu}_{\rm tot}$. $F^{\mu\nu}_{\rm micr}$ and $j^{\mu}_{\rm micr}$ are extracted from $F^{\mu\nu}_{\rm tot}$ and $j^{\mu}_{\rm tot}$ with the help of additional definitions. It follows from (18) and (19) as a mathematical identity that

$$\partial_\mu F^{\mu\nu} = -j^{\nu} \ , \qquad (21)$$

where

$$F^{\mu\nu} \equiv F^{\mu\nu}_{\rm tot} - F^{\mu\nu}_{\rm micr} \ . \qquad (22)$$

Equation (21) has the mathematical form of the Maxwell equation (18) in the whole space-time. However, it is physically relevant only in the regions of the space-time where $F^{\mu\nu}_{\rm micr}=0$ because there $F^{\mu\nu}=F^{\mu\nu}_{\rm tot}$. It is precisely Eq. (21) which is considered in the electromagnetics of external currents in a vacuum.

In the case of the world of color charged particles and non-Abelian fields, the analogs of Eqs. (18) and (20) are, respectively,

$$D^a_\mu(B_{\rm tot}) F^{\mu\nu}(B_{\rm tot}) = -j^{a\nu}_{\rm tot} \qquad (23)$$

and

$$D^a_\mu(B_{\rm micr}) F^{a\mu\nu}(B_{\rm micr}) = -j^{a\nu}_{\rm micr} \ , \qquad (24)$$

where $B^{a\nu}_{\rm micr}$ and $j^{a\nu}_{\rm micr}$ are extracted from $B^{a\nu}_{\rm tot}$ and $j^{a\nu}_{\rm tot}$ with the help of additional definitions which we do not specify here. Analogously to the electromagnetic case, $B^{a\nu}_{\rm micr}$ and $j^{a\nu}_{\rm micr}$ are due to particles which form an arm transporting a non-Abelian charge along a fixed route with a given velocity and direction in color space. The fact that such an arm is possible in the world of color charges follows from Wong's equations of motion[7] for color charged particles. It follows from (23) and (24) that

$$D^a_\mu(B_{\rm tot}) F^{\mu\nu}(B_{\rm tot}) - D^a_\mu(B_{\rm micr}) F^{\mu\nu}(B_{\rm micr}) = -j^{a\nu} \ , \qquad (25)$$

where

$$j^{a\nu} \equiv j^{a\nu}_{\rm tot} - j^{a\nu}_{\rm micr} \ . \qquad (26)$$

Now, the question is whether Eq. (25) can be given the mathematical form of a Yang-Mills equation, i.e., whether the LHS of Eq. (25) can be written, for all points in Minkowski space-time, as

$$D^a_\mu(A) F^{\mu\nu}(A) \ ,$$

where $A^{a\mu}$ is a new non-Abelian gauge potential. If the answer was in the affirmative we would obtain the desired physical justification for Yang-Mills equation (1). Unfortunately, it seems that in general the answer is in the negative because no superposition principle for solutions of Yang-Mills equations has been found until now. Only in rather particular cases such potential $A^{a\mu}$ exists, e.g., when the gauge potentials $B^{a\mu}_{\rm tot}$, $B^{a\mu}_{\rm micr}$ have constant and identical directions in the color space (then the situation is essentially the same as in the electromagnetic case and $A^{a\mu}=B^{a\mu}_{\rm tot}-B^{a\mu}_{\rm micr}$).

Thus, when we prepare the experimental setup which produces the fixed external color current $j^{a\mu}$, the corresponding equation for Yang-Mills fields will be (25) which does not have the form of Yang-Mills equation (1). Because of the nonlinearity of the equation it is not possible to subtract the microscopic field.

For the sake of completeness of our analysis of possible origins of Eq. (1) we would like to remark that an external four-current appears also in quantized non-Abelian gauge theory in generating functionals for Green's functions. However, in this case the external current is a purely mathematical, auxiliary device and it is not expected to have any physical meaning. For instance, in order to obtain physically relevant Green's functions, that current has to be set at zero. Therefore, in this framework we do not find any motivation for considering Eq. (1).

Finally, let us remark that the external sources considered in our paper should not be confused with noncommuting external sources discussed within the framework of algebraic chromodynamics proposed in Ref. 8. In those papers the components $j^{a\mu}$ of the color charge density as well as the components $A^{a\mu}$ of the gauge potential are assumed to be finite-dimensional matrices. In the classical theory which we consider in the present paper $A^{a\mu}(x)$ and $j^{a\mu}(x)$ are real numbers; in a quantized theory they are, roughly speaking, infinite-dimensional matrices; the theory considered in Ref. 8 is situated somewhere inbetween. The noncommuting external sources have been proposed in Ref. 8 without actually deriving them from the quantum chromodynamics. Therefore in the case of the noncommuting external sources we encounter a problem analogous to the one discussed in the present paper.

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