Stochastic quantization and random surface approach to Polyakov string theory

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Polyakov string theories are quantized by stochastic methods. Langevin equations for a string coordinate and two-dimensional metric are invariant under general coordinate transformation and Weyl scaling. In these methods, the conformal anomaly cancels in 26 dimensions and scattering amplitudes of N tachyons in the tree level are computed. As a first step of analytic computation in the random surface approach of a Polyakov string, we evaluate the tachyonic mass and the tachyonic scattering amplitude for right triangles.

I. INTRODUCTION

String theories[1] seem to be an attractive framework for describing all the forces of nature. A viable quantum theory of gravity may be achieved by finite superstring theories. Much work is now in progress for multiloop computations to understand the quantum behavior of bosonic and superstring theories. One expects that nonperturbative aspects of string theories will play a crucial role in our understanding. To these, the string field theory and random surface approach to string theory will provide insights. The latter can be put to numerical computations by Monte Carlo methods. Gauge theories have been successfully studied by Monte Carlo techniques.[2] These procedures inspire a new quantization technique through Langevin equations.[3]

In this paper, we study stochastic quantization and the random surface approach to the Polyakov string theory.

In Sec. II Langevin equations for the string coordinate $\phi(\tau,\sigma)$ and two-dimensional metric $g_{\alpha\beta}(\tau,\sigma)$ are written in a general-coordinate- and Weyl-scaling-invariant way and solved perturbatively using heat-kernel methods. The conformal anomaly computed in the stochastic quantization formulation cancels in 26 dimensions. Tachyonic scattering amplitudes are reproduced in the tree diagram.

For the purpose of implementing a Polyakov string into a numerical Monte Carlo procedure, a random surface approach is examined in Sec. III. However, it is extremely difficult to perform an analytic computation. A first step towards an analytic calculation of tachyonic mass and scattering amplitudes is taken. Finally we discuss methods of implementing the random surface approach in numerical computations.

II. STOCHASTIC QUANTIZATION

Stochastic quantization[3] is formally equivalent to canonical or path-integral quantizations. Green's functions are evaluated over asymptotic distributions of configurations evolving along the fictitious time governed by the Langevin equation. Gauge theories have been treated successfully by this method with and without gauge fixing,[4] and a further numerical computation has been performed.

The non-necessity of gauge fixing and the possibility of implementing into numerical work initiate applications of stochastic quantization techniques to string theories. We propose a stochastic quantization procedure for the Polyakov string theory which preserves both general-coordinate and Weyl-scaling invariances. We briefly comment on stochastic quantization of string field theory in the light cone.

A. General-coordinate- and weyl-scale-transformation-invariant Langevin equation

The Polyakov string action[5] for ϕ and $g_{\alpha\beta}$ of given topology is

$$S=\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{g}\,g^{\alpha\beta}\partial_\alpha\phi_\beta\phi,\quad g=\det(g_{\alpha\beta})\ , \tag{1}$$

where α' is the Regge slope carrying negative-two mass dimensions (we set $2\pi\alpha'=1$) and ξ^α is a two-dimensional coordinate (τ,σ).

The standard procedure of stochastic quantization gives the Langevin equation for ϕ with fictitious time t as

$$\frac{\partial\phi(\xi;t)}{\partial t}=\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\,g^{\mu\nu}\partial_\nu)\phi(\xi;t)+\eta_\phi(\xi;t)\ , \tag{2}$$

To obtain Langevin equation for $g_{\alpha\beta}$, we split $g_{\alpha\beta}$ into a t-independent background $\hat{g}_{\alpha\beta}$ and t-dependent $h_{\alpha\beta}$ as

$$g_{\alpha\beta}(\xi;t)=\hat{g}_{\alpha\beta}(\xi)+h_{\alpha\beta}(\xi;t)\ , \tag{3}$$

its equation is proposed as

$$\frac{\partial h_{\alpha\beta}(\xi;t)}{\partial t}=(\delta_\alpha^\sigma\delta_\beta^\rho-\tfrac{1}{2}g_{\alpha\beta}g^{\sigma\rho})\partial_\sigma\phi\partial_\rho\phi+\eta_{\alpha\beta}(\xi;t)\ . \tag{4}$$

[We have not yet rigorously proved that Eqs. (2) and (4) give the correct quantum theory of Eq. (1).] Here η_ϕ and $\eta_{\alpha\beta}$ are the stochastic noise of ϕ and $h_{\alpha\beta}$, respectively. For simplicity, we require that ϕ, $\hat{g}_{\alpha\beta}$, $h_{\alpha\beta}$, η_ϕ, and $\eta_{\alpha\beta}$ satisfy the boundary condition of a closed string.

Ensemble averages of these noises are defined by

$$\langle \eta_\phi(\xi;t)\eta_\phi(\xi';t')\rangle = 2\frac{1}{\sqrt{\hat{g}}}\delta^2(\xi,\xi')\delta(t-t')$$
$$= \frac{\int [D\eta_\phi]\eta_\phi(\xi;t)\eta_\phi(\xi';t')\exp\left[-\frac{1}{2}\int d^2\xi\, dt\sqrt{\hat{g}}\,\eta^2(\xi;t)\right]}{\int [D\eta_\phi]\exp\left[-\frac{1}{2}\int d^2\xi\, dt\sqrt{\hat{g}}\,\eta^2(\xi;t)\right]}, \tag{5}$$

$$\langle \eta_{\alpha\beta}(\xi;t)\eta_{\sigma\beta}(\xi';t')\rangle = 2\hat{g}_{\alpha(\beta}\hat{g}_{\sigma)\rho}\frac{1}{\sqrt{\hat{g}}}\delta^2(\xi,\xi')\delta(t-t')$$
$$= \frac{\int [D\eta_{\alpha\beta}]\eta_{\alpha\beta}(\xi;t)\eta_{\sigma\rho}(\xi';t')\exp\left[-\frac{1}{2}\int d^2\xi\sqrt{\hat{g}}\hat{g}^{\alpha(\beta}\hat{g}^{\sigma)\rho}\eta_{\alpha\beta}(\xi;t)\eta_{\sigma\rho}(\xi;t)\right]}{\int [D\eta_{\alpha\beta}]\exp\left[-\frac{1}{2}\int d^2\xi\sqrt{\hat{g}}\hat{g}^{\alpha(\beta}\hat{g}^{\sigma)\rho}\eta_{\alpha\beta}(\xi;t)\eta_{\sigma\rho}(\xi;t)\right]}. \tag{6}$$

The stochastic ensemble average (6) should not contain the metric $g_{\alpha\beta}$ in the exponent, since $\eta_{\alpha\beta}$ is the noise driving Langevin equation for the metric $g_{\alpha\beta}$. This situation is circumvented by splitting as in Eq. (3).

Alternatively, one may set up the Langevin equation for $g_{\alpha\beta}$ without splitting as in Eq. (3). The exponent of the noise average in Eq. (6) contains $g_{\alpha\beta}$ and is not Gaussian. See Ref. 6 for a discussion of non-Gaussian noise in stochastic quantization of Einstein gravity.

The noises η_ϕ and $\eta_{\alpha\beta}$ are a scalar and second-rank tensor, respectively, and Eqs. (2)–(6) have the proper general-coordinate transformation.

Under the transformations

$$\begin{aligned} t' &= e^{\sigma(\xi)}t, \\ \phi'(\xi;t') &= \phi(\xi;t), \\ g'_{\alpha\beta}(\xi;t') &= e^{\sigma(\xi)}g_{\alpha\beta}(\xi;t), \\ \eta'_\phi(\xi;t') &= e^{-\sigma(\xi)}\eta_\phi(\xi;t), \\ \eta'_{\alpha\beta}(\xi;t') &= \eta_{\alpha\beta}(\xi;t). \end{aligned} \tag{7}$$

Equations (2)–(6) are covariant. It is worthwhile to note that the fictitious time t transforms also under Weyl scaling. In fact, this transformation is nothing but the local Weyl symmetry in the context of stochastic quantization.

It is difficult to solve coupled Eqs. (2) and (4) analytically. However, these can be treated perturbatively order by order in $h_{\alpha\beta}$. To the zeroth order in $h_{\alpha\beta}$, Eq. (2) can be integrated exactly for a t-independent background $\hat{g}_{\alpha\beta}$ [see also Eqs. (13) and (22)]. The solution of Eq. (2) is

$$\phi(\xi;t) = \int d^2\xi'\sqrt{\hat{g}(\xi')}\int_0^\infty dt'\theta(t-t')\langle\xi|e^{(t-t')\Box_{\hat{g}}}|\xi'\rangle \times\eta_\phi(\xi';t'). \tag{8}$$

For the case of a constant curvature metric for genus ≥ 2, the heat kernel[7] in Eq. (8) can be constructed through the Poincaré series as

$$\begin{aligned} K^{(t)}(\xi,\xi') &= \langle\xi|e^{t\Box_{\hat{g}}}|\xi'\rangle \\ &= \sum_{\gamma\in\Gamma} g_0^{(t)}(\xi,\gamma\xi'). \end{aligned} \tag{9}$$

Here $\Box_{\hat{g}}$ denotes the two-dimensional Laplacian in constant curvature $\hat{g}_{\alpha\beta}$. In complex coordinate $z=\tau+i\sigma$, the action of $\gamma=\binom{a\ b}{c\ d}$ with $ad-bc=1$ on z is $\gamma z=(az+b)/(cz+d)$. The explicit form of $g_0^{(t)}(z,z')$ in Eq. (9) is well known:

$$g_0^{(t)}(z;z') = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}\int_d^\infty db\frac{be^{-b^2/4t}}{(\cosh b-\cosh d)}, \tag{10}$$

where

$$\cosh d = 1+\frac{|z-z'|^2}{2\,\mathrm{Im}z\,\mathrm{Im}z'}.$$

The propagator for ϕ is worked out as

$$\lim_{t\to+\infty}\langle\phi(\xi;t)\phi(\xi';t)\rangle = \int_0^\infty dt' K^{t'}(\xi,\xi'), \tag{11}$$

by using Eqs. (5), (8), and (9).

Actually at the tree level (i.e., Riemann surface with zero genus), it is much easier to work as in Ref. 4 with the gauge-fixed action

$$S_{\text{eff}} = \int dz\,d\bar{z}(\partial_z\phi\partial_{\bar{z}}\phi + b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}), \tag{12}$$

where b and c are antighost and ghost fields. As a ghost and antighost, in fact, decouple from the string coordinates, we only need the Langevin equation for $\phi(z,\bar{z},;t)$:

$$\frac{\partial}{\partial t}\phi(z,\bar{z};t) = \partial_z\partial_{\bar{z}}\phi(z,\bar{z};t)+\eta(z,\bar{z};t)$$

with

$$\langle\eta(z,\bar{z};t)\eta(z',\bar{z}';t')\rangle = 2\delta^2(z-z')\delta(t-t').$$

B. Conformal anomaly in stochastic quantization

As an application of stochastic quantization, we compute the conformal anomaly. The gauge-fixed action (12) contains anticommuting Faddeev-Popov ghost fields. These ghosts are bosonized as in Ref. 8 by replacing conformal dimension-one current $j=2bc$ with bosonic current $j=\partial\chi$. The resulting action reads

$$\mathcal{L} = \tfrac{1}{2}\partial\phi\partial\phi + \frac{1}{2\pi}(\partial\chi\partial\chi + i\tfrac{3}{2}R\chi). \tag{13}$$

The trace of the momentum-energy tensor is

$$\langle T\rangle=\frac{1}{2}\left[1-\frac{d}{2}\right]\langle\partial\phi\partial\phi\rangle+\frac{1}{2\pi}\left[1-\frac{d}{2}\right]\langle\partial\chi\partial\chi\rangle+\frac{3}{4\pi}\left[1-\frac{d}{2}\right]\langle R\chi\rangle-\frac{3}{4\pi}i(1-d)\langle\Box\chi\rangle\ . \tag{14}$$

The Langevin equation for string coordinates ϕ is given in Eq. (2), while the bosonized ghost Langevin equation is

$$\frac{\partial\chi(\xi,t)}{\partial t}=\frac{1}{\pi}\Box_{\hat{g}}\chi-\frac{3i}{4\pi}R+\eta_\chi$$

with stochastic noise η_χ.

To compute the nonvanishing momentum-energy trace, we note that the heat kernel in Eq. (8) for neighboring points ξ and ξ' is given for an arbitrary curved background[9] as

$$K^t(\xi,\xi')=\frac{D^{1/2}(\xi,\xi')}{(4\pi t)^{d/2}}\exp\left[-\frac{\sigma(\xi,\xi')}{2t}\right]\Lambda(\xi,\xi';t)\ . \tag{15}$$

Here $\sigma(\xi,\xi')$ is the geodesic length defined between ξ and ξ'. Definitions of D and Λ are

$$D(\xi,\xi')=\det\left[-\frac{\partial^2}{\partial\xi^\alpha\partial\xi'^\beta}\sigma(\xi,\xi')\right], \tag{16a}$$

$$\Lambda(\xi,\xi';t)=\sum_{n=0}^{\infty}a_n(\xi,\xi')t^n\ , \tag{16b}$$

with

$$a_0(\xi,\xi)=1\ ,\quad a_1(\xi,\xi)=\tfrac{1}{6}R(\xi),\ldots \tag{16c}$$

in the coincidence limit.

We use the dimensional regularization $d=2+\epsilon$ to compute the nonvanishing trace of the momentum-energy tensor (14). The first two terms in Eq. (14) do not vanish only if $\langle\partial\phi\partial\phi\rangle$ and $\langle\partial\chi\partial\chi\rangle$ are singular. Only a_1t in Λ (16b) expanded as a power series of t gives rise to a singular term. Contributions from the first two terms are

$$\langle T\rangle=\frac{1}{48\pi}R(\tilde{D}+1)\ , \tag{17}$$

where $\tilde{D}$ is the dimension of ϕ. The third term in Eq. (14) vanishes, since $\langle R\chi\rangle$ is regular. The last term in Eq. (14) contributes $(R/48\pi)(-27)$. Thus, the total anomaly

$$\langle T\rangle=\frac{1}{48\pi}R(\tilde{D}+1-27)$$

cancels in $\tilde{D}=26$.

C. Tachyonic scattering amplitude

As another sample computation, we compute the N-tachyon scattering amplitude to the zeroth order of $h_{\alpha\beta}$ in perturbation expansion. The vertex operator is given as

$$V^t(z)=g_0e^{ip\cdot\phi(z,\bar{z};t)}\ . \tag{18}$$

Here the centered dot denotes the scalar product in 26 dimensions and t is fictitious time. By averaging over stochastic noise, the scattering amplitude of N tachyons is given by

$$A(p_1,\ldots,p_N)=\lim_{t\to\infty}\left\langle\int\prod_{i=1}^{N}d^2z_i\sqrt{\hat{g}_i}V^t(z_i)\cdots V^t(z_N)\right\rangle$$
$$=\lim_{t\to+\infty}g_0{}^N\frac{\int[d\eta]\int\prod_{i=1}^{N}d^2z_i\sqrt{\hat{g}_i}\exp\left[-\int d^2z\sqrt{\hat{g}}d\tau[\frac{1}{2}\eta^2(z;\tau)-iK^{(t-\tau)}(z_i,z)\eta(z;\tau)]\right]}{\int[d\eta]\exp[-\frac{1}{2}\int d^2z\sqrt{\hat{g}}d\tau\,\eta^2(z;\tau)]} \tag{19}$$

By noting

$$\int d^2z\sqrt{\hat{g}}K^t(z_i,z)K^t(z,z_j)=K^{2t}(z_i,z_j)\ , \tag{20}$$

the scattering amplitude is reduced to

$$A(p_1,\ldots,p_N)=\lim_{t\to+\infty}g_0{}^N\int\prod_{i=1}^{N}d^2z_i\sqrt{\hat{g}_i}\exp\left[-\frac{1}{2}\sum_{i,j}\int d\tau\,\theta(t-\tau)p_i\cdot p_jK^{2(t-\gamma)}(z_i,z_j)\right]\ . \tag{21}$$

Since the kernel for the flat background is

$$K^\tau(z,z')=\int\frac{d^2k}{(2\pi)^2}e^{-\tau k^2}e^{ik(z-z')}\theta(\tau)\ , \tag{22}$$

the exponent in Eq. (21) becomes

$$\lim_{t\to+\infty}\left[-\frac{1}{2}\sum_{i,j}p_i\cdot p_j\int_0^\infty d\tau\int\frac{d^2k}{(2\pi)^2}\theta(t-\tau)e^{-2(t-\tau)k^2}e^{ik(z_i-z_j)}\right]=\frac{1}{16\pi}\sum_{i,j}p_i\cdot p_j\ln|z_i-z_j|^2\ . \tag{23}$$

Thus, Eq. (23) is

$$A(p_1,\ldots,p_N)=g^N\int\prod_{i=1}^{N}d^2z_i\prod_{i<j}|z_i-z_j|^{(1/4\pi)p_i\cdot p_j}\ , \tag{24}$$

with g, the renormalized coupling constant, which absorbs the divergences emerging from the coincident limit of $\ln|z_i-z_j|^2$. As usual, SL(2,R) invariance of Eq. (24) is fixed by

$$A(p_1,\ldots,p_N)=g^N\int\prod_{i=1}^{N}d^2z_i\delta^2(z_R-\alpha)\delta^2(z_S-B)\delta^2(z_T-\gamma)\,|(z_R-z_S)(z_R-z_T)(z_S-z_T)|^2\prod_{i<j}|z_i-z_j|^{(1/4\pi)p_i\cdot p_j}\,. \tag{25}$$

From the discussion given at the end of Sec. II A, we can see that Eq. (25) is exact at the tree level.

D. Stochastic quantization of string field theories in the light cone

In this section we will discuss the stochastic quantization of string field theory with the gauge fixed as in Ref. 4. For simplicity, we will work in the light cone, where all the nonphysical degrees of freedom have been eliminated. (For the treatment of covariant string field theory, see Ref. 10.) The action of open-string field theory takes the form

$$S=-\tfrac{1}{2}\int d^{24}x\,\langle\Phi(x)|(\Box-M^2)|\Phi(x)\rangle+\frac{g}{3!}\int\prod_{i=1}^{3}d^{24}x_i\left\langle V^{(3)}(x_1,x_2,x_3)\left|\prod_{j=1}^{3}\right|\Phi(x_j)\right\rangle$$
$$+\frac{g^2}{4!}\int\prod_{i=1}^{4}d^{24}x_i\left\langle V^{(4)}(x_1,x_2,x_3,x_4)\left|\prod_{j=1}^{4}\right|\Phi(x_j)\right\rangle\,. \tag{26}$$

Here $|V^{(3)}\rangle$ and $|V^{(4)}\rangle$ are the well-known three- and four-string interaction vertex operators, respectively, whose explicit forms can be found in Ref. 11. The number operator M^2 in Eqs. (26) acts on H, the Hilbert space of the first-quantized string.

Introducing noise $|\eta(x;t)\rangle$ with Gaussian distribution, its correlation function is then given as

$$\langle\,|\eta(x;t)\rangle\langle\eta(x';t')|\,\rangle_{\rm stoch}=2\delta^{24}(x-x')\delta(t-t')I$$
$$=\frac{\int[D|\eta\rangle]\,|\eta(x;t)\rangle\langle\eta(x';t')|\exp\left[-\frac{1}{2}\int d^{24}x\,dt\,\langle\eta(x;t)|\eta(x;t)\rangle\right]}{\int[D|\eta\rangle]\exp\left[-\frac{1}{2}\int d^{24}x\,dt\,\langle\eta(x;t)|\eta(x;t)\rangle\right]}\,, \tag{27}$$

where I is the identity operator in H. With this noise, the Langevin equation for $|\Phi\rangle$ reads

$$\frac{\partial}{\partial t}|\Phi(x;t)\rangle=(\Box-M^2)|\Phi(x;t)\rangle-\frac{g}{2!}\int d^{24}x_1d^{24}x_2\langle\Phi(x_1;t)\Phi(x_2;t)|V^{(3)}(x,x_1,x_2)\rangle$$
$$-\frac{g^2}{3!}\int d^{24}x_1d^{24}x_2d^{24}x_3\langle\Phi(x_1;t)\Phi(x_2;t)\Phi(x_3;t)|V^{(4)}(x,x_1,x_2,x_3)\rangle+|\eta(x;t)\rangle\,. \tag{28}$$

Formally we can analyze Eq. (28) perturbatively in orders of g by expanding $|\Phi(x;t)\rangle$ as

$$|\Phi(x;t)\rangle=|\Phi_0(x;t)\rangle+g\,|\Phi_1(x;t)\rangle+g^2|\Phi_2(x;t)\rangle+\cdots\,. \tag{29}$$

The zeroth order in g of Eq. (28) is

$$\frac{\partial}{\partial t}|\Phi_0(x;t)\rangle=(\Box-M^2)|\Phi_0(x;t)\rangle+|\eta(x;t)\rangle\,, \tag{30}$$

whose solution is

$$|\Phi_0(x;t)\rangle=\int_0^t d\tau\,e^{(t-\tau)(\Box-M^2)}|\eta(x;\tau)\rangle\,, \tag{31}$$

with the boundary condition

$$|\Phi(x;0)\rangle=0\,. \tag{32}$$

The first-order solution of Eq. (28) is

$$|\Phi_1(x;t)\rangle=-\tfrac{1}{2}\int d^{24}x_1d^{24}x_2\int_0^t d\tau\int_0^\tau d\tau_1\int_0^\tau d\tau_2\langle\eta(x_1,\tau_1)\eta(x_2,\tau_2)|$$
$$\times\exp[(\tau-\tau_1)(\Box_1+M_1{}^2)+(\tau-\tau_2)(\Box_2+M_2{}^2)$$
$$+(t-\tau)(\Box+M^2)]\,|V^{(3)}(x,x_1,x_2)\rangle\,. \tag{33}$$

Here, $\Box_i$ and $M_i{}^2$ act on the ith space in the direct product of Hilbert spaces. The free propagator can be formally calculated as

$$\langle\langle\, |\Phi_0(x)\rangle\langle\Phi_0(x')|\,\rangle\rangle = \lim_{t\to+\infty} \langle\, |\Phi_0(x;t)\rangle\langle\Phi_0(x';t)|\,\rangle_{\text{stoch}}$$
$$= \frac{1}{\Box - M^2}\delta^{24}(x-x')I \;, \tag{34}$$

or, more explicitly,

$$\langle\{\mathbf{x}'\}|\langle\langle\, |\Phi_0(x)\rangle\langle\Phi_0(x)|\,\rangle\rangle|\{\mathbf{x}\}\rangle$$
$$= \int [D\mathbf{x}]\exp\left[-\int_\tau^{\tau'} dT \int_0^\pi d\sigma\{[\partial_\tau \mathbf{x}(T,\sigma)]^2+[\partial_\sigma \mathbf{x}(T,\sigma)]^2\}\right]\delta(\mathbf{x}(\tau',\sigma)-\mathbf{x}'(\sigma))\delta(\mathbf{x}(\tau,\sigma)-\mathbf{x}(\sigma)) \;. \tag{35}$$

Here $|\{\mathbf{x}\}\rangle$ is the eigenstate of the string coordinate $\mathbf{x}(\sigma)$ in the light-cone gauge, and $\tau = x^+$, $\tau' = x'^+$. Equation (35) agrees with the result obtained in Ref. 11.

However, we should point out the problem in Eq. (34). In terms of components

$$\begin{aligned} |\Phi(x;t)\rangle &= \phi(x;t)|0\rangle + A_i(x;t)a_1^{i\dagger}|0\rangle \\ &\quad + h_{ij}(x;t)a_1^{i\dagger}a_1^{j\dagger}|0\rangle + \cdots \;, \\ |\eta(x;t)\rangle &= \eta(x;t)|0\rangle + \eta_i(x;t)a_1^{i\dagger}|0\rangle \\ &\quad + \eta_{ij}(x;t)a_1^{i\dagger}a_1^{j\dagger}|0\rangle + \cdots \;, \end{aligned} \tag{36}$$

Eq. (30) represents an infinite set of Langevin equations for different spins

$$\begin{aligned} &\frac{\partial}{\partial t}\phi(x;t) = (\Box + 2\pi)\phi(x;t) + \eta(x;t) \;, \\ &\frac{\partial}{\partial t}A_i(x;t) = \Box A_i(x;t) + \eta_i(x;t) \;, \\ &\frac{\partial}{\partial t}h_{ij}(x;t) = (\Box - 2\pi)h_{ij}(x;t) + \eta_{ij}(x;t) \;. \end{aligned} \tag{37}$$

[A component expansion of Eq. (28) will also include couplings between different spins.] The operator $-(\Box+2\pi)$ in the first equation of Eq. (37) is not positive-definite. The correlation function of $\tilde{\phi}(k;t)$, Fourier transform of $\phi(x;t)$,

$$\langle\tilde{\phi}(k;t)\tilde{\phi}(k';t)\rangle_{\text{stoch}}$$
$$= \int_0^t d\tau\, e^{2(t-\tau)(-k^2+2\pi)}(2\pi)^{24}\delta^{24}(k+k') \;, \tag{38}$$

diverges in the limit $t \to +\infty$ for $k^2 < 2\pi$, leading to the breakdown of stochastic quantization. This is not surprising, because the tachyon cannot be quantized consistently. All of the higher-spin equations in Eq. (37) lead to a sensible asymptotic limit. The well-known indefinite problem of higher-spin field action in the covariant formulation[10] does not occur. This is one of the advantages of the light-cone gauge.

In principle, one can calculate the scattering amplitude in this formulation. However, it is technically very difficult.

III. RANDOM SURFACE APPROACH TO POLYAKOV STRING

To implement the Polyakov string theory further into numerical computations, one triangulates the two-dimensional surface. One may speculate that the fundamental structure of the string world sheet may be a triangulated random surface.

To keep symmetries of theory as much as possible in random surface formulations, we follow the Regge calculus[12] idea for a two-dimensional surface. For notations and completeness we review the basic ideas of a random surface approach below and in Secs. III A and III B. In continuum two-dimensional theory, the metric $g_{\alpha\beta}(\tau,\sigma)$ and string coordinate $\phi(\tau,\sigma)$ are defined at each point (τ,σ) of the two dimensions. In the Regge gravity approach,[12–14] one point (τ,σ) is represented by a triangle. The metric at the point (τ,σ) is represented by the length of the sides of the triangle [see Eq. (43)], and is constant inside the triangle. The affine connection is nonzero on the links, and the curvature is concentrated on the vertices given as deficit angle. The string coordinate at (τ,σ) is now represented by ϕ_i defined on the vertices.

One recalls the Euler theorem which gives the relationship between the number of vertices (N_0), links (N_1), and triangles (N_2) in terms of g holes as

$$N_0 - N_1 + N_2 = 2(1-g) \;. \tag{39}$$

A link is shared by two triangles, giving

$$2N_1 = 3N_2 \;. \tag{40}$$

For convenience, we embed a triangulated two-dimensional surface into three-dimensional Euclidean space. For a particular triangle with vertices $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, parametrization is given as

$$\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2 + \gamma\mathbf{x}_3, \quad \alpha+\beta+\gamma = 1, \quad \alpha,\beta,\gamma \ge 0 \;. \tag{41}$$

Similarly the string coordinate inside the triangle is given as

$$\phi = \alpha\phi_1 + \beta\phi_2 + \gamma\phi_3 \;, \tag{42}$$

where ϕ_1, ϕ_2, and ϕ_3 are 26-dimensional vectors at three vertices of triangle.

The metric for this triangle can be calculated as

$$g_{\alpha\beta} = \partial_\alpha \mathbf{x}\cdot\partial_\beta \mathbf{x}$$
$$= \begin{vmatrix} l_{13}{}^2 & \frac{1}{2}(l_{13}{}^2 + l_{23}{}^2 - l_{12})^2 \\ \frac{1}{2}(l_{13}{}^2 + l_{23}{}^2 - l_{12}{}^2) & l_{23}{}^3 \end{vmatrix} \;, \tag{43}$$

where $|l_{ij}|$ is length of the link between vertices i and j. Defining

$$H_{\alpha\beta} = \partial_\alpha\phi^i\partial^i_\beta\phi^i$$
$$= \begin{vmatrix} (\phi_1-\phi_3)^2 & (\phi_1-\phi_3)(\phi_2-\phi_3) \\ (\phi_1-\phi_3)(\phi_2-\phi_3) & (\phi_2-\phi_3)^2 \end{vmatrix} \;, \tag{44}$$

we can rewrite action (1) on the simplicial surface as[13]

$$S=\sum_{\text{sum over triangles}} \int d\alpha\, d\beta \sqrt{g} g^{\alpha\beta} H_{\alpha\beta}$$

$$=\frac{1}{2} \sum_{\text{sum over triangles}} \frac{1}{\Delta_{ijk}} (\phi_i l_{jk} + \phi_j l_{ui} + \phi_u l_{ij})^2 , \qquad (45)$$

where Δ_{iju} is the area of the triangle with vertices i, j, and k. For a surface consisting of equilateral triangles, Eq. (45) is simplified as

$$S=\frac{1}{2\sqrt{3}} \sum_{\text{links}} (\phi_i - \phi_j)^2 . \qquad (46)$$

We can extend random surface approach to string in curved target space with action

$$S=\int d^2\xi \sqrt{g} g^{\alpha\beta} \partial_\alpha \phi^m \partial_\beta \phi^n G_{mn}(\phi) , \qquad (47)$$

where $G_{mn}(\phi)$ is the metric of target space. Equation (45) is modified to

$$S=\sum_{\text{sum over triangles}} \int \frac{d\alpha\, d\beta}{\Delta_{ijk}} (\phi_i^m l_{jk} + \phi_j^m l_{ki} + \phi_k^m l_{ij}) G_{mn}(\phi) \cdot (\phi_i^n l_{jk} + \phi_j^n l_{ki} + \phi_k^n l_{ij}) . \qquad (48)$$

A. Symmetries of simplicial action

We now discuss symmetries of action (45). One can choose a different coordinate α' and β' for ϕ and $g_{\alpha\beta}$ inside a particular triangle. One choice of coordinates α and β of a triangle is expressed in the coordinates α' and β' of another triangle. In both cases, α' and β' are the general coordinate transformation of α and β. Under this, Eqs. (43) and (44) are covariant and the action (45) is invariant.[12] By general-coordinate transformation, one can always go to the conformal gauge.

Next, we turn to discuss whether the action (45) has Weyl-scale-transformation invariance. By definition, the Weyl-scale-transformed metric $g'_{\alpha\beta}$ should be of the same form as $g_{\alpha\beta}$:

$$g'_{\alpha\beta} = e^{2\sigma_\Delta} g_{\alpha\beta} , \qquad (49)$$

where σ_Δ is a constant given for each triangle. This rescales three sides of the triangle by the same factors e^{σ_Δ}. Since each link is shared by two triangles, scaling in one particular triangle would necessitate the same rescaling for neighboring triangles. Thus, one has only a global scale transformation.[15]

B. Quantization of the random surface model

In the path-integral quantization of action (1), one sums over all possible surfaces and ϕ. Similar summations[16] are performed for the quantization of Eq. (45). The measure for ϕ is simply $\prod_{i \in \text{vertices}} d\phi_i$, but determination of measure[17] for $g_{\alpha\beta}$ requires care.

Surfaces are classified by the number of holes g. The surface of genus g is approximated by N_2 triangles. [Note that N_0 and N_1 are fixed by Eqs. (39) and (40).] Different surfaces are obtained by linking the vertices in all possible, but distinct ways. Link lengths are allowed to vary with constraint of triangle inequality.

Measure for $g_{\alpha\beta}$ is proposed as

$$\prod_{\tau,\sigma} dg_{\alpha\beta}(\tau,\sigma) = \sum_{\substack{\text{Surfaces given by different}\\ \text{linkage of vertices}}} \prod_{i,j \in \text{links}} dl_{ij}^{\,2} . \qquad (50)$$

For a given surface of genus g and particular linkage of vertices, $dg_{\alpha\beta}$ at one triangle is given as a wedge product of the link length $dl_{ij}^{\,2}$ of three sides. Sharing of a link by two triangles is taken care of by identifying the sides of adjacent triangles by a Dirac δ function.

C. Approximate treatment of tachyonic mass and amplitudes in the random surface approach

An important problem in the random surface approach is whether Eq. (45) gives the correct continuum limit where the number of triangles is very large. For example, do theories given by Eqs. (45) and (1) provide the same phase transitions? These require deeper analyses which we will not undertake here. However, since symmetries (general coordinate transformation and local Weyl scaling) of action (1) are larger than those (general coordinate transformation and global scaling) of action (45), it is not clear at all that actions (1) and (45) have the same limit. As a preliminary step, we compute the tree amplitude of tachyon scattering and the ground-state mass in the random surface formulation of a Polyakov string in the path-integral approach.

We take our random surface as shown in Fig. 1, which is a collection of right triangles with lengths a and b. Since we have not been able to sum over other two-dimensional surfaces analytically yet, our computations in this section should be taken as a first step towards calculations of tachyonic mass and scattering amplitudes in the random surface approach.

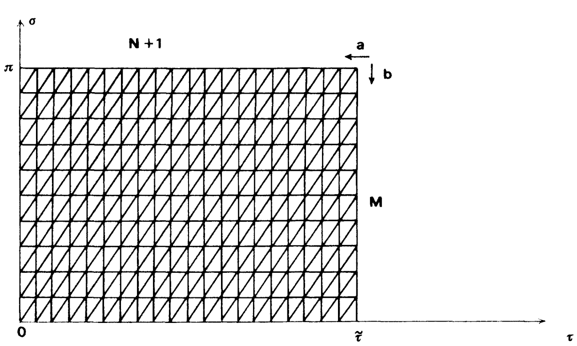

FIG. 1. Two-dimensional surface made of right triangles with sides a and b.

For right triangles in Fig. 1, the metric (43) is diagonal:

$$g_{\alpha\beta}=\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}, \tag{51}$$

and Eq. (44) is simplified as

$$S=\sum_{\substack{i,i'\\ j,j'}} \phi_{i,j}(\delta_{i,i'}A_{jj'}+B_{ii'}\delta_{j,j'})\phi_{i',j'} . \tag{52}$$

Here $\phi_{i,j}$ denotes the field on site (i,j) in Fig. 1. [Accidently, Eq. (52) happens to agree with the result in Ref. 18 where the lattice approach is taken.] The matrices A and B for a closed string are

$$A=\frac{1}{a^2}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix}, \tag{53}$$

$$B=\frac{1}{b^2}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix}. \tag{54}$$

Tachyonic mass can be evaluated by the same method as in the continuum theory with the path-integral approach which is described in Appendix A. Eigenvalues $(\pi l/\tau)^2$ and $(2n)^2$ in Eq. (A7) are replaced in the random surface approach by those of A and B matrices, which are

$$\frac{4}{a^2}\sin^2\frac{n\pi}{2(N+1)},\quad n=1,2,\ldots,N ,$$

$$\frac{4}{b^2}\sin^2\frac{m\pi}{M},\quad m=0,1,2,\ldots,M-1 ,$$

respectively. Identical computations as in the Appendix give a contribution of a particular triangle of Fig. 1 to the tachyonic (mass)^2 as

$$D\cdot\left[\frac{2G}{\pi ab}-\frac{\pi}{6}\right],\quad G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} .$$

However, one should sum over all different lengths of triangles in Eq. (45), and these have not yet been calculated.

The tachyonic scattering amplitude in the path-integral quantization is

$$A(p_1,\ldots,p_n)=\int\prod_{t=1}^{n}\sum_{\text{triangles}}\tfrac{1}{2}g_0 ab e^{ip_t\cdot\phi_{i_t,j_t}}e^{-S}\prod_{i,j}d\phi_{i,j} , \tag{55}$$

here g_0 is the coupling constant. The ϕ is integrated out formally to give

$$A(p_1,\ldots,p_n)=(\tfrac{1}{2}g_0 ab)^n\sum_{\text{triangles}}\exp\left[\sum_{s,t}p_t D_{(i_t,j_t)(i_s,j_s)}p_s\right], \tag{56}$$

where $D_{(i_t,j_t)(i_s,j_s)}$ is the propagator obtained from the action (52). This propagator is hard to obtain. However, for certain extreme cases with $a \ll b$, one can expand $D_{(i,j)(i',j')}$ as

$$\frac{1}{I\otimes A+B\otimes I}=\left[I\otimes I-B\otimes\frac{1}{A}+B\otimes\frac{1}{A}\cdot B\otimes\frac{1}{A}+\cdots\right]\frac{1}{A}, \tag{57}$$

where

$$\frac{1}{A}=\frac{a^2}{N+1}\begin{bmatrix} 1\times N & 1\times(N-1) & 1\times(N-2) & \cdots & 1\times 2 & 1\times 1 \\ 1\times(N-1) & 2\times(N-1) & 2\times(N-2) & \cdots & 2\times 2 & 2\times 1 \\ 1\times(N-2) & 2\times(N-2) & 3\times(N-2) & \cdots & 3\times 2 & 3\times 1 \\ & & & \cdots & & \\ 1\times 2 & 2\times 2 & 3\times 2 & \cdots & (N-1)\times 2 & (N-1)\times 1 \\ 1\times 1 & 2\times 1 & 3\times 1 & \cdots & (N-1)\times 1 & N\times 1 \end{bmatrix}. \tag{58}$$

These computations should be taken with the qualification made earlier.

D. Implementation of random surface into numerical computations

The Polyakov string can be simulated by Monte Carlo numerical methods such as lattice gauge theories. One can ask nonperturbative questions such as whether (1) a negative mass-squared state in the tree-level spectra of a bosonic theory persists even in the full quantum theory, or (2) whether there are phase transitions in string theories.

Starting with some initial triangulated surfaces with ϕ at vertices, link lengths and string coordinates are updated by Metropolis algorithms. In the process of iteration, configurations equivalent up to global scale transformation may occur several times, since our action is not gauge fixed. Because the stochastic process is guaranteed to generate configurations, which have the equal action, with the same probability, gauge inequivalent configurations will be duplicated with the same number of times. Thus, even though the global scale transformation group is noncompact, gauge fixing is not necessary. It is not clear at the present moment whether Eq. (25) has the correct continuum limit. Numerical simulations of action (25) are planned.

Note also that curved target space action (47) does not present a new problem, even though integration is impossible analytically. Recently, there have been a number of works[19] in this direction.

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APPENDIX

In this appendix we compute the tachyonic mass for the closed bosonic string in the path-integral approach.

The transition amplitude between the configurations $\phi(\sigma)$ at $t=0$ and $\tilde{\phi}(\sigma)$ at $t=\tilde{\tau}$ is given by

$$\langle\{\tilde{\phi}(\sigma)\},\tilde{\tau}\,|\,\{\phi(\sigma)\},0\rangle = [\det(-\partial^2)]^{-(D-2)/2}\exp(-S_{\rm cl}) \qquad (A1)$$

with $\det(-\partial^2)$ evaluated on the strip $[0,\pi]\times[0,\tilde{\tau}]$. The power $D-2$ results from the standard argument for the string coordinates and ghosts. In the case $\tilde{\phi}=\phi=0, S_{\rm cl}$ vanishes; (A1) reduces to

$$\langle\{0\},\tilde{\tau}\,|\,\{0\},0\rangle = [\det(-\partial^2)]^{-(D-2)/2} . \qquad (A2)$$

The determinant $\det(-\partial^2)$ can be evaluated as a product of the eigenvalues of $-\partial^2$, which is given by

$$\xi_{n,l} = \left[\frac{\pi l}{\tilde{\tau}}\right]^2 + (2n)^2 , \qquad l=1,2,\ldots , \quad n=0,1,2,\ldots , \qquad (A3)$$

that is

$$[\det(-\partial^2)]^{-1} = \prod_{l=1}^{\infty}\frac{\tilde{\tau}}{\sqrt{\pi l}}\prod_{n,l=1}^{\infty}\frac{\pi}{\left[\frac{\pi l}{\tilde{\tau}}\right]^2+(2n)^2} . \qquad (A4)$$

Using

$$\prod_{l=1}^{\infty}\frac{a^2}{l^2+b^2} = \frac{\pi b}{\sinh \pi b}\,\frac{1}{2\pi a}, \quad \sum_{l=1}^{\infty} l = -\tfrac{1}{12} ,$$

one obtains

$$[\det(-\partial^2)]^{-(D-2)/2} = \left[\frac{1}{2\tilde{\tau}}\right]^{(D-2)/2}\exp\left[\frac{(D-2)\tilde{\tau}}{12}\right]\prod_{n=1}^{\infty}(1-e^{-4n\tilde{\tau}})^{-(D-2)/2} . \qquad (A5)$$

The amplitude (A2) can also be computed in Hamiltonian approach as

$$\begin{aligned}\langle\{0\},\tilde{\tau}\,|\,\{0\},0\rangle &= \langle\{0\}\,|\,e^{-\tilde{\tau}L_0}\,|\,\{0\}\rangle \\ &= \sum_{\{N\}}\int\frac{d^{24}p}{(2\pi)^{24}}\,|\langle\{0\}\,|\,\{N\},p\rangle|^2 e^{-(\tilde{\tau}/\pi)(p^2+m_{\{N\}}{}^2)} \\ &= \left[\frac{1}{2\tilde{\tau}}\right]^{12}\sum_{\{N\}}e^{-(\tilde{\tau}/\pi)m_{\{N\}}{}^2}\,|\langle\{0\}\,|\,\{N\}\rangle|^2 .\end{aligned} \qquad (A6)$$

Comparison of (A5) with (A6) results in the extraction of the ground-state mass

$$m_0{}^2 = -\frac{D-2}{12}\pi = -2\pi . \qquad (A7)$$

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