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Neutrino oscillations and the Landau-Zener formula

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(Received 9 February 1987)

We discuss solar-neutrino oscillations and the Landau-Zener probability using a heuristic picture in analogy with an electron spin in a time-dependent magnetic field. The extreme nonadiabatic resonant oscillation is also briefly investigated.

The two-neutrino system obeys, as it propagates in matter, the Schrödinger equation[1–9]

$$i\frac{d}{dt}\begin{bmatrix}\nu_e(t)\\ \nu_\mu(t)\end{bmatrix}=\frac{\delta m^2}{4E}\begin{bmatrix}a(t)-\cos2\theta & \sin2\theta\\ \sin2\theta & -a(t)+\cos2\theta\end{bmatrix}\begin{bmatrix}\nu_e(t)\\ \nu_\mu(t)\end{bmatrix}\equiv H(t)\,|\,\psi\rangle \tag{1}$$

with θ, δm^2, and E the mixing angle, (mass)^2 difference, and energy of the neutrino, respectively. In Eq. (1)

$$\frac{\delta m^2}{2E}a(t)\equiv\sqrt{2}N_e(r(t))G_{\text{Fermi}} \tag{2}$$

is the matter-induced ν_e-ν_μ splitting with $N_e(r(t))$ the electron number density at distance r from the Sun's core.

A simplication of the geometric aspects of the problem occurs if we consider neutrinos born at $t=0$ in the Sun's core so that $r=t$. Although this simplication is not essential we will use it for now.

Level "crossing" occurs at r_0 where

$$a(r_0)=\cos2\theta\ . \tag{3}$$

The diagonal elements in the 2×2 Hamiltonian of Eq. (1) are then equal [vanishing due to the overall phase subtraction implicit in Eq. (1)]. Diagonalizing the matrix $H(t)$ yields two "instantaneous" levels:

$$\begin{aligned}\mathcal{E}_2(t)&=-\mathcal{E}_1(t)\\ &=\tfrac{1}{2}\{[a(t)-\cos2\theta]^2+(\sin2\theta)^2\}^{1/2}\frac{\delta m^2}{2E}\ .\end{aligned} \tag{4}$$

In vacuum, the two mass eigenstates

$$\begin{aligned}|\,\nu_1\rangle_{\text{vac}}&=\cos\theta\,|\,\nu_e\rangle+\sin\theta\,|\,\nu_\mu\rangle\ ,\\ |\,\nu_2\rangle_{\text{vac}}&=-\sin\theta\,|\,\nu_e\rangle+\cos\theta\,|\,\nu_\mu\rangle\end{aligned} \tag{5}$$

are represented, as shown in Fig. 1, by two unit vectors

$$\pm\hat{\mathbf{n}}_0=\pm(\cos2\theta\hat{\mathbf{e}}_z+\sin2\theta\hat{\mathbf{e}}_x)\ , \tag{6}$$

respectively. In Fig. 1 the $\hat{\mathbf{e}}_z$ $(-\hat{\mathbf{e}}_z)$ direction corresponds to the ν_e (ν_μ) state. We note that $\pm\hat{\mathbf{n}}_0$ are tilted by twice the angle, i.e., by 2θ. The time evolution of ν_e (i.e., $\nu_e-\nu_\mu$ oscillation in vacuum) can be described by the precession of a unit vector $\hat{\boldsymbol{\mu}}_0$, which is taken to coincide initially with $\hat{\mathbf{e}}_z$, around the axis $\hat{\mathbf{n}}_0$ with the frequency $\omega_0=\delta m^2/2E$ (Fig. 1).

Upon defining an angle β_0 as $\cos2\beta_0=\hat{\boldsymbol{\mu}}_0\cdot\hat{\mathbf{e}}_z$, the probability that ν_e remains as ν_e is given by

$$P(\nu_e\to\nu_e;\text{vac})=\cos^2\beta_0=\tfrac{1}{2}(1+\hat{\boldsymbol{\mu}}_0\cdot\hat{\mathbf{e}}_z)\ . \tag{7}$$

In a coordinate system in which $\hat{\mathbf{n}}_0=\hat{\mathbf{e}}'_z$ and $\hat{\mathbf{e}}_y=\hat{\mathbf{e}}'_y$, we have

$$\hat{\mathbf{e}}_z=\cos2\theta\hat{\mathbf{e}}'_z-\sin2\theta\hat{\mathbf{e}}'_x\ , \tag{8a}$$

$$\hat{\boldsymbol{\mu}}_0=\cos2\theta\hat{\mathbf{e}}'_z-\sin2\theta(\cos\omega_0t\hat{\mathbf{e}}'_x+\sin\omega_0t\hat{\mathbf{e}}'_y)\ . \tag{8b}$$

Substituting Eq. (8) into Eq. (7), we obtain the familiar result for vacuum oscillations

$$P(\nu_e\to\nu_e;\text{vac})=1-\sin^2(2\theta)\sin^2(\omega_0t/2)\ . \tag{9}$$

For ν_e produced in the Sun, the mass eigenstates in matter

$$|\nu_1(t)\rangle_{mat}=\cos\alpha(t)\,|\nu_e\rangle+\sin\alpha(t)\,|\nu_\mu\rangle\,,$$
$$|\nu_2(t)\rangle_{mat}=-\sin\alpha(t)\,|\nu_e\rangle+\cos\alpha(t)\,|\nu_\mu\rangle\,, \tag{10}$$

are represented by the two instantaneous unit vectors

$$\pm\hat{\mathbf{n}}(t)=\pm(\cos2\alpha\hat{\mathbf{e}}_z+\sin2\alpha\hat{\mathbf{e}}_x)\,. \tag{11}$$

The time evolution of ν_e is now described by the precession of $\hat{\boldsymbol{\mu}}$ around the instantaneous $\hat{\mathbf{n}}(t)$axis [see Fig. 2(a)] with the frequency

$$\omega(t)=\mathscr{E}_2(t)-\mathscr{E}_1(t)\,. \tag{12}$$

(Its vacuum value is of course $\omega_0=\delta m^2/2E$.) Note that $\omega(t)\hat{\mathbf{n}}$ is a vector sum of $\omega_0\hat{\mathbf{n}}_0$ and $-a\omega_0\hat{\mathbf{e}}_z$. Our picture is equivalent to that of precession of an electron spin around a fictitious time-dependent magnetic field $\mathbf{B}(t)=\omega(t)\hat{\mathbf{n}}(t)$ (Ref. 10). A relationship between the vacuum mixing angle θ and the effective mixing angle in matter, α, can easily be found from Fig. 1 as $\tan2\alpha=\sin2\theta/(\cos2\theta-a)$.

As ν_e propagates in the Sun, the vector $\omega\hat{\mathbf{n}}$ migrates upward toward the vacuum vector $\omega_0\hat{\mathbf{n}}_0$ with its x component fixed as $\omega_0\sin2\theta$. At the crossover point (which for convenience we shift to $t=0$) $\omega(t)$ has its *minimal* value

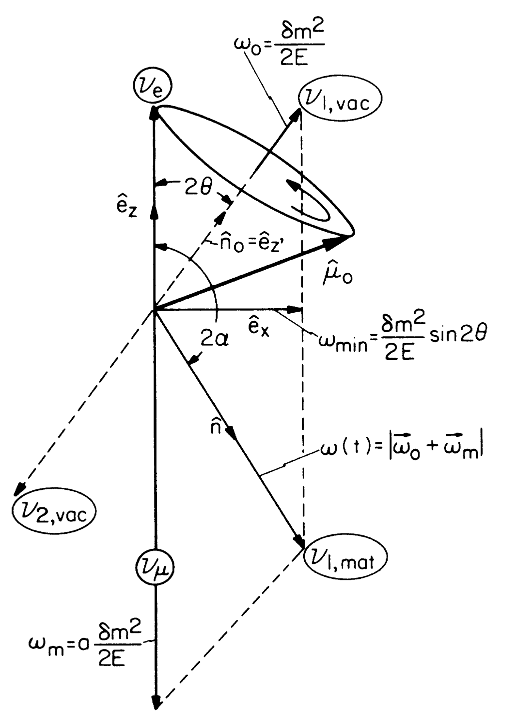

FIG. 1. The precession analog of neutrino oscillations. The $\hat{\mathbf{e}}_z(-\hat{\mathbf{e}}_z)$ direction corresponds to $\nu_e(\nu_\mu)$. The vacuum mass eigenstates $\nu_{1,vac}$ and $\nu_{2,vac}$ are represented by $\pm\hat{\mathbf{n}}_0$. Neutrino oscillations in vacuum correspond to the precession of the state vector $\hat{\boldsymbol{\mu}}_0$ around $\hat{\mathbf{n}}_0$ with the vacuum precession frequency $\omega_0=\delta m^2/2E$. In matter the instantaneous state vector (which is not displayed in Fig. 1) precesses around $\hat{n}$ with a frequency ω. $\boldsymbol{\omega}$ is given by the vector sum of $\boldsymbol{\omega}_0$ and $-(\delta m^2/2E)a(t)\hat{\mathbf{e}}_z$ which is the matter contribution.

$$\omega(0)=(\sin2\theta)\frac{\delta m^2}{2E}\,. \tag{13}$$

The rate of migration of the axis $\hat{\mathbf{n}}(t)$ can be read off from Fig. 1:

$$\omega_m=\frac{1}{2}\left|\frac{d}{dt}\left[\arctan\frac{\sin2\theta}{\tilde{a}}\right]\right|$$
$$=\frac{1}{2}\frac{|a'(t)|}{\sin2\theta}\frac{1}{1+[\tilde{a}(t)/\sin2\theta]^2}\,. \tag{14}$$

For simplicity we will adopt the linear approximation in which the vector $\boldsymbol{\omega}(t)$ travels with constant speed along the vertical line in Fig. 1, i.e.,

$$\tilde{a}(t)\equiv a(t)-\cos2\theta=a'(0)t \tag{15}$$

in which case $a'(t)=a'(0)=$ const. At crossover $(t=0)$ $\tilde{a}(t)$ vanishes and the expression (14) for ω_m obtains its *maximal* value

$$\omega_m(0)=\tfrac{1}{2}\,|a'(0)|\,/\sin2\theta\,. \tag{16}$$

Intuitively we expect the ratio of precession frequency to migration angular velocity to be a measure of the goodness of the adiabatic approximation. This ratio obtains its minimal value at $t=0$,

$$r\equiv\frac{\omega(0)}{\omega_m(0)}=\frac{2\sin2\theta}{|a'(0)|}\frac{\delta m^2}{2E}\,, \tag{17}$$

and hence we expect the probability of level jumping at crossover to be a monotonically decreasing function of r.

The adiabatic approximation applies when ω_m is much smaller than the precession $\omega(t)$, i.e., large r, so that the system "can follow" the changing $\mathbf{B}(t)$. Thus, if initially on the upper branch, $\mathscr{E}_2(t)$ say, the system will remain on the same branch.

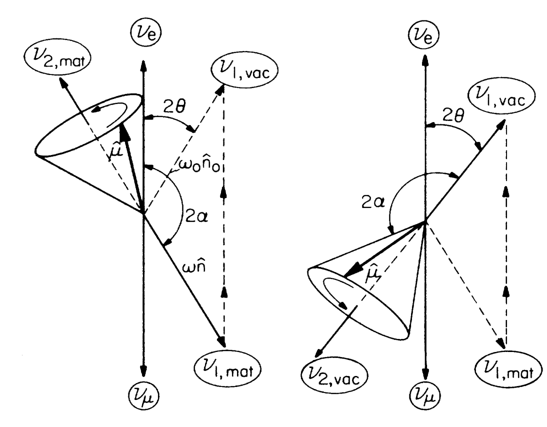

FIG. 2. Neutrino oscillations in matter. When ν_e is produced in the Sun, the state vector $\hat{\boldsymbol{\mu}}$ precesses around $\hat{\mathbf{n}}$ as shown in (a). In the adiabatic case, $\hat{\boldsymbol{\mu}}$ continues to precess around $\hat{\mathbf{n}}(t)$ until $\hat{\mathbf{n}}(t)$ becomes $\hat{\mathbf{n}}_0$. (This is when the neutrino leaves the Sun.) When observed at the Earth, $\hat{\boldsymbol{\mu}}$ is precessing around $\hat{\mathbf{n}}_0$, as shown in (b).

In our picture, the solar-neutrino oscillation may be described as follows. When ν_e is produced in the Sun, the vector $\hat{\boldsymbol{\mu}}$ is precessing around the axis $\hat{\mathbf{n}}$ as shown in Fig. 2(a). By the time when ν_e leaves the Sun, $\hat{\mathbf{n}}$ has migrated to $\hat{\mathbf{n}}_0$ and when this process is adiabatic, $\hat{\boldsymbol{\mu}}$ continues to precess around $\hat{\mathbf{n}}$ which eventually becomes $\hat{\mathbf{n}}_0$ as shown in Fig. 2(b). The adiabatic process means that the opening angle of the cone formed by the precessing $\hat{\boldsymbol{\mu}}$ remains the same. The probability that ν_e survives as ν_e is given, from Eq. (7), by

$$P(\nu_e \to \nu_e) = \tfrac{1}{2}(1 + \hat{\boldsymbol{\mu}}\cdot\hat{\mathbf{e}}_z) , \tag{18}$$

where $\hat{\mathbf{e}}_z$ and $\hat{\boldsymbol{\mu}}$ are given by Eqs. (8a) and (8b) with θ in Eq. (8b) replaced by α. When averaged over time, Eq. (18) leads to the well-known result[7,8]

$$P(\nu_e \to \nu_e) = \tfrac{1}{2}(1 + \cos 2\theta \cos 2\alpha) . \tag{19}$$

In general, if $H_{12}(t)$ is a perturbation effecting $\mathscr{E}_2 \to \mathscr{E}_1$ transitions, we have, to lowest order in H_{12}, a transition amplitude

$$A_{12} \simeq \int_{-\infty}^{\infty} dt\, H_{12}(t) \exp\left[i \int_{-\infty}^{t} \mathscr{E}_2(t')dt'\right] \exp\left[i \int_{t}^{\infty} \mathscr{E}_1(t')dt'\right] \tag{20a}$$

$$= e^{i\Phi} \int_{-\infty}^{\infty} dt\, H_{12}(t) \exp\left[i \int_{0}^{t} \omega_{21}(t')dt'\right] , \tag{20b}$$

where the exponentials in (20a) represent the phase factors picked by the system prior to and after the $2 \to 1$ jump at time t and $\omega_{21} = \mathscr{E}_2 - \mathscr{E}_1$. In Eq. (20b) we factorized a constant phase $e^{i\Phi}$ which does not affect the transition probability given by, to lowest order,

$$P_{12} \equiv |A_{12}|^2 . \tag{21}$$

The migration of the axis $\hat{\mathbf{n}}(t)$ can be viewed as precession around the $\hat{\mathbf{e}}_y$ direction with frequency ω_m. This is equivalent to an off-diagonal matrix element of a Hamiltonian for the equation of motion of ν_1 and ν_2:

$$H_{12}(t) = \omega_m(t) = \frac{1}{2} \frac{|a'(0)|}{\sin 2\theta} \frac{1}{1 + [\tilde{a}(t)/\sin 2\theta]^2} . \tag{22}$$

Substituting Eq. (22) into (20b), we finally obtain

$$A_{12}^{(1)} = e^{i\Phi} \int_{-\infty}^{\infty} dt \frac{1}{2} \frac{|a'(0)|}{\sin 2\theta} \frac{\exp\left[i \int_0^t \{[a'(0)t']^2 + \sin^2 2\theta\}^{1/2} \frac{\delta m^2}{2E} dt'\right]}{1 + [a'(0)t]^2/\sin^2 2\theta} , \tag{23}$$

with the (1) suffix indicating first-order perturbation.

Defining

$$\sinh z \equiv \frac{|a'|t}{\sin 2\theta} , \tag{24}$$

we can rewrite Eq. (23) as

$$A_{12}^{(1)} = \tfrac{1}{2} e^{i\Phi} \int_{-\infty}^{\infty} dz\, e^{f(z)} , \tag{25}$$

where

$$f(z) = \frac{ir}{4}(z + \tfrac{1}{2}\sinh 2z) - \ln \cosh z . \tag{26}$$

Calculating Eq. (25) with use of the saddle-point method (stationary-phase approximation), we find

$$A_{12}^{(1)} \simeq e^{i\Phi} \left[\frac{\pi}{6}\right]^{1/2} e^{1/3} e^{-\pi r/8} ,$$

or

$$P_{12}^{(1)} \simeq \left[\frac{\pi}{6} e^{2/3}\right] e^{-\pi r/4} \simeq e^{-\pi r/4} , \tag{27}$$

which is the Landau-Zener (LZ) formula.[11,12]

The LZ formula is a special case of the general WKB formula

$$A_{12}^{\rm WKB} = e^{i\Phi} \exp\left[\int_0^{t_0} \omega_{21}(t')dt'\right] , \tag{28}$$

where t_0 satisfies $\omega_{21}(t_0) = 0$ and is the complex crossing point nearest to the real axis. Equation (28) corresponds to our Eq. (20b). In the stationary-phase approximation, Eq. (28) allows us to generalize to the Zener formula beyond the linear approximation (15) to treat three-level crossings, etc.[13]

Finally, we note that though the route yielding Eq. (27) in the treatment of Zener[12] and Dar *et al.*[9] [by reducing Eq. (1) to an exactly solvable Weber equation and investigating the asmptotic behavior of these solutions] appears to be very different, it is actually very similar. The asymptotic behavior of special functions are generally obtained by using the WKB (stationary phase) approximation on integral representations anyway.

Let us next discuss P_{12} in the extreme nonadiabatic case with the r parameter very small; i.e., precession which is slow compared with the migration of the axis. Consider the evolution of the system between two times symmetric relative to the crossover $t = 0$ which, in order to include the bulk of the crossover region, are taken to correspond to $\boldsymbol{\omega}$ vectors at $\pm 45°$ relative to the x axis. [See Fig. 3(a).] In the limit of small net precession, the individual rotations $\boldsymbol{\omega}(t)dt$ can be vectorially added to yield

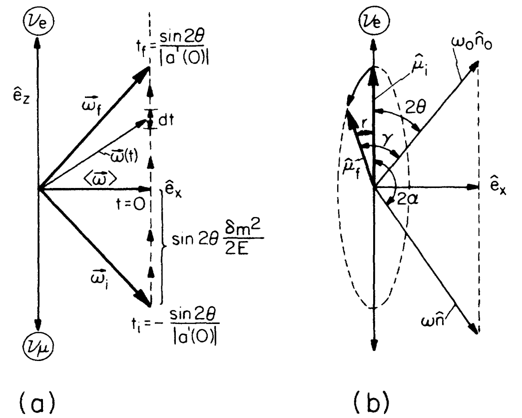

FIG. 3. The transition through the crossover region from $\omega_i = (\delta m^2/2E)\sin 2\theta(\hat{e}_x - \hat{e}_z)$ to $\omega_f = (\delta m^2/2E)\sin 2\theta(\hat{e}_x + \hat{e}_z)$ at time $t_{i,f} = \mp \sin 2\theta / |a'(0)|$, respectively. The intermediate $\omega(t)$ vector travels as indicated by the arrow at constant speed $d\omega_z/dt = a'(0)$ along the line parallel to the z axis. $\langle \omega \rangle$ is the average value of ω. The effect of the $\hat{e}_x$ rotation by angle r around the x axis on the initial state vector $\hat{\mu}_i$ is shown in (b).

a resultant rotation

$$\int_{t_i}^{t_f} \omega(t)dt = \langle \omega \rangle (t_f - t_i)$$
$$= \frac{(\sin 2\theta) 2(\sin 2\theta)}{|a'(0)|} \frac{\delta m^2}{2E} \hat{e}_x$$
$$= r\hat{e}_x \ , \qquad (29)$$

with r defined in Eq. (17).

The corrections due to noncommutativity of rotation will be only $O(r^2)$ and by symmetry we expect that the net rotation will be along the x axis as in Eq. (29).

That is, in contrast with the adiabatic case where the vector $\hat{\mu}_i$ representing ν_e precesses continuously around $\hat{n}(t) = \cos 2\alpha \hat{e}_z + \sin 2\alpha \hat{e}_x$, in the extreme nonadiabatic case, since $\hat{n}$ migrates rapidly (compared with the precession) to $\hat{n}_0$, $\hat{\mu}_i$ is simply rotated to $\hat{\mu}_f$ by an angle r [see Fig. 3(b)] according to Eq. (29).

The probability $P(\nu_e \to \nu_e)$ in this case is given by

$$P_{\rm NA}(\nu_e \to \nu_e) = \tfrac{1}{2}(1 + \cos\gamma \cos 2\theta) \ , \qquad (30)$$

where the subscript NA denotes the nonadiabatic case, and γ is the angle between $\hat{\mu}_f$ and $\hat{n}_0$, i.e., $\cos\gamma = \cos r \cos 2\theta$. For nonadiabatic processes, Eq. (19) is modified to

$$P_{\rm NA}(\nu_e \to \nu_e) = \tfrac{1}{2}[1 + (1 - 2P_{12})\cos 2\theta \cos 2\alpha] \ , \qquad (31)$$

where P_{12} is the transition probability from level 2 to level 1. Comparing Eq. (30) with Eq. (31), we obtain P_{12} applicable to the extreme nonadiabatic case as[14]

$$P_{12} = \frac{1}{2}\left[1 - \cos r \frac{\cos 2\theta}{\cos 2\alpha}\right] \simeq 1 - \frac{r^2}{4} \ , \qquad (32)$$

for $\theta \simeq 0$ and $\alpha \simeq 90°$. This is in agreement with the result obtained analytically in Ref. 13. We note that the result in Eq. (32) is different from $P_{12} \simeq 1 - \pi r/4$ which would have resulted from a naive (and unjustified) extrapolation of the LZ formula in Eq. (27) to small r values.

One of the authors (C.W.K.) would like to thank G. Feldman for valuable discussions. This work was supported in part by the National Science Foundation.

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