Chiral-symmetry breaking in lattice QCD with two and four fermion flavors

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We study the thermodynamics of lattice quantum chromodynamics for two and four flavors of staggered fermions at temperature $T=1/4a$, where a is the lattice spacing. For four flavors with mass below $0.05/a$ we find evidence for a first-order chiral-symmetry-breaking phase transition. For two flavors there is an increasingly rapid crossover as the mass is lowered, but we find no conclusive evidence of a first-order transition for quark masses down to $0.0125/a$. A rough estimate is made of the zero-mass transition temperature for two flavors of quarks.

I. INTRODUCTION

The phase structure of lattice gauge theory for finite temperature is important for an understanding of quantum chromodynamics and has potential implications for both astrophysics and nuclear physics. This problem has been studied extensively for pure SU(3) lattice gauge theory[1] and results have recently been reported by several groups for SU(3) lattice gauge theory with four flavors of dynamical, staggered fermions.[2-5] In this paper we present further data for four flavors of staggered fermions as well as results for two light flavors.

The partition function for the theory we study can be written in the form

$$\begin{aligned} Z &= \int [\delta U] e^{-S_W(U)} \det M(U)^{N_f/4} \\ &= \int [\delta U] e^{-S_W(U)} \det[P_e M^\dagger(U) M(U) P_e]^{N_f/4} \\ &= \int [\delta U] e^{-S_{\rm eff}} . \end{aligned} \tag{1}$$

S_W is the Wilson action for pure gauge theory, and the fermion matrix $M(U)$ is given by

$$M(U)_{i,j} = 2ma\delta_{i,j} + \sum_\mu \frac{n_{i,\mu}}{\alpha_\mu} (U_{i,\mu}\delta_{i,j-\mu} - U^\dagger_{i-\mu,\mu}\delta_{i,j+\mu}) . \tag{2}$$

i and j refer to lattice points and μ is a unit vector on the lattice. $U_{i,\mu}$ is the SU(3) matrix associated with the link leaving the ith lattice point in the μ direction. m is the quark mass and $\eta_{i,\mu}$ are the usual staggered-fermion phases. The lattice spacings in the space and time directions are given by $a\alpha_s$ and $a\alpha_t$, respectively, with $\alpha_\mu=\alpha_s$ for μ pointing in any of the spatial directions and $\alpha_\mu=\alpha_t$ for μ pointing in the time direction. We shall be interested in the limit $\alpha_s,\alpha_t \to 1$. The number of quark flavors is N_f. The matrix M has been raised to the $N_f/4$ power since M itself describes four flavors of quarks. In the second line of Eq. (1) we have used the well-known fact that $M^\dagger M$ does not connect even and odd lattice sites, and has the same determinant on both sublattices.[6] P_e is a projection operator onto even sites. Its introduction prevents another doubling of flavors.

In order to carry out our simulation we wish to generate a set of configurations of the U's with a probability distribution proportional to $\exp(-S_{\rm eff})$. We do so by making use of the hybrid-molecular-dynamics algorithms described in detail in Ref. 7. In order to employ these algorithms we must extend the effective action by introducing an auxiliary field that plays the role of the momentum in a standard molecular-dynamics calculation. Integration of the molecular-dynamics equations moves the system along surfaces of constant probability for the extended effective action. In integrating these equations numerically we introduce a finite step size Δt. We define one physical time unit of the simulation to consist of $1/\Delta t$ molecular-dynamics steps plus at least one heat-bath updating of the auxiliary "momentum" field. The time histories we present are all measured in these units.

In Sec. II we present results for four flavors of quarks with masses in the range $0.5 \geq am \geq 0.025$. Most of our work was on $8^3\times 4$ lattices with some checks of finite-size effects on $10^3\times 4$ lattices. Although for all masses we find a range of gauge couplings in which the plaquette and the Polyakov loop undergo rapid change, we find strong evidence for a first-order phase transition only for $am=0.025$. Our results are in good agreement with those of other authors.[2-5]

In Sec. III we set out the formalism needed to evaluate the energy and pressure of the two-flavor theory. Then in Sec. IV we present our two-flavor results. We again find a region of rapid crossover, but do not find evidence of a first-order phase transition for $am \geq 0.0125$. It is, of course, possible that a first-order transition does exist at a finite mass lower than the ones we have studied, or that

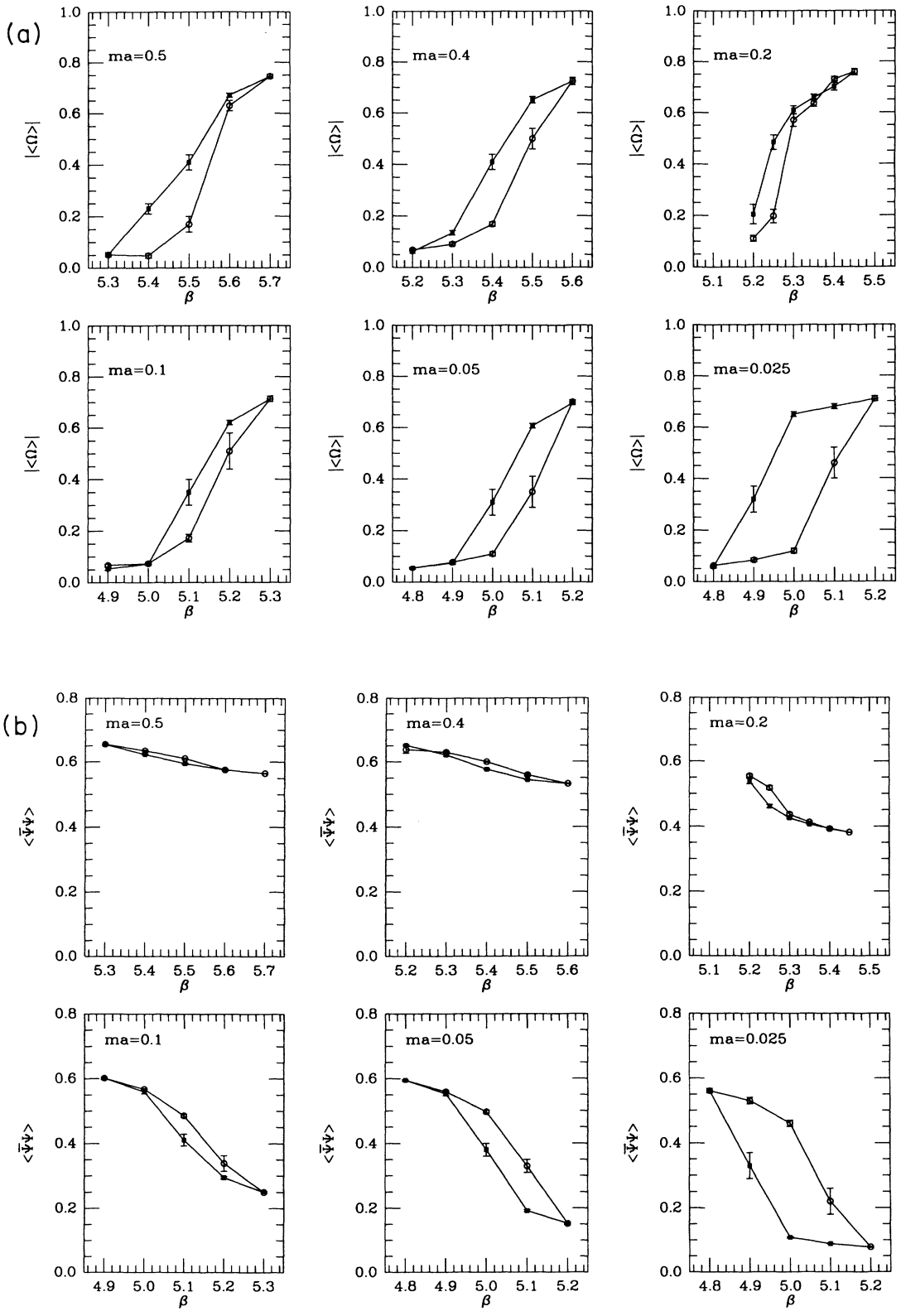

FIG. 1. Hysteresis loops for (a) $|\Omega|$ and (b) $\bar{\psi}\psi$. Open circles are those obtained while increasing the coupling constant. Solid squares are those obtained while decreasing the coupling. One hundred time steps were run at each value of the coupling except for $ma = 0.025$, for which 50 time steps were run.

the transition is second order and does not occur until there is a zero quark mass. We give a rough estimate of the zero-mass transition temperature.

II. FOUR-FLAVOR RESULTS

In this section we discuss the thermodynamics of QCD with four flavors of equal-mass staggered fermions. Except for the results shown in Fig. 3, all data were taken on an $8^3\times4$ lattice. For this part of our study we used the Φ algorithm of Ref. 7 in which the fermion determinant is eliminated by the introduction of an auxiliary scalar field.

It is known that in the limit of large quark mass the theory has a first-order phase transition associated with color deconfinement.[1] However, it has been observed that this transition weakens as the quark mass is lowered, and finally disappears.[2,3] For a sufficiently small quark mass, evidence has been found for a first-order transition associated with chiral-symmetry breaking.[4,5] The range of masses over which the transition is absent is large—about an order of magnitude.

We began our study of this problem by making runs for a variety of masses in which the gauge coupling $\beta=6/g^2$ was varied over a region in which measured quantities changed rapidly. We began with an equilibrated lattice on one side of the crossover region and slowly changed β, going through the crossover region and then returning. The coupling was incremented or decremented in steps of $\Delta\beta=0.1$ for all mass values except $ma=0.2$, for which steps were $\Delta\beta=0.05$. One hundred time units were run at each value of the coupling except for $ma=0.025$, for which 50 time units were run. The resulting hysteresis loops for the magnitude of the Polyakov loop, $|\Omega|$, and the chiral order parameter $\bar{\psi}\psi$ are shown in Figs. 1(a) and 1(b), respectively. In these graphs the open circles are those points obtained while increasing the coupling constant, and the solid squares are those obtained while decreasing the coupling. It will be noted that there is a region of rapid variation in $|\Omega|$ for all masses, and that the overall jump in this quantity does not vary appreciably as a function of mass. Not surprisingly, there is significant variation in $\bar{\psi}\psi$ only for small values of ma. These results are in agreement with those of Ref. 2.

The hysteresis loops seen in Fig. 1 can arise either because of an increase in the equilibration time in the vicinity of the crossover region, or because of the presence of a first-order phase transition. In order to investigate this point we searched for the coexistence of two different states. This was done by equilibrating the lattice at values of β above and below the crossover region, and then following the time history of $|\Omega|$ after the coupling had been set to the crossover value. In Fig. 2 we see that for short runs approximately coexisting states were found for all values of ma except $ma=0.2$. However, upon extending the runs, we found that the system evolved into one or the other states for all values of ma except $ma=0.025$. Results of the longer runs are illustrated in Fig. 3 where we show data for $ma=0.05$ on both $8^3\times4$ and $10^3\times4$ lattices. It will be noted that the values of $|\Omega|$ in the two states are virtually independent of volume. At a slightly larger value of β, both hot and cold lattices quickly equilibrate to the larger values of $|\Omega|$, while at

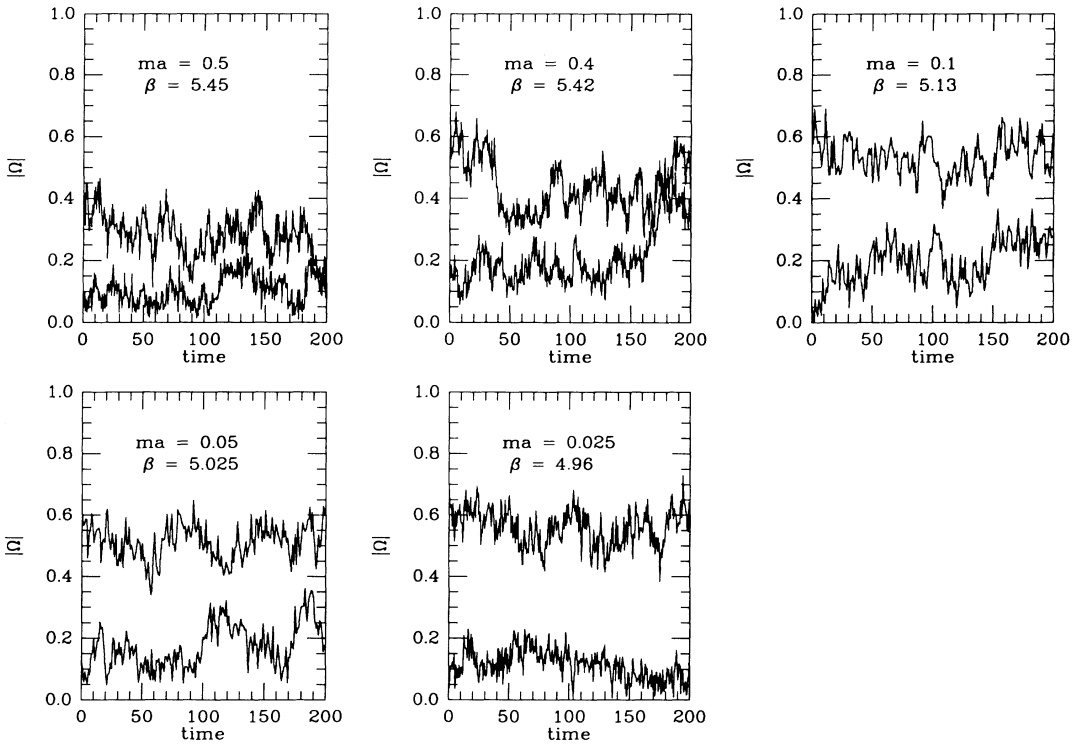

FIG. 2. Approximately coexisting states. Each plot shows two time histories for $|\Omega|$ coming from the same system, but with different initial conditions.

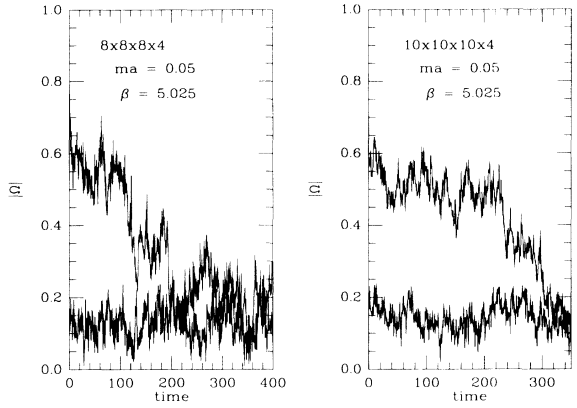

FIG. 3. Longer time histories of states with different initial conditions for $ma=0.05$. One plot is for an $8^3\times4$ lattice, the other is for a $10^3\times4$ lattice.

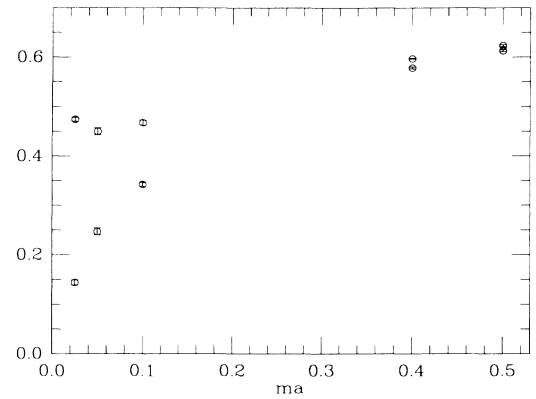

FIG. 5. A plot of the values of $\langle\bar{\psi}\psi\rangle$ immediately above and below the crossover region as a function of mass.

$\beta=5.00$ they quickly equilibrate to the lower value. There is a sharp crossover region, with a rather long equilibration time at the transition coupling, but there is no conclusive evidence for a first-order transition at $ma=0.05$.

In Fig. 4 we show the time history of the Polyakov loop for $ma=0.025$ and $\beta=4.96$ from hot and cold starts. We interpret the rapid change in the state with a large value of $|\Omega|$ at $t=380$ as a genuine tunneling. We believe our results are indicative of a first-order phase transition at $ma=0.025$ in accordance with the conclusions of Refs. 4 and 5. However, we have not conclusively ruled out the possibility that the crossover simply becomes sharper and the equilibration time longer at smaller masses, and that there is no phase transition for finite ma.

Our value for the critical coupling constant at $ma=0.025$ is $\beta_c=4.96\pm0.03$. We have measured the discontinuity in the magnitude of the Polykov loop, the chiral order parameter, and the plaquette P in the two phases at $ma=0.025$ to be

$$\begin{aligned}\Delta\langle|\Omega|\rangle&=0.45\pm0.02\ ,\\ \Delta\langle\bar{\psi}\psi\rangle&=0.33\pm0.01\ ,\\ \Delta\langle P\rangle&=0.136\pm0.006\ .\end{aligned}\tag{3}$$

All of these results are in agreement[8] with those reported in Ref. 5.

In Fig. 5 we plot the values of the chiral order parameter $\langle\bar{\psi}\psi\rangle$ immediately above and below the crossover region as a function of mass. The order parameter in the symmetric phase extrapolates toward zero at $ma=0.0$, while in the broken-symmetry phase it approaches a finite value. The error bars in Fig. 5 are statistical, there may also be systematic errors arising from uncertainty in the position of the crossover points. Finally in Fig. 6 we plot the value of β at the crossover or transition point as a function of quark mass. The curve extrapolates to a zero mass value of $\beta=4.90\pm0.03$.

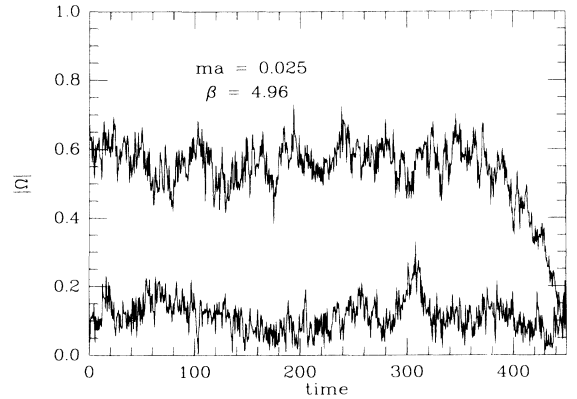

FIG. 4. Time history of $|\Omega|$ for $ma=0.025$ and $\beta=4.96$ from hot and cold starts.

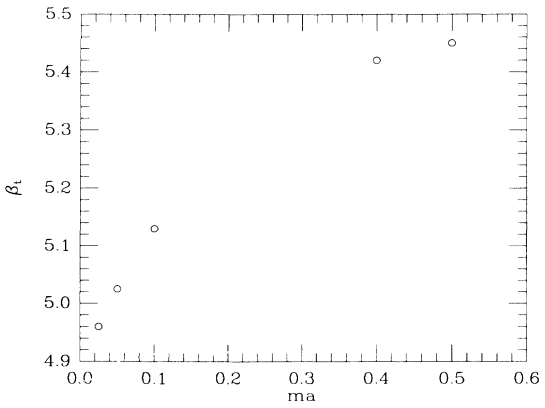

FIG. 6. The value of β at the crossover or transition point as a function of quark mass for four flavors of fermions.

III. ENERGY AND PRESSURE OF LATTICE QCD

The energy density and pressure are computed from the expressions

$$\epsilon = -\frac{1}{V}\frac{\partial \ln Z}{\partial \beta}\bigg|_V ,$$
$$p = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}\bigg| , \tag{4}$$

where the partition function Z is defined in Eq. (1). If we denote the gauge and fermion contributions to the energy density by ϵ_g and ϵ_f, respectively, and introduce similar notation for the pressure, then

$$a^4\epsilon_g = \frac{1}{N_s{}^3N_t}\sum_{\text{sites}}\left\langle \left[-\frac{6}{g^2}+6\frac{\partial g_s{}^{-2}}{\partial\xi}\right]P_{\text{ss}} + \left[\frac{6}{g^2}+6\frac{\partial g_t{}^{-2}}{\partial\xi}\right]P_{\text{st}}\right\rangle ,$$
$$3a^4p_g = \frac{1}{N_s{}^3N_t}\sum_{\text{sites}}\left\langle \left[-\frac{6}{g^2}+6\frac{\partial g_s{}^{-2}}{\partial\xi}+6\frac{\partial g_s{}^{-2}}{\partial \ln\alpha_s}\right]P_{\text{ss}} + \left[\frac{6}{g^2}+6\frac{\partial g_t{}^{-2}}{\partial\xi}+6\frac{\partial g_t{}^{-2}}{\partial \ln\alpha_s}\right]P_{\text{st}}\right\rangle . \tag{5}$$

Here P_{ss} and P_{st} are the space-space and space-time plaquettes and g_s and g_t are their respective couplings. $\xi=\alpha_s/\alpha_t$, and we have set $\alpha_s=\alpha_t=1.0$ after performing the differentiation.[9]

Using the fact that $\text{tr}A=\langle R^*AR\rangle_R$, where R is a Gaussian random vector, we can write the fermion contributions to the energy density and pressure in the form

$$a^4\epsilon_f = \frac{N_f}{4N_s{}^3N_t}\sum_{\text{sites}}\langle (X^*P_eM_t^\dagger R+\text{H.c.})\rangle + \frac{N_f}{2N_s{}^3N_t}a\frac{\partial m}{\partial\xi}\sum_{\text{sites}}\langle (X^*P_eR+\text{H.c.})\rangle_R ,$$
$$3a^4p_f = -\frac{N_f}{4N_s{}^3N_t}\sum_{\text{sites}}\langle (X^*P_eM_s^\dagger R+\text{H.c.})\rangle + \frac{N_f}{2N_s{}^3N_t}a\left[\frac{\partial m}{\partial\xi}+\frac{\partial m}{\partial \ln\alpha_s}\right]\times\sum_{\text{sites}}\langle (X^*P_eR+\text{H.c.})\rangle_R , \tag{6}$$

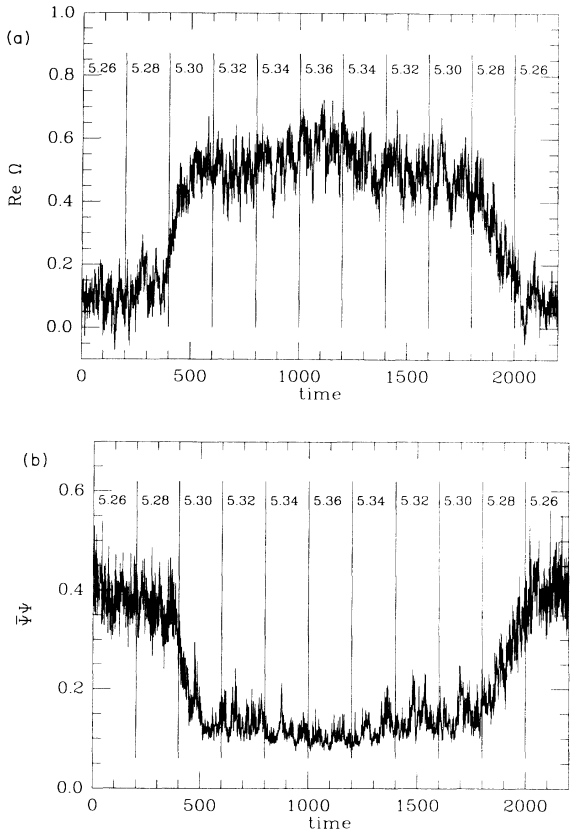

FIG. 7. Unfolded hysteresis loops for (a) ReΩ and (b) $\bar{\psi}\psi$ for two flavors of fermions with $ma=0.025$ on an $8^3\times4$ lattice. The run was started at $\beta=5.26$ and β was changed by 0.02 every 200 time units.

where $X=(1/M)R$, and M_s and M_t are the spatial and temporal hopping terms in the fermion matrix, Eq. (2).

A perturbative calculation of $\partial g_s{}^{-2}/\partial\xi$ and $\partial g_t{}^{-2}/\partial\xi$ has been carried out for the pure gauge theory by Karsch[9] and extended to zero fermion mass by Trinchero.[10] These calculations have been done to one-loop order so we must also use the one-loop β function in Eq. (6). At zero temperature, or on an N^4 lattice, the space-space and space-time plaquettes are equal, and M_t and M_s give the same contributions. Euclidean invariance requires that $\epsilon=-p$ at zero temperature. This in turn requires that, neglecting

TABLE I. Parameters used in the conjugate-gradient calculations for two flavors of quarks. ma is the quark mass in units of the lattice spacing, Δt is the microcanonical step size, R the conjugate gradient residual, V the volume of the lattice, and N_{cg} the average number of conjugate-gradient iterations. For masses 0.1, 0.05, and 0.025 we give N_{cg} for β immediately above and below the crossover region. For 0.0125 we given N_{cg} exactly at the crossover.

ma	Δt	$(R^2/V)^{1/2}$	N_{cg}
0.1	0.04	0.005	55 (β=5.3) 41 (β=5.5)
0.05	0.04	0.005	93 (β=5.26) 65 (β=5.38)
0.025	0.02	0.005	149 (β=5.26) 86 (β=5.34)
0.0125	0.01	0.0025	240(β=5.275)

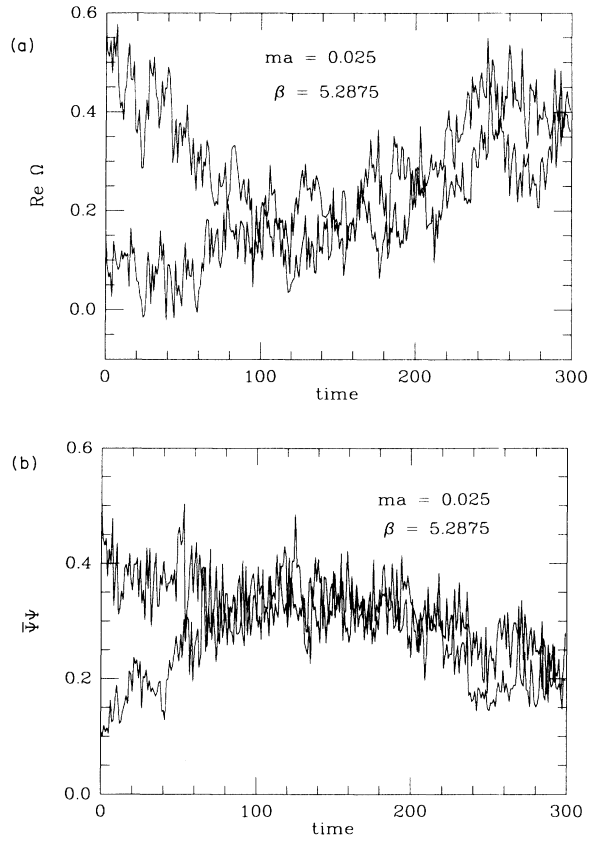

FIG. 8. Time histories for (a) ReΩ and (b) $\bar{\psi}\psi$ for two flavors of fermions with $ma=0.025$ and $\beta=5.2875$ on an $8^3\times4$ lattice. The initial configurations were equilibrated lattices at $\beta=5.2$ and 5.36.

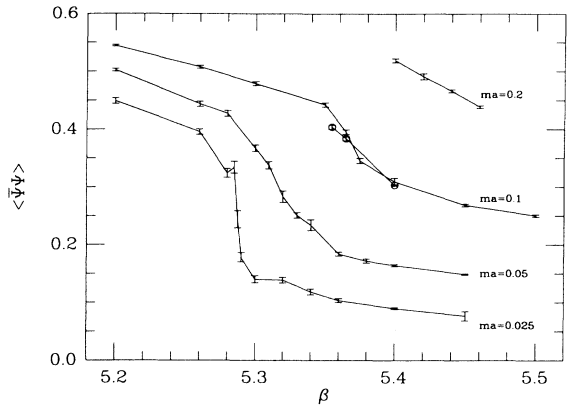

FIG. 9. $\langle\bar{\psi}\psi\rangle$ vs β for various masses. All points were obtained on an $8^3\times4$ lattice except those indicated with circles, which were obtained on a $10^3\times4$ lattice.

the quark mass,

$$\frac{\partial g_s^{-2}}{\partial\xi^2}+\frac{\partial g_t^{-2}}{\partial\xi^2}=\frac{1}{2}\frac{\partial g^{-2}}{\partial\ln a} \tag{7}$$

which has been previously deduced by Karsch from the requirement of invariance of the string tension.[9] Using the two-loop β function but one-loop asymmetry coefficients destroys this relation and leads to violation of Euclidean invariance at zero temperature. (Strictly speaking, we do not have exact Euclidean invariance on our N^4 lattice since we are using antiperiodic boundary conditions for the fermions in the time direction and periodic boundary conditions in the spatial directions, but this effect appears to be small.) However, it would be inconsistent to ignore the derivatives of g^{-2} completely. As can be seen in Eq. (5), this would constrain the energy and pressure so that $\epsilon=3p$, and we could not have a discontinuous energy with a continuous pressure as we cross the transition. For the energy to jump with the pressure continuous, we must have contributions of the same order from the derivative terms and the $6/g^2$ terms. This implies that the dependence of $P_{ss}-P_{st}$ on temperature is higher order in g^2 than the dependence of $P_{ss}+P_{st}$. Because $\partial m/\partial\xi$ and $\partial m/\partial\ln\alpha_s$ are proportional to the quark mass, we neglect them.

Since g is not small in our simulation, the perturbative calculations of β and $\partial g_{s,t}^{-2}/\partial\xi$ may not be good approximations. From the pure gauge theory we know that the perturbative β function does not properly describe the scaling of g_c with N_T for $N_T=4$. In contrast, the perturbative calculation of the asymmetry coefficients works remarkably well at $N_T=4$ (Ref. 11). We cannot sort out the complicated issue of the accuracy of perturbation theory here, so we will proceed as best we can. Looking ahead to our results, we find that the pressure is remark-

TABLE II. Values of the space-space plaquette P_{ss}, the space-time plaquette P_{st}, and the chiral order parameter $\langle\bar{\psi}\psi\rangle$, as a function of β on an $8^3\times4$ lattice for $ma=0.1$.

β	P_{ss}	P_{st}	$\langle\bar{\psi}\psi\rangle$
5.200	1.3695±0.0020	1.3725±0.0019	0.544 48±0.001 29
5.260	1.4230±0.0032	1.4221±0.0027	0.507 54±0.002 15
5.300	1.4527±0.0045	1.4561±0.0042	0.478 91±0.002 66
5.350	1.5003±0.0021	1.5017±0.0017	0.442 36±0.004 79
5.365	1.5314±0.0040	1.5352±0.0044	0.397 61±0.006 78
5.375	1.5629±0.0052	1.5687±0.0051	0.347 51±0.008 19
5.400	1.5891±0.0053	1.5972±0.0060	0.309 45±0.008 17
5.450	1.6274±0.0018	1.6383±0.0028	0.268 36±0.002 77
5.500	1.6525±0.0029	1.6616±0.0033	0.250 22±0.002 61

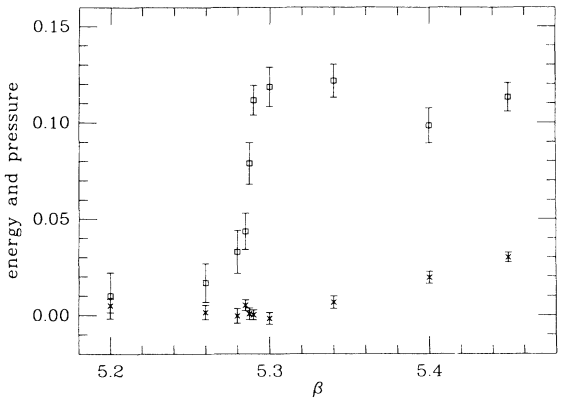

FIG. 10. The energy and pressure (in units of $1/a^4$) on an $8^3\times4$ lattice with $ma=0.025$. The zero-temperature contribution has been subtracted out as described in the text. The subtractions at $\beta=5.285$, 5.2875, and 5.29 were obtained by interpolating between $\beta=5.28$ and 5.3 on the 8^4 lattice. This is justified since the 8^4 lattice is behaving smoothly in this region.

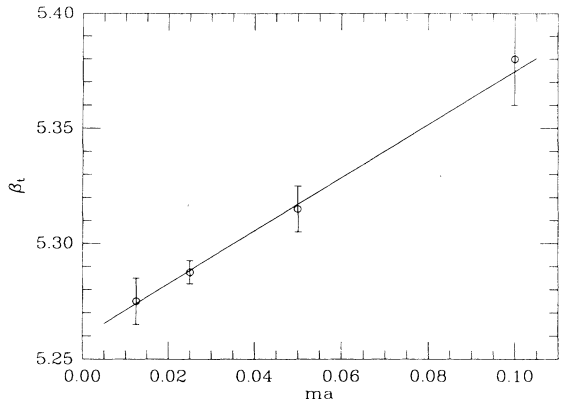

FIG. 11. The value of β at the crossover point as a function of quark mass for two flavors of fermions.

ably constant across the transition region which provides an internal check.

IV. TWO-FLAVOR RESULTS

We simulated QCD with two flavors of quarks using the "R algorithm" described in Ref. 7. Most of our simulations were done on $8^3\times4$ lattices, but we also ran on some 8^4 lattices to get the zero-temperature energy and pressure.

We began our study by checking the effects of the finite step size in the molecular dynamics steps and the effect of different stopping residuals in the conjugate gradient on $\bar{\psi}\psi$ and the average plaquette. These results will be reported in more detail elsewhere. Based on these results we adopted the parameters in Table I. Here Δt is the microcanonical time step, R is the conjugate-gradient residual, and V is the volume of the lattice. Since the random source vector has a magnitude proportional to the lattice volume, we normalize the residual by dividing by V.

We have carried out thermal cycles for various masses as in the case of four flavors. We typically ran a few hundred time steps at each value of β. Figure 7 shows the time history of the Polyakov loop and $\bar{\psi}\psi$ for such an "unfolded" loop at $ma=0.025$. The abscissa on this plot is actually the molecular dynamics time, and β was changed every 200 trajectories as indicated on the plot. Notice that the measured quantities change dramatically over a small range of β just as in the case of four flavors. We next made some long runs in the apparent transitional range of β with hot and cold starting lattices. Figure 8 is the time history of such a pair of runs for $ma=0.025$. In all cases we found that the hot and cold starts evolved slowly together, as in Fig. 8, over a period of as much as 200 time units. This is in contrast with the behavior expected at a first-order transition where if the time histories converged at all they would be expected to do so in a "tunneling" event. As we decreased the quark mass we found that the correlation time of the system increased and the dependence of β became sharper. For example, Fig. 9 is a plot of $\langle\bar{\psi}\psi\rangle$ against β for several masses. It is clear from these results that we are approaching a phase transition. However, we cannot say whether this transition is a first-order transition appearing for quark masses smaller than those we have simulated or a higher-order transition, which would be smoothed out for any finite quark mass.

We measured equilibrium thermodynamic quantities from the final parts of the runs after the system had

TABLE III. Values of the fermion component of the energy and pressure, ϵ_f and p_f, as a function of β on an $8^3\times4$ lattice for $ma=0.1$.

β	$a^4\epsilon_f$	a^4p_f
5.200	0.363 02±0.001 19	−0.359 12±0.000 59
5.260	0.362 90±0.000 82	−0.362 09±0.000 70
5.300	0.369 72±0.001 86	−0.360 75±0.000 87
5.350	0.372 56±0.002 84	−0.361 07±0.001 08
5.365	0.377 57±0.001 66	−0.360 59±0.000 35
5.375	0.388 37±0.001 83	−0.358 39±0.000 66
5.400	0.396 30±0.002 39	−0.357 64±0.001 09
5.450	0.400 38±0.002 39	−0.357 13±0.000 97
5.500	0.403 12±0.000 91	−0.357 48±0.000 47

TABLE IV. Values of the total energy and pressure and the quantity $\epsilon-3p$ as a function of β on an $8^3\times4$ lattice for $ma=0.1$.

β	$a^4\epsilon$	a^4p	$a^4(\epsilon-3p)$
5.200	0.8791±0.0026	−0.8585±0.0012	3.4513±0.0041
5.260	0.8817±0.0020	−0.8858±0.0019	3.5362±0.0049
5.300	0.9187±0.0054	−0.8900±0.0034	3.5860±0.0090
5.350	0.9298±0.0117	−0.9104±0.0038	3.6585±0.0014
5.365	0.9575±0.0056	−0.9182±0.0011	3.7097±0.0070
5.375	0.9888±0.0051	−0.9251±0.0019	3.7620±0.0078
5.400	1.0170±0.0088	−0.9310±0.0016	3.8081±0.0073
5.450	1.0485±0.0115	−0.9407±0.0032	3.8692±0.0036
5.500	1.0525±0.0080	−0.9525±0.0027	3.9086±0.0047

equilibrated. We tabulated results for a number of quantities at all measured masses, and we present those for $ma=0.1$ in Tables II–IV. Because of the correlations among different quantities the error bars on the total energy and the pressure and on $\epsilon-3p$ are less than those which would be obtained from simply combining the errors of their individual components.

In studying the energy and pressure we wished to isolate the finite-temperature contributions, since they are the ones accessible to experiment. We therefore measured the energy and pressure on an 8^4 lattice with $ma=0.025$. In order to cancel as much of the systematic error as possible, we used the same microcanonical time step, the same accuracy per site in the conjugate gradient, and the same spatial size in studying the "zero"-temperature and high-temperature lattices. The energy and pressure with the zero-temperature values subtracted are plotted in Fig. 10. Notice that the pressure is much smoother than the energy density across the transition region. This is reassuring since we expect the pressure to be continuous across any transition.

Before beginning this study we did not know if the real world would be better approximated by two or by three flavors of low-mass quarks since the strange-quark mass is roughly equal to the temperature range of interest. Having done the calculation we see that the chiral transition occurs only for very small or vanishing quark masses. Therefore we would not expect the inclusion of a 150-MeV quark to qualitatively change the results. Of course it would shift the crossover value of β.

In Fig. 11 we plot the dependence of β_t, the value of the coupling at which the behavior changes most rapidly, on ma. With $N_T=4$ our temperature is $aT=\frac{1}{4}$. Clearly we would like to know the critical temperature in MeV, and to do so we need to measure a known mass to set the scale. We have only recently started to measure hadron masses, but Billoire and Marinari[12] have reported hadron mass measurements with two flavors of Kogut-Susskind fermions on a $6^3\times24$ lattice at $\beta=5.4$, and their results can be used to make a rough estimate of the transition temperature. Using measurements at quark masses of 0.1 and 0.2 from two different operators coupling to the ρ meson they extrapolate the ρ mass in units of the lattice spacing to be 1.17 or 1.30. If we use the perturbative β function to scale these masses to $\beta=5.26$, we obtain estimates for T_c of 134–121 MeV. However, in the pure gauge theory the effective β function for $N_T\approx4$ was about one-half of the perturbative β function, and there is no reason to expect better behavior with fermions. However, we have not scaled over a very large range of β, and if we arbitrarily use one-half of the perturbative β function these estimates change to 146–132 MeV. These numbers are very uncertain. They contain the uncertainty in our estimate of the critical β extrapolated to zero quark mass as well as the uncertainties in the mass measurements of Billoire and Marinari. We have repeated the calculations of Billoire and Marinari for the π and ρ masses, and found agreement with their results. Calculations for smaller quark masses and larger lattices will be reported elsewhere. Both our high-temperature calculation and the mass calculation suffer from the effects of a coarse lattice spacing and a small spatial size of the lattice. Certainly hadron mass calculations in the pure gauge theory show large systematic errors when measured on lattices as small as this.

Although the transition with dynamical quarks is significantly softer than in the pure gauge theory, perhaps even nonexistent for finite-mass quarks, the energy at least rises very abruptly over a small range of temperature. Certainly QCD with dynamical fermions undergoes a dramatic qualitative change as the temperature is raised.

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