Operator regularization and one-loop Green's functions

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In this paper we present an alternate way of computing amplitudes in quantum field theory in the context of background-field quantization. We concentrate mainly on one-loop effects. The Feynman diagrams of the usual perturbation series are avoided by first performing the functional integration and then using a perturbative expansion due to Schwinger. In this approach we regulate operators rather than the initial Lagrangian. To one-loop order our scheme reduces to a perturbative expansion of the well-known ζ function associated with the superdeterminant of an operator. This technique preserves all symmetries present in the initial theory and does not lead to any explicit divergences as the regulating parameter approaches its limiting value. For illustration, we apply our approach to a toy $(\phi^3)_6$ scalar theory, to Yang-Mills theory in the covariant gauge, and to quantum electrodynamics. This method reproduces the usual axial anomaly in the three-point functions VVA and AAA. Operator regularization is used in a dimensionally regulated theory reproducing the usual results obtained in the dimensionally regulated Feynman-diagram approach. An outline of how operator regularization is applied beyond one-loop order is provided. Other possible applications of operator regularization are discussed.

I. INTRODUCTION

The fundamental entity of quantum field theory is the generating functional of Green's functions. Expanding this quantity in powers of Planck's constant $\hbar$, yielding the so-called "loop expansion," has provided a way of studying the properties of many field-theory models. Regulating divergences which occur in the evaluation of this generating functional in a manner that respects symmetries appearing in the initial Lagrangian is a problem that has received much attention. In the usual Feynman perturbation series these divergences are regulated by the introduction of a new parameter into the initial Lagrangian. In the case of gauge theories, and especially, supersymmetry, it has proved very difficult to find a successful regularization procedure. Any procedure for regulating supersymmetric theories which involves moving away from four dimensions runs into difficulties with the treatment of γ_5—a matrix inherently connected to four dimensions.

Operator regularization is a regulating procedure which avoids the difficulty of regulating Feynman integrals in a symmetry-preserving way by regulating the determinants of operators and inverses of operators which occur directly in the generating functional. In this paper we restrict our attention to the one-loop generating functional—this involves the determinants of operators. The two-loop, or $O(\hbar^2)$, generating functional, which involves the determinants of operators, will be considered in a separate paper.[1] However the basic regularization technique is the same. We regularize the logarithm of an operator $\ln A$ and from this we can deduce the regulated form of $\det A$ appropriate to the $O(\hbar)$ generating functional *and* the regulated form of A^{-1} appropriate to the $O(\hbar^2)$ generating functional.

At the one-loop level the method of operator regularization reduces to ζ-function regularization[2–5] of functional determinants. However the ζ function has been used heretofore in computations of one-loop Euclidean effective actions within the context of a heat-kernel expansion.[6–8] In this work we propose a different use for the ζ function as a means to compute perturbatively one-loop Euclidean Green's functions without encountering any divergences.

Expanding the generating functional in powers of the classical background field in background-field quantization leads directly to the one-particle-irreducible (1PI) Green's functions. In the operator regularization method a perturbative expansion introduced by Schwinger,[9] but rarely used finds a natural setting. At the one-loop level this perturbative expansion can be applied directly to the ζ function. Expressions for the regulated functions are then obtained by inspection—the evaluation of the resulting expressions involves well-defined finite momentum integrals. The evaluation leads to the finite Green's functions. It is a feature of the general method and not just of the one-loop cases treated in this paper that no divergent integrals are explicitly encountered.

It is important to note that operator regularization can be applied to all orders of the loop expansion, in principle, not just to one-loop order. Thus it can be interpreted as a generalization to all orders of the ζ function regularization at the one-loop level. However, we must emphasize that our use of ζ function regularization is intrinsically different from that of other authors.[2–5,10–12] We use the ζ function as a regularization of Green's functions in perturbation theory.

It is instructive to apply operator regularization to various models and to compute Green's functions in these models. This serves the dual purpose of showing how the procedure works and comparing the operator-regulated Green's functions with those obtained in the more conventional approaches. We apply the method in this paper to the computation of one-loop Green's functions in models with external bose fields only—a $(\phi^3)_6$ scalar field theory (a toy model used to illustrate the details of the calculation of Green's functions), pure Yang-Mills theory, and a massless spinor model with Abelian vector and axial-vector fields—and in a model with external bose fields and external Fermi fields: namely, massless QED.

In Yang-Mills theory we treat he vacuum polarization in an arbitrary covariant gauge. We explicitly compute this Green's function in a particular gauge. This calculation is similar to computations using "proper time" integration in Ref. 7, but here no divergences occur. The operator regularization prescription renders all integrals well behaved.

The inclusion of external Fermi fields as well as external Bose fields involves the use of supermatrices.[13,14] In particular at the one-loop level the generating functional is the superdeterminant of a supermatrix.[13–15] It is known that the superdeterminant has two representations in terms of the even and odd blocks of a supermatrix. The use of a perturbative expansion in the regularized versions of the two different representations of the superdeterminant gives rise to two different calculational paths to the Green's functions. In principle we are free to use either calculational path, but in practice it turns out that one proves to be more convenient than the other. For purposes of illustration, we compute the fermion propagator and 1PI vertex function in each approach. While the results in the two approaches are not the same, they are related by finite renormalizations.

In Euclidean space the occurrence of Abelian axial-vector fields, either quantum or classical, coupled to spinor fields, is problematical as gauge invariant couplings are not Hermitian. To overcome this problem, and restore Hermiticity, we continue the axial-vector field to imaginary values in Euclidean space.[10,16] Unfortunately this is at the expense of axial-gauge invariance. The interesting Green's functions in the axial model are the $\langle VVA \rangle$ and $\langle AAA \rangle$ three-point functions. By examining the ζ functions associated with these three-point functions we show that at the one-loop-level operator regularization reproduces automatically the usual anomaly in the divergence of the Abelian axial-vector currents involved. A peculiar feature, however, is that the two-point function $\langle AA \rangle$ is not transverse. Perhaps this is not a surprising result as we have dispensed with axial-gauge invariance. Transversability of $\langle AA \rangle$ can be restored by a finite renormalization.

While the main purpose of this paper is to discuss operator regularization as the regularization procedure in quantum field theory, we also examine the use of dimensional regularization in conjunction with the Schwinger expansion and operator regularization. It is interesting to note that if the Dirac algebra and the momentum integrals are done in (arbitrary) n dimensions with a view to ending up in (integer) d dimensions, then operator regularization as a regularization procedure is redundant—it just reduces to the usual representations for $\ln A$ and A^{-1}. Furthermore, for most (but not all) of the Green's functions which we have computed using dimensional regularization precisely the same results are obtained as in the dimensionally regulated Feynman diagram approach—including any divergences. Thus if dimensional regularization is used in the Schwinger expansion it is necessary to provide a subtraction procedure. It is a remarkable fact that the results obtained for the $\langle VVA \rangle$ and $\langle AAA \rangle$ three-point functions should prove to be exceptions to the above rule.

For these Green's functions we find that the dimensionally regulated Schwinger expansion reproduces the usual anomaly in the divergence of the Abelian axial-vector currents in n dimensions irrespective of whether we treat γ_5 as being totally anticommuting or according to the 't Hooft-Veltman prescription.[17]

This paper is arranged as follows. In Sec. II we set out the essential features of operator regularization and of the perturbative procedure for calculating Euclidean Green's functions. In this section a sample calculation in a toy $(\phi^3)_6$ model is examined in detail to illustrate the procedure. Section III contains a discussion of the vacuum polarization in pure Yang-Mills theory. Section IV is devoted to the inclusion of external Fermi fields in the context of massless QED. In Sec. V we show how operator regularization can be applied to models with external axial-vector currents. In Sec. VI we use dimensional regularization in the Schwinger perturbative expansion, while Sec. VII contains a discussion of the results obtained.

There are two appendixes. In Appendix A we consider the approach of Lee and Rim[15] to the inclusion of Fermi fields. We show that the two forms of the one-loop generating functional used in Sec. IV can also be obtained in this approach. In Appendix B we write out the full regulated one-loop generating functional appropriate to the approach of Sec. IV.

II. FORMALISM

The background-field method in the context of path-integral quantization[6,18,19] is the starting point of our procedure. We consider the general case where a field $\phi_i(x)$ is split into a classical part $f_i(x)$ and a quantum part $h_i(x)$,

$$\phi_i(x)=f_i(x)+h_i(x) \ . \tag{2.1}$$

The field $\phi_i(x)$ may be either fermionic or bosonic.

The generating functional for Green's functions in the theory in the presence of a source function $J_i(x)$ is given by

$$Z[f_j,J_j]=\int dh_k \exp\left[\int dx\,[\mathscr{L}(f_j+h_j)+J_i h_i]\right] . \tag{2.2}$$

We will be dealing exclusively with theories defined in Euclidean space. The Lagrangian $\mathscr{L}$ is composed of the classical Lagrangian $\mathscr{L}_0$ plus any gauge-fixing $\mathscr{L}_{\rm gf}$ and ghost terms $\mathscr{L}_g$ that may be required. The general form of $\mathscr{L}(f_i+h_i)$ that we will consider is

$$\mathscr{L}(f_i,h_i)=\tfrac{1}{2}h_iM_{ij}(f_j)h_j+\frac{1}{3!}a_{ijk}(f_j)h_ih_jh_k + \frac{1}{4!}b_{ijkl}h_ih_jh_kh_l \ . \tag{2.3}$$

Since in this paper we are dealing exclusively with "one-loop" effects we restrict our attention to those terms in Eq. (2.3) that are bilinear in h_i. (By the word "one loop" we mean "to first order in $\hbar$." Since we are not dealing with the Feynman perturbation series, we do not strictly speaking encounter the "loops" that occur in Feynman diagrams.) We thus consider only

$$\mathscr{L}^{(2)}=\tfrac{1}{2}h_iM_{ij}(f_j)h_j \ . \tag{2.4}$$

Upon substituting Eq.(2.4) into Eq. (2.2) we arrive at the one-loop generating functional

$$Z[f_j,0]=\int dh_k \exp\left[\int dx\,[\tfrac{1}{2}h_iM_{ij}(f_j)h_j]\right] . \tag{2.5}$$

Evaluation of the functional integral in Eq. (2.5) involves the "superdeterminant" of M_{ij}, as h_i may be either fermionic or bosonic.[20,13,14] To see how the superdeterminant arises, let us consider a general integral of the form

$$I=\int db\,df \exp[\tfrac{1}{2}(bM_{bb}b+fM_{fb}b + bM_{bf}f+fM_{ff}f)] \ . \tag{2.6}$$

In the argument of the exponential on the right-hand side of Eq. (2.6) we can complete the square in either the bosonic variable b or the fermionic variable f. This leads to either

$$I=\int db\,df \exp\{\tfrac{1}{2}[b(M_{bb}-M_{bf}M_{ff}{}^{-1}M_{fb})b+(f+bM_{bf}M_{ff}{}^{-1})M_{ff}(M_{ff}{}^{-1}M_{fb}b+f)]\} \tag{2.7a}$$

or

$$I=\int db\,df \exp\{\tfrac{1}{2}[(b+fM_{fb}M_{bb}{}^{-1})M_{bb}(M_{bb}{}^{-1}M_{bf}f+b)+f(M_{ff}-M_{fb}M_{bb}{}^{-1}M_{bf})f]\} \ . \tag{2.7b}$$

In Eq. (2.7a) we shift the variable of integration

$$f\rightarrow f'=f+M_{ff}{}^{-1}M_{fb}b \ ,$$

to obtain

$$I=\int db \exp\{\tfrac{1}{2}[b(M_{bb}-M_{bf}M_{ff}{}^{-1}M_{fb})b]\} \times\int df \exp[\tfrac{1}{2}(fM_{ff}f)] \ . \tag{2.8a}$$

Similarly, if in Eq. (2.7b) we make the shift

$$b\rightarrow b'=b+M_{bb}{}^{-1}M_{bf}f \ ,$$

we obtain

$$I=\int db \exp[\tfrac{1}{2}(bM_{bb}b)] \times\int df \exp\{\tfrac{1}{2}[f(M_{ff}-M_{fb}M_{bb}{}^{-1}M_{bf})f]\} \ . \tag{2.8b}$$

The standard integrals[20]

$$\det{}^{-1/2}A=\int db \exp[\tfrac{1}{2}(bAb)] \ , \tag{2.9a}$$

$$\det{}^{1/2}B=\int df \exp[\tfrac{1}{2}(fBf)] \tag{2.9b}$$

can now be applied to Eqs. (2.8a) and (2.8b) to show that

$$I=\mathrm{sdet}^{-1/2}M \ , \tag{2.10}$$

where

$$M=\begin{bmatrix} M_{bb} & M_{bf} \\ M_{fb} & M_{ff}\end{bmatrix}$$

with either

$$\mathrm{sdet}M=\det(M_{bb}-M_{bf}M_{ff}{}^{-1}M_{fb})\det{}^{-1}M_{ff} \tag{2.11a}$$

or

$$\mathrm{sdet}M=\det M_{bb}\det{}^{-1}(M_{ff}-M_{fb}M_{bb}{}^{-1}M_{bf}) \ . \tag{2.11b}$$

Equation (2.5) shows that the one-loop generating functional for Green's functions is given by

$$Z_1[f_j,0]=\mathrm{sdet}^{-1/2}[M_{ij}(f_j)] \ . \tag{2.12}$$

We now see by Eqs. (2.11) and (2.12) that Z_1 is given by the ratio of determinants of operators. In our approach it is these operators that are regulated. We regulate the logarithm of an operator H according to

$$\ln H=\lim_{s\rightarrow 0}\left[-\frac{d^m}{ds^m}\left[\frac{s^{m-1}}{m!}H^{-s}\right]\right] \quad (m=1,2,3,\ldots) \ . \tag{2.13}$$

We use this equation to deduce the regulated form of both $\det H$ and H^{-1} as

$$\det H=\exp(\mathrm{tr}\ln H)=\exp\left\{\mathrm{tr}\lim_{s\rightarrow 0}\left[-\frac{d^m}{ds^m}\left[\frac{s^{m-1}}{m!}H^{-s}\right]\right]\right\} \tag{2.14a}$$

and

$$H^{-1}=\frac{d}{dH}\ln H=\lim_{s\rightarrow 0}\left[\frac{d^m}{ds^m}\left[\frac{s^m}{m!}H^{-s-1}\right]\right] . \tag{2.14b}$$

It is Eq. (2.14a) with $m=1$ that we will use to regulate the determinants occurring in Eq. (2.12).

If we now rewrite H^{-s} as

$$H^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\exp(-Ht) \tag{2.15}$$

in Eq. (2.14a) we arrive at the result

$$\det H = \exp[-\zeta'(0)] , \tag{2.16a}$$

where we have defined the ζ function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \operatorname{tr} \exp(-Ht) . \tag{2.16b}$$

This is the usual ζ-function regularization of the determinant of an operator.[2–5]

In Ref. 9 Schwinger does not employ the ζ function, but rather uses the standard integral

$$\ln(A/B) = -\int_0^\infty \frac{dt}{t}(e^{-At} - e^{-Bt})$$

to write

$$\det H = \exp\left[-\operatorname{tr} \int_0^\infty \frac{dt}{t} e^{-Ht}\right] . \tag{2.17}$$

[An infinite constant independent of H has been discarded in Eq. (2.17).]

We note from Eq. (2.17) that

$$(\det H)_{\text{Schwinger}} = \exp\{\lim_{s\to 0}[-\Gamma(s)\zeta(s)]\}$$
$$\equiv \exp[-\Gamma(0)\zeta(0)] . \tag{2.18}$$

Thus Schwinger's $\det H$ is an unregulated expression; it is necessary to provide a regularization to obtain well-defined results. In Sec. VI we make use of dimensional regularization in this context.

Equation (2.16) has been used before[2–4] to discuss the one-loop generating functional directly. We propose to use Eq. (2.16) to compute one-loop Green's functions. This allows us to use the parameter s in Eq. (2.16) to regulate the theory *without* having to insert a regulating parameter into the initial Lagrangian. Two appealing features of this approach are that symmetries of the theory are not explicitly broken and that no explicit ultraviolet divergences are ever encountered. Both of these points will be illustrated in subsequent sections.

The procedure by which we extract one-loop Green's functions from Eq. (2.12) is completely distinct from the Feynman perturbation expansion. We rely rather on a perturbative expansion due to Schwinger[9] for $\operatorname{tr} e^{-Ht}$ in Eq. (2.16b). Our first step is to separate H into

$$H = H_0 + H_I , \tag{2.19}$$

where H_0 is independent of the background field f_i and H_I is at least linear in f_i. The expansion is

$$\operatorname{tr} e^{-Ht} = \operatorname{tr}\Bigg[e^{-H_0 t} + (-t)e^{-H_0 t}H_I + \frac{(-t)^2}{2}\int_0^1 du\, e^{-(1-u)H_0 t}H_I e^{-uH_0 t}H_I$$
$$+ \frac{(-t)^3}{3}\int_0^1 du\, u \int_0^1 dv\, e^{-(1-u)H_0 t}H_I e^{-u(1-v)H_0 t}H_I e^{-uvH_0 t}H_I + \cdots\Bigg] . \tag{2.20}$$

Using Eq. (2.20) in Eq. (2.16) we obtain an expression for $\zeta(s)$ that can be used to generate one-loop, 1PI Green's functions. In the expression

$$\det H = \exp\Bigg\{-\lim_{s\to 0}\frac{d}{ds}\Bigg[\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1} tr\Bigg[e^{-H_0 t} - te^{-H_0 t}H_I + \frac{t^2}{2}\int_0^1 du\, e^{-(1-u)H_0 t}H_I e^{-uH_0 t}H_I$$
$$- \frac{t^3}{3}\int_0^1 du\, u\int_0^1 dv\, e^{-(1-u)H_0 t}H_I e^{-u(1-v)H_0 t}H_I e^{-uvH_0 t}H_I + \cdots\Bigg]\Bigg]\Bigg\} \tag{2.21}$$

we need only consider those terms of order n in the background field to obtain the n-point one-loop 1PI Green's functions. From Eq. (2.21) it is apparent that our identification of H_0 is constrained by the requirement that it has positive eigenvalues to ensure convergence of the t integrals.

To illustrate this procedure, let us consider the massless ϕ^3 scalar theory in six dimensions. The Lagrangian for this model is

$$\mathscr{L} = -\tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{3!}\phi^3 . \tag{2.22}$$

Upon making the expansion of Eq. (2.1) in a classical field f and a quantum field h, $\phi = f + h$, we see that the bilinear term in the Lagrangian is

$$\mathscr{L}^{(2)} = -\tfrac{1}{2}h(x)[p^2 + \lambda f(x)]h(x) . \tag{2.23}$$

(We have used the notation $p_\mu = -i\partial_\mu$.) Equations (2.11) and (2.12) for this model reduce to a particularly simple form for the generating functional Z_1:

$$Z_1 = \det{}^{-1/2}(p^2 + \lambda f) . \tag{2.24}$$

The absence of fermions in this model considerably reduces the complexity of the superdeterminant we have to deal with.

If we now identify the operators H_0 and H_I in Eq. (2.19) with p^2 and λf, respectively, then by Eq. (2.21), Eq. (2.24) can be written

$$Z_1=\exp\left[-\frac{1}{2}\left\{-\lim_{s\to 0}\frac{d}{ds}\left[\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\mathrm{tr}\left[e^{-p^2t}-te^{-p^2t}(\lambda f)+\frac{t^2}{2}\int_0^1 du\, e^{-(1-u)p^2t}(\lambda f)e^{-up^2t}(\lambda f)\right.\right.\right.\right.$$
$$-\frac{t^3}{3}\int_0^1 du\, u\int_0^1 dv\, e^{-(1-u)p^2t}(\lambda f)e^{-u(1-v)p^2t}$$
$$\left.\left.\left.\left.\times(\lambda f)e^{-uvp^2t}(\lambda f)+\cdots\right]\right]\right\}\right] . \tag{2.25}$$

To one-loop order this series plays the same role as Feynman rules in the usual perturbation theory. If, for example, we want to evaluate the one-loop correction to the two-point function in this model, we restrict our attention to the term bilinear in f on the right-hand side of Eq. (2.25). This leaves us with

$$Z_{1ff}=\exp\left\{-\frac{1}{2}\left[-\lim_{s\to 0}\frac{d}{ds}\left\{\frac{\lambda^2}{\Gamma(s)}\int_0^\infty dt\frac{t^{s+1}}{2}\mathrm{tr}\int_0^1 du\, e^{-(1-u)p^2t}fe^{-up^2t}f\right\}\right]\right\} . \tag{2.26}$$

The next step is to compute the functional trace

$$K=\mathrm{tr}(e^{-(1-u)p^2t}f\, e^{-up^2t}f) . \tag{2.27}$$

Schwinger[9] has pointed out that such traces are most easily evaluated in momentum space. We introduce a complete orthonormal set of states $|p\rangle$ that are eigenstates of the operator p_μ, where, in n dimensions,

$$\langle x\,|\,p\rangle=e^{ip\cdot x}/(2\pi)^{n/2} \tag{2.28a}$$

and

$$\langle p\,|\,f\,|\,q\rangle=f(p-q)/(2\pi)^{n/2} . \tag{2.28b}$$

On the right-hand side of Eq. (2.28b), $f(p-q)$ is the Fourier transform of $f(x)$:

$$f(p-q)=\int\frac{d^nx}{(2\pi)^{n/2}}f(x)e^{-ix\cdot(p-q)} . \tag{2.29}$$

Equation (2.27) takes the form

$$K=\int d^6p\, d^6q\, d^6r\, d^6s\langle p\,|\,e^{-(1-u)p^2t}\,|\,q\rangle$$
$$\times\langle q\,|\,f\,|\,r\rangle\langle r\,|\,e^{-up^2t}\,|\,s\rangle\langle s\,|\,f\,|\,p\rangle \tag{2.30}$$

upon inserting $1=\int d^6p\,|p\rangle\langle p|$ at the appropriate places. Using Eq. (2.28), we rewrite Eq. (2.30) as

$$K=\int\frac{d^6p\, d^6q}{(2\pi)^6}e^{-[(1-u)p^2t+uq^2t]}f(p-q)f(q-p) . \tag{2.31}$$

After shifting the variable of integration $p\to p+q$, Eq. (2.31) becomes

$$K=\int\frac{d^6p\, d^6q}{(2\pi)^6}e^{-[(1-u)(p+q)^2+uq^2]t}f(p)f(-p) . \tag{2.32}$$

Finally, in Eq. (2.32), we shift the variable of integration $q\to q+(1-u)p$ so that

$$K=\int\frac{d^6p\, d^6q}{(2\pi)^6}e^{-[q^2+u(1-u)p^2]t}f(p)f(-p) . \tag{2.33}$$

This procedure for computing the trace in Eq. (2.27) can be easily adapted to any term in Eq. (2.25).

Upon substituting Eq. (2.33) into Eq. (2.26), we find that

$$Z_{1ff}=\exp[\tfrac{1}{2}\zeta'_{ff}(0)] , \tag{2.34a}$$

where

$$\zeta_{ff}(s)=\frac{\lambda^2}{2\Gamma(s)}\int_0^\infty dt\, t^{s+1}\int_0^1 du\int d^6p\, f(p)f(-p)\int\frac{d^6q}{(2\pi)^6}e^{-[q^2+u(1-u)p^2]t} . \tag{2.34b}$$

We use Eq. (2.15) to integrate over t:

$$\zeta_{ff}(s)=\frac{\lambda^2}{2\Gamma(s)}\int d^6p\, f(p)f(-p)\int_0^1 du\int\frac{d^6q}{(2\pi)^6}\frac{\Gamma(s+2)}{[q^2+u(1-u)p^2]^{s+2}} . \tag{2.35}$$

[We see at this stage that the parameters $u, v, \ldots$ in Eq. (2.25) play roles akin to the Feynman parameters in the usual approach.]

The standard integral

$$\int\frac{d^nq}{(2\pi)^n}\frac{(q^2)^r}{(q^2+c^2)^m}=\frac{1}{(16\pi^2)^{n/4}}(c^2)^{(n/2)+r-m}\frac{\Gamma(r+n/2)\Gamma(m-r-n/2)}{\Gamma(n/2)\Gamma(m)} \tag{2.36}$$

gives

$$\zeta_{ff}(s)=\frac{\lambda^2\Gamma(s+2)}{2\Gamma(s)}\int d^6p\, f(p)f(-p)\int_0^1 du\left[\frac{[u(1-u)p^2]^{1-s}}{(4\pi)^3}\frac{\Gamma(s-1)}{\Gamma(s+2)}\right]. \tag{2.37}$$

Equation (2.37) can also be arrived at by first integrating over q in Eq. (2.34b) using the standard integrals

$$\int \frac{d^nq}{(2\pi)^n}e^{-tq^2}=\frac{t^{-n/2}}{(4\pi)^{n/2}} \tag{2.38}$$

and then using Eq. (2.15) to integrate over t. In Eq. (2.37) the u integral can now be evaluated as it is of the form

$$\int_0^1 du\, u^{r-1}(1-u)^{s-1}[au+b(1-u)]^{-r-s} = a^{-r}b^{-s}\frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} \tag{2.39}$$

where $a=b=1$. We consequently find that

$$\begin{aligned}\zeta_{ff}(s)&=\frac{\lambda^2}{2}\frac{\Gamma(s-1)\Gamma^2(2-s)}{\Gamma(s)\Gamma(4-2s)}\times\int\frac{d^6p}{(4\pi)^3}f(p)f(-p)(p^2)^{1-s}\\&\simeq\frac{\lambda^2}{2}\int\frac{d^6p}{(4\pi)^3}f(p)f(-p)\times\left[-\frac{p^2}{6}[1+s(\tfrac{8}{3}-\ln p^2)+\cdots]\right].\end{aligned} \tag{2.40}$$

At no stage in the calculation have we encountered any explicitly divergent integrals, and we see that $\zeta'_{ff}(0)$ is well defined.

Substitution of Eq. (2.40) into Eq. (2.34a) yields our final expression for Z_{1ff}:

$$Z_{1ff}=\exp\left[\frac{\lambda^2}{24}\int\frac{d^6p}{(4\pi)^3}f(p)f(-p)p^2(\ln p^2-\tfrac{8}{3})\right]. \tag{2.41}$$

We see that we have recovered the usual result based on computing regularized Feynman diagrams followed by a renormalization to excise infinities that arise as the regulating parameter (e.g., n, the dimension of space-time) approaches its limiting value.

It is straightforward to adapt the above procedure to compute one-loop 1PI n-point Green's functions in any theory. Our procedure is quite distinct from the usual Feynman approach (although some of the integrals we must evaluate are superficially similar to the usual Feynman integrals.) It is not possible to generate our perturbation series by simply taking unregulated Feynman integrals and inserting the regulating parameter s in some prescribed way. Consequently our procedure is not some special case of the approach of either Lee and Milgram[21] or Speer.[22]

We now list the steps in our procedure for computing one-loop 1PI n-point Green's functions.

(1) Identify the term in the Lagrangian that is bilinear in the quantum fields [cf. Eq. (2.5)].

(2) Determine which form of the superdeterminant for the one-loop generating functional is most convenient for the process to be computed [cf. Eqs. (2.11) and (2.12)]. If the only external fields present are bose, these choices are identical.

(3) Having written the one-loop generating functional as a ratio of determinants of operators

$$Z=\frac{\det^{1/2}A}{\det^{1/2}B} \tag{2.42}$$

regulate these determinants through use of the ζ function yielding

$$Z=\frac{\exp-\frac{1}{2}\zeta'^{A}(0)}{\exp-\frac{1}{2}\zeta'^{B}(0)} \tag{2.43}$$

[cf. Eq. (2.16)].

(4) Expand each ζ function in powers of that part of the operator that depends on the external fields using Eq. (2.20).

(5) Select those terms in the expansion appropriate for the n-point Green's function we wish to compute. We denote these by $\zeta_n^A(s)$ and $\zeta_n^B(s)$.

(6) Evaluate the trace of the terms selected in momentum space using Eq. (2.28).

(7) Perform the relevant momentum and parameter integrations using Eqs. (2.36), (2.38), and (2.39) to obtain the final expressions for $\zeta_n^A(s)$ and $\zeta_n^B(s)$.

(8) The amplitude corresponding to the one-loop 1PI n-point Green's function is given by

$$-\tfrac{1}{2}[\zeta_n^{A'}(0)-\zeta_n^{B'}(0)]$$

as the full one-loop generating functional is given by

$$Z=\exp\left[-\tfrac{1}{2}\sum_n[\zeta_n^{A'}(0)-\zeta_n^{B'}(0)]\right].$$

III. YANG-MILLS THEORY

The Lagrangian for a classical Yang-Mills field W_μ^a in Euclidean space is

$$\mathscr{L}_0=\tfrac{1}{4}F_{\mu\nu}^a(W)F_{\mu\nu}^a(W), \tag{3.1}$$

where

$$F_{\mu\nu}^a(W)=\partial_\mu W_\nu^a-\partial_\nu W_\mu^a+gf^{abc}W_\mu^bW_\nu^c.$$

The expansion of W_μ^a into a classical piece V_μ^a and a quantum piece Q_μ^a,

$$W_\mu^a=V_\mu^a+Q_\mu^a, \tag{3.2}$$

leads to the bilinear effective Lagrangian

$$\mathscr{L}=-\tfrac{1}{2}Q^a_\mu\left[D^{2ab}(V)\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right]D^{ap}_\mu(V)D^{pb}_\nu(V)+2gf^{apb}F^p_{\mu\nu}(V)\right]Q^b_\nu-\bar{c}^{\,a}D^{2ab}(V)c^b\ , \qquad (3.3)$$

provided we work in the background gauge-covariant Honerkamp gauge[7]

$$D^{ab}_\mu(V)Q^b_\mu=0\ . \qquad (3.4)$$

Here we have defined

$$D^{ab}_\mu(V)\equiv\partial_\mu\delta^{ab}+gf^{apb}V^p_\mu$$

and c^a and $\bar{c}^{\,a}$ are complex anticommuting scalar ghost fields.

From Eqs. (3.3) and (2.12) we see immediately that the generating functional for all one-loop 1PI n-point Green's functions is

$$Z_1=\det[-D^{2ab}(V)]\times\det{}^{-1/2}\left\{-\left[D^{2ab}(V)\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right]D^{ap}_\mu(V)D^{pb}_\nu(V)+2gf^{apb}F^p_{\mu\nu}(V)\right]\right\}\ . \qquad (3.5)$$

For purposes of illustration, in this paper we will focus our attention on the two-point amplitude.

We can now regulate the determinants in Eq. (3.5) by Eq. (2.21). First of all, if $H_0=p^2\delta^{ab}$,

$$H_I=-igf^{apb}(p\cdot V^p+V^p\cdot p)-g^2f^{apr}f^{rqb}V^p\cdot V^q\ ,$$

then, to second order in the background field V^a_μ,

$$\operatorname{tr}e^{D^2(V)t}=\operatorname{tr}\left[e^{-p^2t}\delta^{ab}+(-t)e^{-p^2t}(-g^2f^{apr}f^{rqb}V^p\cdot V^q)+\frac{(-t)^2}{2}\int_0^1du\,e^{-(1-u)p^2t}[-igf^{apr}(p\cdot V^p+V^p\cdot p)]e^{-up^2}[-igf^{rqb}(p\cdot V^q+V^q\cdot p)]+\cdots\right]\ . \qquad (3.6)$$

The trace in Eq. (3.6) can be evaluated using the methods explained in the preceding section. A straightforward calculation leads to the second-order expression for $\zeta'_{VV}(s)$ associated with $\det[-D^2(V)]$:

$$\zeta^1_{VV}(s)=\frac{g^2C_2\delta^{ab}}{(4\pi)^2}\int d^4p\,V^a_\mu(p)V^b_\nu(-p)\frac{\Gamma(2-s)\Gamma(1-s)}{\Gamma(4-2s)}(p^2)^{-s}(-\delta_{\mu\nu}p^2+p_\mu p_\nu)\ . \qquad (3.7)$$

In the second determinant occurring in Eq. (3.5), we restrict our attention for the purposes of calculational simplicity to the gauge given by $\alpha=1$. A computation similar to the above shows that the second-order term in the ζ function associated with $\det[-\delta_{\mu\nu}D^{2ab}(V)-2gf^{apb}F^p_{\mu\nu}(V)]$ is

$$\zeta_{VV}{}^2(s)=\frac{4g^2C_2\delta^{ab}}{(4\pi)^2}\int d^4p\,V^a_\mu(p)V^b_\nu(-p)\left[\frac{\Gamma(2-s)\Gamma(1-s)}{\Gamma(4-2s)}-\frac{\Gamma^2(1-s)}{\Gamma(2-2s)}\right](p^2)^{-s}(-\delta_{\mu\nu}p^2+p_\mu p_\nu)\ . \qquad (3.8)$$

Combining Eqs. (2.16), (3.5), (3.7), and (3.8) we find that, for the two-point function,

$$\begin{aligned}Z_{1VV}&=\exp[-\zeta^{1'}_{VV}(0)+\tfrac{1}{2}\zeta^{2'}_{VV}(0)]\\&=\exp\left[\frac{-g^2C_2\delta^{ab}}{2(4\pi)^2}\int d^4p\,V^a_\mu(p)V^b_\nu(-p)(-p^2\delta_{\mu\nu}+p_\mu p_\nu)(\tfrac{64}{9}-\tfrac{11}{3}\ln p^2)\right]\ .\end{aligned} \qquad (3.9)$$

We thus see that the usual "$\frac{11}{3}\ln p^2$" term in the vacuum-polarization tensor is recovered. The factor of $\frac{64}{9}$ is peculiar to the regularization procedure being employed and can be changed by a finite renormalization.

Kallosh's theorem[18] leads us to expect that this coefficient of $\ln p^2$ in Eq. (3.9) is independent of the gauge condition used. It is therefore of interest to see how the computation of the vacuum polarization with α arbitrary differs from the $\alpha=1$ computation given above. For $\alpha\neq1$ the H_0 associated with the second determinant in Eq. (3.5) must be identified with $p^2\delta_{\mu\nu}-(1-1/\alpha)p_\mu p_\nu$. The nondiagonal nature of H_0 forces us to consider integrals of the form

$$I=\int d^4q\int_0^1du\int_0^\infty dt\,P_{\lambda\mu,\sigma\nu}(q,t)\exp\left[-t\left\{u\left[(q+p)^2\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right](q+p)_\mu(q+p)_\nu\right]+(1-u)\left[q^2\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right]q_\mu q_\nu\right]\right\}\right]\ , \qquad (3.10)$$

where $P_{\lambda\mu,\sigma\nu}(q,t)$ is a complicated, field-dependent function of q_α. The exponential factor in Eq. (3.10) can be simplified using the complete set of orthonormal projection operators:

$$T_{\mu\nu}(q)=(\delta_{\mu\nu}-q_\mu q_\nu/q^2)\ , \tag{3.11a}$$

$$L_{\mu\nu}(q)=q_\mu q_\nu/q^2\ . \tag{3.11b}$$

These allow us to write

$$\exp\left\{-u\left[q^2\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right]q_\mu q_\nu\right]t\right\}=\sum_{n=0}^{\infty}\frac{1}{n!}\left[-uq^2t\left[T_{\mu\nu}+\frac{1}{\alpha}L_{\mu\nu}\right]\right]^n=e^{-utq^2}T_{\mu\nu}+e^{-utq^2/\alpha}L_{\mu\nu}\ . \tag{3.12}$$

The projection operators can now be absorbed into the polynomial $P_{\lambda\mu,\sigma\nu}$ in Eq. (3.10) and the integral over q can be evaluated in a straightforward (albeit tedious) fashion.

It is not immediately apparent how noncovariant gauges are to be handled in our approach. For example, in the planar gauge we are confronted with expressions of the form

$$\exp Q\equiv\exp\left[-u\left[q^2\delta_{\mu\nu}-q_\mu q_\nu+\frac{1}{\alpha}q^2n_\mu n_\nu\right]t\right]\ . \tag{3.13}$$

The expansion of Eq. (3.12) is not feasible, as the matrix in the exponent of Eq. (3.13) cannot be decomposed into orthogonal projection operators. It is necessary to diagonalize Q but this leads to apparently intractable momentum integrals. Similar considerations apply in the axial, light-cone, and Coulomb gauges.

IV. QED AND THE INCLUSION OF EXTERNAL FERMIONS

In this section we extend operator regularization to the case where both external bosons and external fermions appear. We do this to one-loop order in the context of massless QED in Euclidean space, described by a classical Lagrangian

$$\mathcal{L}_0=\bar{\chi}(-i\partial\!\!\!/-eW\!\!\!\!/\,)\chi+\tfrac{1}{4}(\partial_\mu W_\nu-\partial_\nu W_\mu)^2\ . \tag{4.1}$$

We use Hermitian γ matrices satisfying $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$. The background-field expansion is

$$W_\mu=V_\mu+Q_\mu\ , \tag{4.2a}$$

$$\chi=\eta+\psi\ , \tag{4.2b}$$

where V_μ and η are the classical fields and Q_μ and ψ are the quantum fields. We use an arbitrary covariant gauge for the quantization of Q_μ. To compute the one-loop 1PI generating functional we need only consider the terms in the Lagrangian bilinear in the quantum fields: namely,

$$\mathcal{L}^{(2)}=\bar{\psi}D\!\!\!\!/\,\psi+\tfrac{1}{2}Q_\mu\left[p^2\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right]p_\mu p_\nu\right]Q_\nu-e\bar{\eta}Q\!\!\!\!/\,\psi-e\bar{\psi}Q\!\!\!\!/\,\eta\ , \tag{4.3}$$

where

$$D\!\!\!\!/\equiv\gamma^\mu(-i\partial_\mu-eV_\mu)=p\!\!\!/-eV\!\!\!\!/\ .$$

The formalism of Sec. II cannot be directly applied to the bilinear Lagrangian $\mathcal{L}^{(2)}$ of Eq. (4.3) as ψ and $\bar{\psi}$ are independent quantum fields in the associated path integral. However, it is possible to rewrite $\mathcal{L}^{(2)}$ in the form of Eq. (2.4) by the following device. We introduce the notation

$$\theta=\begin{bmatrix}\psi\\ \bar{\psi}^T\end{bmatrix},\quad \theta^T=[\psi^T,\bar{\psi}] \tag{4.4}$$

and we identify the quantum field

$$\begin{bmatrix}h_1\\ h_2\\ \vdots\\ h_h\end{bmatrix}$$

of Eq. (2.4) with $\binom{Q_\mu}{\theta}$. It is now clear that the bilinear Lagrangian can be written as

$$\mathcal{L}^{(2)}=\tfrac{1}{2}(Q_\mu,\psi^T,\bar{\psi})\begin{bmatrix}p^2\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right]p_\mu p_\nu & -e\bar{\eta}\gamma_\mu & e\eta^T\gamma_\mu^T\\ e\gamma_\nu^T\bar{\eta}^T & 0 & -D\!\!\!\!/^T\\ -e\gamma_\nu\eta & D\!\!\!\!/ & 0\end{bmatrix}\begin{bmatrix}Q_\nu\\ \psi\\ \bar{\psi}^T\end{bmatrix}\equiv\tfrac{1}{2}h_i^TM_{ij}h_j\ . \tag{4.5}$$

It is clear that the Bose-Bose, Bose-Fermi, and Fermi-Fermi blocks of the supermatrix M have the following symmetry properties:

$$M_{bb}^T=M_{bb},\quad M_{ff}^T=-M_{ff},\quad M_{bf}^T=-M_{fb} \tag{4.6}$$

and that M_{bb} and M_{ff} are bosonic while M_{bf} and M_{fb} are fermionic. Evaluation of the path integral (2.5) leads at once to the one-loop generating functional

$$Z_1=\text{sdet}^{-1/2}M\ . \tag{4.7}$$

As explained in Sec. II, sdet$M$ has two representations. It is only when M_{bf} and M_{fb} are nonzero that we have two representations, i.e., when external fermion fields are involved. Consequently there are two distinct calculational paths which can be used—approach A when we use Eq. (2.11a) and approach B when we use Eq. (2.11b). We now consider approaches A and B in turn.

In approach A, using Eq. (2.11a) (see Appendix A) we find

$$Z_1=\frac{\det \not{D}}{\det^{1/2}\left[p^2\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right]p_\mu p_\nu-e^2\bar{\eta}\gamma_\mu \not{D}^{-1}\gamma_\nu\eta-e^2\bar{\eta}\gamma_\nu\not{D}^{-1}\gamma_\mu\eta\right]}\,, \tag{4.8}$$

where we have used the result

$$\det\begin{bmatrix}0 & -\not{D}^T\\ \not{D} & 0\end{bmatrix}=\det{}^2\not{D}\ .$$

In Appendix A, we outline an alternative derivation of Eq. (4.8) based on the approach of Ref. 15. Each of the determinants occurring in Eq. (4.8) requires regularization and a corresponding ζ function. However the numerator contributes to Green's functions with only external boson lines. As we are interested in this section in Green's functions with external fermion lines we focus our attention on the denominator of Eq. (4.8).

The regularized one-loop generating functional is, then, by Eq. (2.43),

$$Z_1=\exp[\tfrac{1}{2}\zeta'(0)]\,, \tag{4.9a}$$

where

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\mathrm{tr}\exp\left\{-t\left[p^2\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right]p_\mu p_\nu-e^2\bar{\eta}\gamma_\mu\frac{1}{\not{p}-e\not{V}}\gamma_\nu\eta-e^2\bar{\eta}\gamma_\nu\frac{1}{\not{p}-e\not{V}}\gamma_\mu\eta\right]\right\}. \tag{4.9b}$$

In Eq. (4.9b) it is understood that the exponential is tr[exp($-tH$)], where

$$H_{\mu\nu}\equiv p^2\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right]p_\mu p_\nu-e^2\bar{\eta}\gamma_\mu\frac{1}{\not{p}-e\not{V}}\gamma_\nu\eta-e^2\bar{\eta}\gamma_\nu\frac{1}{\not{p}-e\not{V}}\gamma_\mu\eta$$

$$\equiv H_{0\mu\nu}+H_{I\mu\nu}\ . \tag{4.10}$$

As in Eq. (3.12), we find

$$(e^{-tH_0})_{\mu\nu}=e^{-tp^2}T_{\mu\nu}+e^{-tp^2/\alpha}L_{\mu\nu}\ . \tag{4.11}$$

Our next step is to expand $\not{D}^{-1}$ in powers of the background field in the ζ function (4.9b):

$$\not{D}^{-1}=\frac{1}{\not{p}-e\not{V}}=\frac{1}{\not{p}}+\frac{1}{\not{p}}e\not{V}\frac{1}{\not{p}}+\frac{1}{\not{p}}e\not{V}\frac{1}{\not{p}}e\not{V}\frac{1}{\not{p}}+\cdots\ .$$

It is now straightforward to apply the perturbative expansion of Eq. (2.20) to this ζ function and to select from the expansion those terms appropriate for any particular Green's function. For example, for the fermion two-point function and the fermion-fermion boson three-point function we find

$$\zeta(s)\simeq\frac{2e^2}{\Gamma(s)}\int_0^\infty dt\, t^s\mathrm{tr}\left[\bar{\eta}\gamma_\mu\left[\frac{1}{\not{p}}+\frac{1}{\not{p}}e\not{V}\frac{1}{\not{p}}\right]\gamma_\nu\eta(e^{-tp^2}T_{\mu\nu}+e^{-tp^2/\alpha}L_{\mu\nu})\right]. \tag{4.12}$$

We are now in a position to complete the computation of the Green's functions following steps six, seven, and eight as outlined in Sec. II. This is straightforward. Before discussing the results of these computations, we return to the calculational path based on the other representation of the superdeterminant.

In approach B, using Eq. (2.11b) we find

$$Z_1=\frac{\det^{1/2}\begin{bmatrix}e^2\gamma_\nu\eta\Pi^{\alpha-1}_{\nu\mu}(\gamma_\mu\eta)^T & \not{D}-e^2\gamma_\nu\eta\Pi^{\alpha-1}_{\nu\mu}\bar{\eta}\gamma_\mu\\ -(\not{D}-e^2\gamma_\nu\eta\Pi^{\alpha-1}_{\nu\mu}\bar{\eta}\gamma_\mu)^T & e^2(\bar{\eta}\gamma_\nu)^T\Pi^{\alpha-1}_{\nu\mu}\bar{\eta}\gamma_\mu\end{bmatrix}}{\det^{-1/2}(\Pi^\alpha_{\mu\nu})}\,, \tag{4.13}$$

where we have used the notation

$$\Pi^{\alpha}_{\mu\nu}=p^2\delta_{\mu\nu}-\left[1-\frac{1}{\alpha}\right]p_\mu p_\nu \ , \tag{4.14a}$$

$$\Pi^{\alpha-1}_{\mu\nu}=\frac{1}{p^2}\left[\delta_{\mu\nu}-(1-\alpha)\frac{p_\mu p_\nu}{p^2}\right] . \tag{4.14b}$$

As $\Pi^{\alpha}_{\mu\nu}$ is independent of the classical background fields the denominator of Eq. (4.13) can be absorbed into the normalization of Z_1. However this is a special feature of Abelian gauge theory. For non-Abelian gauge theories $\det\Pi^{\alpha}_{\mu\nu}$ will depend on classical background fields and hence cannot be absorbed in this way.

It is interesting to note that the use of a noncovariant gauge for quantization purposes can be easily incorporated in approach B for an Abelian gauge theory. It is not immediately apparent how noncovariant gauges can be incorporated into approach A even for an Abelian gauge theory.

The numerator of Eq. (4.13) is the determinant of any antisymmetric matrix of the form

$$\begin{aligned}\Lambda&=\begin{bmatrix} P & R\\ -R^T & Q\end{bmatrix}\\ &=\begin{bmatrix} P & 0\\ -R^T & 1\end{bmatrix}\begin{bmatrix} 1 & P^{-1}R\\ 0 & Q+R^TP^{-1}R\end{bmatrix} ,\end{aligned} \tag{4.15}$$

where $P^T=-P$, $Q^T=-Q$ are identified as

$$P=e^2\gamma_\nu\eta\Pi^{\alpha-1}_{\nu\mu}(\gamma_\mu\eta)^T \ , \tag{4.16a}$$

$$Q=e^2(\bar\eta\gamma_\nu)^T\Pi^{\alpha-1}_{\nu\mu}\bar\eta\gamma_\mu \ , \tag{4.16b}$$

and

$$R=\not{D}-e^2\gamma_\nu\eta\Pi^{\alpha-1}_{\nu\mu}\bar\eta\gamma_\mu \ . \tag{4.16c}$$

It is easily shown by Eq. (4.15) that

$$Z_1=\det{}^{1/2}\Lambda=\det{}^{1/2}R\,\det{}^{1/2}(R+PR^{-1T}Q) \ . \tag{4.17}$$

As we are interested in computing the fermion two-point function and the vertex function and as $PR^{-1T}Q$ can only contribute to Green's functions with at least four external fermions, we restrict our attention to

$$\begin{aligned}Z_1&\simeq\det{}^{1/2}(R^2)\\ &\simeq\det{}^{1/2}(\not{D}^2-e^2\{\not{D},\gamma_\mu\eta\Pi^{\alpha-1}_{\mu\nu}\bar\eta\gamma_\nu\})\\ &\equiv\det{}^{1/2}H \ .\end{aligned} \tag{4.18}$$

In Appendix B we write out explicitly the contributions which we ignore here.

The regularized one-loop generating functional is, then, in approach B,

$$Z_1=\exp[-\tfrac{1}{2}\zeta'(0)] \ , \tag{4.19a}$$

where

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\mathrm{tr}[\exp-t(H_0+H_I)] \ . \tag{4.19b}$$

To use the Schwinger expansion as in Sec. II we have separated H as

$$H_0=p^2 \tag{4.20a}$$

and

$$\begin{aligned}H_I&=-e\{\not{p},\not{V}\}-e^2\{\not{p},\gamma_\mu\eta\Pi^{\alpha-1}_{\mu\nu}\bar\eta\gamma_\nu\}\\ &\quad+e^2V^2+e^3\{\not{V},\gamma_\mu\eta\Pi^{\alpha-1}_{\mu\nu}\bar\eta\gamma_\nu\}\\ &\equiv H_1+H_2+H_2'+H_3 \ .\end{aligned} \tag{4.20b}$$

The Schwinger perturbative expansion can now be applied to the ζ function (4.19b) in a straightforward manner. The terms in this expansion appropriate to the computation of the fermion two-point function and the vertex function are

$$\begin{aligned}\zeta(s)=&-\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^s\mathrm{tr}[e^{-tp^2}(H_2+H_3)]\\ &+\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s+1}\mathrm{tr}(e^{-(1-u)p^2t}H_2e^{-up^2t}H_1) \ .\end{aligned} \tag{4.21}$$

The first term alone contributes to the fermion two-point function, but both terms contribute nontrivially to the vertex function. The calculations of the fermion two-point function and the vertex function are straightforward, following the procedural steps outlines in Sec. II.

Equations (4.12) and (4.21) are the ζ functions appropriate to the fermion propagator and the vertex function in massless QED in the two approaches outlined above. In approach A we compute from Eq. (4.12) this ζ function in the limit of zero momentum transfer to the photon:

$$\zeta(s)=\frac{e^2}{8\pi^2}\int d^4p\left[\frac{1}{p^2}\right]^s\frac{\Gamma(1-s)}{\Gamma(3-s)}\left[2+(\alpha^{s+1}-1)\frac{(2-s)}{(1+s)}\right]\bar\eta(p)\left[\not{p}+\frac{e}{4\pi^2}\left[\not{V}(0)-2s\frac{\not{p}}{p^2}p\cdot V(0)\right]\right]\eta(-p) \ .$$

By Eq. (4.9a) the contributions to the one-loop generating functional will be

$$Z_1\simeq\exp\left\{\frac{1}{2}\left[\frac{e^2}{8\pi^2}\int d^4p(\tfrac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2)\bar\eta(p)\left[\not{p}+\frac{e}{4\pi^2}\not{V}(0)\right]\eta(-p)-\alpha\frac{e^3}{16\pi^4}\int d^4p\,\bar\eta(p)\not{p}\eta(-p)\frac{p\cdot\not{V}(0)}{p^2}\right]\right\} .$$

The one-loop two-point function and vertex function are obtained from $\ln Z_1$ by appropriate functional differentiations

It is interesting to note the unusual dependence of these Green's functions on the gauge-fixing parameter α. We can trace this dependence back to Eq. (4.11), where the second term contained an extra factor of $1/\alpha$ in the exponential. The

$\alpha \ln\alpha$ terms in Z_1 could, however, be removed by a finite renormalization. Their presence in Z_1 indicates that the implicit renormalization in the operator regularization method differs from the conventional renormalization schemes merely by finite renormalizations.

Turning now to approach B, we compute from Eq. (4.21) the corresponding ζ function,

$$\zeta(s) = -\frac{e^2}{4\pi^2}\alpha \int d^4p \left[\frac{1}{p^2}\right]^s \frac{\Gamma(2-s)}{\Gamma(3-s)} \bar{\eta}(p) \left[\not p + \frac{e}{4\pi^2}\left[\not V(0) - 2s\frac{\not p}{p^2} p\cdot V(0)\right]\right] \eta(-p) ,$$

giving rise, by Eq. (4.19a), to the one-loop generating functional

$$Z_1 \simeq \exp\left\{\frac{1}{2}\left[\frac{e^2}{8\pi^2}\alpha \int d^4p (\tfrac{1}{2} - \ln p^2) \bar{\eta}(p) \left[\not p + \frac{e}{4\pi^2}\not V(0)\right]\eta(-p) - \frac{e^3}{16\pi^4}\alpha \int d^4p\, \bar{\eta}(p) \not p \eta(-p) \frac{p\cdot V(0)}{p^2}\right]\right\} .$$

We see by inspection the approaches A and B lead by different calculational paths to Green's functions related to one another by α-dependent finite renormalizations.

V. AXIAL MODELS

In the preceding sections we have seen how our procedure leads to finite Green's functions without breaking any symmetries present in the initial Lagrangian. Since it is *operators* that are being regulated rather than the initial Lagrangian, it is well worth considering how anomalies occur in our approach. As the Lagrangian is not being altered through the insertion of a regulating parameter that may fail to respect a symmetry initially present, it is not immediately clear how anomalies arise.

Let us examine the axial anomaly in an Abelian theory. We consider a classical axial-vector field A_μ and a classical vector field V_μ coupled to a quantum Dirac spinor ψ with the Euclidean space gauge-invariant Lagrangian

$$\mathcal{L}^G = \bar{\psi}(\not p - e\not V - g\not A\gamma_5)\psi . \tag{5.1}$$

We first note that, in order to apply our technique, we must work in Euclidean space so that the t integral in Eq. (2.21) converges as $t\to\infty$. Since we are in Euclidean space it is possible to work with a Hermitian representation for γ_μ and γ_5. We thus see that $\bar{\psi}(\not p - e\not V)\psi$ in Eq. (5.1) is Hermitian provided $\bar{\psi} = \psi^\dagger$ (and *not* $\bar{\psi} = \psi^\dagger\gamma_4$ as in Minkowski space). This means, however, that the term $-g\bar{\psi}\not A\gamma_5\psi$ in Eq. (5.1) is not Hermitian.

If we were to restore Hermiticity in Eq. (5.1) by replacing A_μ by iA_μ the Lagrangian of the model becomes

$$\mathcal{L}^H = \bar{\psi}(\not p - e\not V - ig\not A\gamma_5)\psi . \tag{5.2}$$

We note that $\mathcal{L}^G$ is invariant under the local gauge transformations

$$\psi \to e^{i\alpha\gamma_5}\psi,\quad \bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5},\quad A_\mu \to A_\mu - \frac{1}{g}\partial_\mu\alpha , \tag{5.3}$$

but $\mathcal{L}^H$ is not.

Let us examine the three-point functions $\langle VVA\rangle$ and $\langle AAA\rangle$ using $\mathcal{L}^H$ in Eq. (5.2). Equation (2.12) gives the one-loop generating functional

$$Z_1^H = \det(\not p - e\not V - ig\not A\gamma_5) . \tag{5.4}$$

In order to apply our method for computing Green's functions we rewrite Z_1^H in Eq. (5.4) as

$$\begin{aligned} Z_1^H &= \det{}^{1/2}[(\not p - e\not V - ig\not A\gamma_5)^2] \\ &= \det{}^{1/2}[p^2 - e(\not p\not V + \not V\not p) - ig(\not p\not A - \not A\not p)\gamma_5 \\ &\quad + e^2V^2 + g^2A^2 + ieg(\not V\not A - \not A\not V)\gamma_5] . \end{aligned} \tag{5.5}$$

In computing the anomaly in the divergence of the axial-vector current, to which A_μ couples, we exploit a trick due to Capper.[23] We denote by $R_{\lambda\mu\nu}(p_1,p_2)$ the Green's function $\langle A_\lambda(-p_1-p_2)V_\mu(p_1)V_\nu(p_2)\rangle$. It is a standard result that the anomalous equations are of the form

$$(p_1+p_2)^\lambda R_{\lambda\mu\nu}(p_1,p_2) = \lambda_1\epsilon_{\mu\nu\alpha\beta}p_1^\alpha p_2^\beta , \tag{5.6a}$$

$$p_1^\mu R_{\lambda\mu\nu}(p_1,p_2) = \lambda_2\epsilon_{\lambda\nu\alpha\beta}p_1^\alpha p_2^\beta , \tag{5.6b}$$

$$p_2^\nu R_{\lambda\mu\nu}(p_1,p_2) = \lambda_3\epsilon_{\lambda\mu\alpha\beta}p_1^\alpha p_2^\beta , \tag{5.6c}$$

where λ_i are constants whose sum is fixed. The Capper trick is to take appropriate derivatives of Eqs. (5.6a)–(5.6c) with respect to external momenta and then to set these momenta equal to zero. We find

$$R_{\lambda\mu\nu}(p,-p) = \lambda_1\epsilon_{\mu\nu\alpha\lambda}p^\alpha , \tag{5.7a}$$

$$R_{\lambda\mu\nu}(0,p) = \lambda_2\epsilon_{\lambda\nu\mu\beta}p^\beta , \tag{5.7b}$$

$$R_{\lambda\mu\nu}(p,0) = \lambda_3\epsilon_{\lambda\mu\alpha\nu}p^\alpha . \tag{5.7c}$$

The standard approach to the anomalous Feynman diagrams for $\langle VVA\rangle$ cannot uniquely fix λ_1, λ_2, and λ_3 without the imposition of vector gauge invariance[24] but in our approach gauge invariance is automatically preserved and their values are uniquely fixed.

Similarly, if $S_{\lambda\mu\nu}(p_1,p_2)$ denotes $\langle A_\lambda(-p_1-p_2)\times A_\mu(p_1)A_\nu(p_2)\rangle$ the corresponding anomalous equations are

$$(p_1+p_2)^\lambda S_{\lambda\mu\nu}(p_1,p_2) = \sigma_1\epsilon_{\mu\nu\alpha\beta}p_1^\alpha p_2^\beta , \tag{5.8a}$$

$$p_1^\mu S_{\lambda\mu\nu}(p_1,p_2) = \sigma_2\epsilon_{\lambda\nu\alpha\beta}p_1^\alpha p_2^\beta , \tag{5.8b}$$

$$p_2^\nu S_{\lambda\mu\nu}(p_1,p_2) = \sigma_3\epsilon_{\lambda\mu\alpha\beta}p_1^\alpha p_2^\beta , \tag{5.8c}$$

where, again, σ_i are constants whose sum is fixed. Capper's trick yields

$$S_{\lambda\mu\nu}(p,-p) = \sigma_1\epsilon_{\mu\nu\alpha\lambda}p^\alpha , \tag{5.9a}$$

$$S_{\lambda\mu\nu}(0,p) = \sigma_2\epsilon_{\lambda\nu\mu\beta}p^\beta , \tag{5.9b}$$

$$S_{\lambda\mu\nu}(p,0) = \sigma_3\epsilon_{\lambda\mu\alpha\nu}p^\alpha . \tag{5.9c}$$

Equations (5.6)—(6.9) show that the (anomalous) divergence of the current associated with an external field in $R(S)$ is proportional to $R(S)$ where the momentum of the particular external field has been set equal to zero.

We now apply Eqs. (2.16b) and (2.20) to compute the ζ function associated with the three-point functions R and S. Straightforward computation leads to the results

$$\zeta_{AAA}(s)=\frac{4g^3}{(8\pi^2)^2}\epsilon_{\alpha\beta\gamma\delta} \times \int d^4p[p^\alpha A^\beta(p)A^\gamma(0)A^\delta(-p)(p^2)^{-s}] \times\left[-4\frac{\Gamma(1-s)\Gamma(2-s)\Gamma(1+s)}{\Gamma(s)\Gamma(4-2s)}\right], \tag{5.10a}$$

$$\zeta_{VVA}(s)=\frac{4e^2g}{(8\pi^2)^2}\epsilon_{\alpha\beta\gamma\delta} \times \int d^4p[p^\alpha V^\beta(p)A^\gamma(0)V^\delta(-p)(p^2)^{-s}] \times\left[-2\frac{\Gamma^2(1-s)\Gamma(1+s)}{\Gamma(s)\Gamma(2-2s)}\right], \tag{5.10b}$$

where we have set the momentum of A^γ equal to zero. It is immediately apparent that $\zeta'(0)$ in Eqs. (5.10a) and (5.10b) when used in conjunction with Eqs. (5.7) and (5.9) automatically fixes

$$\sigma_1=\sigma_2=\sigma_3=\tfrac{1}{3}\lambda_1\left[\frac{g^2}{e^2}\right]=\frac{1}{3}\frac{g^3}{2\pi^2}\ . \tag{5.11}$$

To fix the values of λ_2 and λ_3 we now compute $\zeta_{VVA}(s)$ with the momentum of V^β equal to zero and we find

$$\lambda_2=\lambda_3=0\ . \tag{5.12}$$

We thus see that the ζ-function approach automatically respects gauge invariance in the vector field V^μ in Eq. (5.1) and, upon using the prescription of Eq. (5.2) we unambiguously obtain the correct divergence of the axial-vector current in both the VVA and AAA three-point functions, the latter being one third the former.

Two features immediately distinguish our approach from the usual analysis of the VVA and AAA Green's functions based on three-point Feynman diagrams. First of all, in the Feynman-diagram approach the two diagrams associated with $\langle VVA\rangle$ and $\langle AAA\rangle$ are superficially identical once one anticommutes two factors of γ_5 in $\langle AAA\rangle$ through other factors of γ_μ until they are adjacent to one another. In the ζ-function approach the computation of $\langle VVA\rangle$ and $\langle AAA\rangle$ are completely distinct; indeed upon substituting Eq. (5.5) into Eq. (2.21) we see that $\langle AAA\rangle$ receives a contribution only from the term of order $H_I{}^3$ while $\langle VVA\rangle$ receives contributions from terms of order $H_I{}^2$ and $H_I{}^3$. The second feature of the ζ-function approach is that regularization of Eq. (5.4) by means of Eq. (2.16) unambiguously fixes the value of the anomaly by Eqs. (5.11) and (5.12). In contrast, the Feynman-diagram approach employs regulators that do not fix the values of λ_i and σ_i in Eqs. (5.6) and (5.8) (Ref. 24); only the sums $\lambda_1+\lambda_2+\lambda_3$ and $\sigma_1+\sigma_2+\sigma_3$ are unambiguous and the values of λ_i and σ_i of Eqs. (5.11) and (5.12) can only be obtained by invoking vector gauge invariance and Bose symmetry.

It is of interest to compute the two-point function $\langle AA\rangle$ using Eq. (5.4). Again, straightforward application of our procedure leads to

$$\zeta_{AA}(s)=\frac{g^2}{4\pi^2}\int d^4p(p^2)^{-s}\frac{\Gamma^2(2-s)}{\Gamma(4-2s)} \times\left[p\cdot A(p)p\cdot A(-p) -\frac{1}{s-1}p^2A(p)\cdot A(-p)\right] \tag{5.13}$$

for the ζ function associated with this amplitude. It is immediately apparent that $\zeta'_{AA}(0)$ is not transverse. As gauge invariance of the form in Eq. (5.3) is not present in Eq. (5.2), perhaps it is not surprising that we do not obtain gauge-invariant (transverse) results for Green's functions involving external A_μ fields. However, a finite counterterm can restore transversality in Eq. (5.3); of course, no such counterterm can eliminate the anomalies of Eq. (5.11).

If we were to naively ignore problems of Hermiticity and use Eq. (5.1) directly, then by Eq. (2.12) we would be faced with the functional

$$Z_1^G=\det(\not p-e\not V-g\not A\gamma_5)\ . \tag{5.14}$$

In order to apply our procedure to the non-Hermitian operator in Eq. (5.14), we may try to replace (5.14) by

$$Z_1^G=\det{}^{1/2}[(\not p-e\not V-g\not A\gamma_5)(\not p-e\not V-g\not A\gamma_5)^\dagger] =\det{}^{1/2}[(\not p-e\not V-g\not A\gamma_5)(\not p-e\not V+g\not A\gamma_5)]\ . \tag{5.15}$$

Applying our approach to compute $\langle VVA\rangle$ and $\langle AAA\rangle$ from Eq. (5.15) we find that these two functions are *zero*. [This is not unexpected as Eq. (5.15) is an even function of g and zero is the only possible answer for the three-point functions that is Bose symmetric and transverse in all three vertices.] Similarly we find that the two-point function $\langle AA\rangle$ is transverse. We thus see that the three-point and two-point functions are gauge invariant if they are computed from the gauge-invariant functional (5.15); unfortunately, we can find no reason to justify computing Green's functions from Eqs. (5.15).

VI. REGULARIZATION IN n DIMENSIONS

The divergent integrals that occur in the Feynman-diagram approach to perturbation theory are conveniently regulated by doing all computations in n dimensions.[17,25] After evaluating all n-dimensional Feynman integrals by use of Eq. (2.3b) the divergences are parametrized as poles of the form $(n-4)^{-1}$; infinities that occur as n approaches 4 must be removed by renormalization.

It is interesting to see what happens when we apply the technique of operator regularization to n-dimensional theories. To illustrate what happens let us examine the two-point function in the ϕ^3 scalar theory examined in Sec. II, but now performing all calculations in n, rather than 6, dimensions. The ζ function associated with the two-point function is then given by

$$\zeta_{ff}(s,n)=\frac{\lambda^2}{2\Gamma(s)}\int d^np\, f(p)f(-p)\int_0^1 du\frac{d^nq}{(2\pi)^n}\frac{\Gamma(s+2)}{[q^2+u(1-u)p^2]^{s+2}} \tag{6.1}$$

in place of Eq. (2.35). Evaluating the integrals over q and u in Eq. (6.1) we find

$$\zeta_{ff}(s,n)=\frac{\lambda^2}{2}\frac{\Gamma(s+2-n/2)\Gamma^2(n/2-s-1)}{\Gamma(s)\Gamma(n-2s-2)}\int\frac{d^np}{(4\pi)^{n/2}}f(p)f(-p)(p^2)^{(n/2-s-2)}\,. \tag{6.2}$$

Upon setting $n=6-2\epsilon$, Eq. (6.2) reduces to

$$\zeta_{ff}(s,n)\simeq\frac{\lambda^2}{2}\int\frac{d^np}{(4\pi)^3}f(p)f(-p)\left[-\frac{p^2}{6}\frac{s}{s+\epsilon}[1+\epsilon(\tfrac{8}{3}-\gamma+\ln 4\pi-\ln p^2)+s(\tfrac{8}{3}-\ln p^2)]\right]. \tag{6.3}$$

From Eq. (6.3) it is immediately apparent that $\zeta'_{ff}(0)$ for $\epsilon\neq 0$ is given by

$$\zeta'_{ff}(0,n)\simeq\frac{\lambda^2}{2}\int\frac{d^np}{(4\pi)^3}f(p)f(-p)\left[-\frac{p^2}{6}\left[\frac{1}{\epsilon}+\frac{8}{3}-\gamma+\ln 4\pi-\ln p^2\right]\right]. \tag{6.4}$$

This gives precisely the same value for the two-point function as is calculated from the dimensionally regulated Feynman diagram for this Green's function; indeed, even the pole at $\epsilon=0$ is recovered.

From Eq. (6.3) we see that

$$\lim_{s\to 0}\frac{d}{ds}\left[\lim_{n\to 6}\zeta_{ff}(s,n)\right]\neq\lim_{n\to 6}\left[\lim_{s\to 0}\frac{d}{ds}\zeta_{ff}(s,n)\right]. \tag{6.5}$$

Indeed, on the left-hand side of Eq. (6.5) we recover the finite result of Eq. (2.40), while on the right-hand side of Eq. (6.5) we recover Green's function that is calculated in the dimensionally regulated Feynman-diagram expansion as illustrated above. It is interesting to note that in any given theory the approach based on operator regularization gives us a finite answer; the divergence in Eq. (6.4) occurs only because the theory is altered by allowing ϵ to approach zero after the calculations have been completed.

We also note that in the n-dimensional theory the original approach of Schwinger based on Eqs. (2.17) and (2.20) yields a result identical to Eq. (6.4). This is equivalent to saying that, in n dimensions,

$$\lim_{s\to 0}[\Gamma(s)\zeta(s,n)]=\lim_{s\to 0}\left[\frac{d}{ds}\zeta(s,n)\right]. \tag{6.6}$$

This equality is due to the fact that no pole occurs at $s=0$ if the tracing operation in Eq. (2.21) is done in $n\neq$integer number of dimensions.

These properties of $(\phi^3)_6$ scalar theory given in Eqs. (6.5) and (6.6) above persist in QED. Explicit computation yields the following results. For the vacuum-polarization tensor we get

$$\zeta_{VV}(s,n)\approx\frac{e^2}{12\pi^2}\int d^np\frac{s}{s+\epsilon}[1+\epsilon(\tfrac{5}{3}-\gamma+\ln 4\pi-\ln p^2)+s(\tfrac{5}{3}-\ln p^2)]$$
$$\times[p^2V(p)\cdot V(-p)-p\cdot V(p)p\cdot V(-p)]\,. \tag{6.7}$$

For the spinor self-energy we get

$$\zeta_{\bar\eta\eta}(s,n)\approx\frac{e^2}{8\pi^2}\int d^np\frac{s}{s+\epsilon}[\alpha+\alpha\epsilon(1-\gamma+\ln 4\pi-\ln p^2)+s(\tfrac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2)]\bar\eta(p)\not p\eta(-p)\,, \tag{6.8a}$$

$$\zeta_{\bar\eta\eta}(s,n)\approx\frac{e^2}{8\pi^2}\int d^np\frac{s}{s+\epsilon}\alpha[1+\epsilon(1-\gamma+\ln 4\pi-\ln p^2)+s(\tfrac{1}{2}-\ln p^2)]\bar\eta(p)\not p\eta(-p) \tag{6.8b}$$

in approaches A and B of Sec. IV, respectively. For the vertex function, with zero momentum transfer to the photon, we find similarly

$$\zeta_{\bar{\eta}V\eta}(s,n) \approx \frac{e^3}{32\pi^4}\int d^pn \frac{s}{s+\epsilon}\left[[\alpha+\alpha\epsilon(1-\gamma+\ln 8\pi^2-\ln p^2)+s(\tfrac{3}{2}+\alpha\ln\alpha-\alpha\ln p^2)]\bar{\eta}(p)V\!\!\!/(0)\eta(-p) - 2\alpha(s+\epsilon)\bar{\eta}(p)p\!\!\!/\eta(-p)\frac{p\cdot V(0)}{p^2}\right], \quad (6.9a)$$

$$\zeta_{\bar{\eta}V\eta}(s,n) \approx \frac{e^3}{32\pi^4}\int d^np \frac{s}{s+\epsilon}\alpha\left[[1+\epsilon(1-\gamma+\ln 8\pi^2-\ln p^2)+s(\tfrac{1}{2}-\ln p^2)]\bar{\eta}(p)V\!\!\!/(0)\eta(-p) - 2(s+\epsilon)\bar{\eta}(p)p\!\!\!/\eta(0)\frac{p\cdot V(0)}{p^2}\right]. \quad (6.9b)$$

We note that in Eqs. (6.7)—(6.9) when we consider (for $\epsilon\neq 0$)

$$\lim_{s\to 0}\frac{d}{ds}\zeta(s,n)$$

we recover precisely the results of the corresponding dimensionally regulated Feynman integrals. Furthermore, we also see that Eq. (6.6) is satisfied. If instead we consider

$$\lim_{s\to 0}\frac{d}{ds}\left[\lim_{n\to 4}\zeta(s,n)\right]$$

then we recover the results of Sec. IV for the spinor self-energy and the vertex function.

Of particular interest are the two and three-point functions in the axial model of Sec. V based on Eq. (5.2). If we first compute the ζ function associated with the two-point function $\langle AA\rangle$ in n dimensions (taking γ_5 to be anticommuting with all γ_μ in n dimensions) our standard procedure gives the result

$$\zeta_{AA}(s,n) = \frac{4g^2}{\Gamma(s)}\int \frac{d^np}{(4\pi)^{n/2}}(p^2n)^{n/2-2-s}\frac{\Gamma^2(n/2-s)\Gamma(s+2-n/2)}{\Gamma(n-2s)} \times\left[p\cdot A(p)p\cdot A(-p)-p^2A(p)\cdot A(-p)\left[\frac{2-n}{2-n+2s}\right]\right]. \quad (6.10)$$

It is easy to see that if $n=4$, Eq. (6.10) reduces to Eq. (5.13) and the Green's function computed from $\zeta'_{AA}(0,4)$ is not transverse. However, if we compute the Green's function $\langle AA\rangle$ in n dimensions by considering $\zeta'_{AA}(0,n)$ we obtain a transverse two-point function identical to the result obtained in the dimensionally regulated Feynman-diagram approach.

The ζ functions associated with the Green's function $\langle VVA\rangle$ and $\langle AAA\rangle$, when one of the external momenta associated with the external axial-vector fields is set equal to zero, are, in $n=4-2\epsilon$ dimensions,

$$\zeta_{AAA}(s,n) \simeq 4g^3\epsilon_{\alpha\beta\gamma\delta}\int\frac{d^np}{(8\pi^2)^{n/2}}[p_\alpha A_\beta(p)A_\gamma(0)A_\delta(-p)(p^2)^{-s-\epsilon}]\left[\frac{\Gamma(1-s-\epsilon)\Gamma(2-s-\epsilon)\Gamma(1+s+\epsilon)}{\Gamma(s)\Gamma(4-2s-2\epsilon)}(-4+8\epsilon)\right], \quad (6.11a)$$

$$\zeta_{VVA}(s,n) \simeq 4e^2g\epsilon_{\alpha\beta\gamma\delta}\int\frac{d^np}{(8\pi^2)^{n/2}}[p_\alpha V_\beta(p)A_\gamma(0)V_\delta(-p)(p^2)^{-s-\epsilon}]\left[\frac{\Gamma(1+s+\epsilon)\Gamma^2(1-s-\epsilon)}{\Gamma(s)\Gamma(2-2s-2\epsilon)}(-2)\right]. \quad (6.11b)$$

For $\langle AAA\rangle$ and $\langle VVA\rangle$, both

$$\lim_{s\to 0}\frac{d}{ds}\left[\lim_{n\to 4}\zeta(s,n)\right] \text{ and } \lim_{n\to 4}\left[\lim_{s\to 0}\frac{d}{ds}\zeta(s,n)\right]$$

are well defined—in fact they are equal to one another. This is due to the fact that $\langle AAA\rangle$ and $\langle VVA\rangle$ are not divergent, in contrast with the Green's functions treated earlier.

In our computations of $\zeta_{AAA}(s,n)$ and $\zeta_{VVA}(s,n)$ we have treated the n-dimensional γ_5 as totally anticommuting with *all* of the γ_μ's and then set

$$T_R(\gamma_\alpha\gamma_\beta\gamma_5)=0\,, \quad (6.12a)$$

$$T_R(\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_5)=4\epsilon_{\alpha\beta\gamma\delta}\,. \quad (6.12b)$$

We have not treated the n-dimensional γ_5 in a special way such as in the Feynman-diagram approach of Refs. 17 and 26 and yet we recover the correct anomaly in the $\langle VVA\rangle$ and $\langle AAA\rangle$ Green's functions. Remarkably, even if γ_5 is defined in n dimensions according to the 't Hooft—Veltman prescription we still recover the results of Eqs. (6.11a) and (6.11b) as the extra contributions vanish identically.

VII. DISCUSSION

In this paper we have given a pedagogical presentation of an alternative way of computing one-loop 1PI Green's functions in quantum field theory. Two features of our approach distinguish it from the usual techniques.

First of all, the perturbative expansion that we have used is based on the Schwinger expansion of Eq. (2.20). This approach does not lead to Feynman diagrams. It is worth recalling the way in which the one-loop 1PI Feynman diagrams are generated in background-field quantization in order to contrast the Feynman perturbation series with what happens when we use Eq. (2.20). In the Feynman approach, $M_{ij}(f)$ in Eq. (2.3) is separated as $M_{ij}^{(0)}+M_{ij}^{(1)}(f)$, where $M_{ij}^{(0)}$ contains all parts of $M_{ij}(f)$ independent of the background field f. Equation (2.5) is then written as

$$Z[f,0]=\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{1}{2}\int dx\, M_{ij}^{(1)}(f)\frac{\delta^2}{\delta J_1\delta J_j}\right]^n\int dh_k \exp\left[\int dx\left(\frac{1}{2}h_iM_{ij}^{(0)}h_j+J_ih_i\right)\right]\right]_{J_i=0} . \tag{7.1}$$

The functional integral over the quantum field h_k can now be evaluated leading to the usual one-loop 1PI Feynman integrals with their associated divergences. In the approach based on the Schwinger expansion we have first integrated over h_k in Eq. (2.5) to obtain Eq. (2.12) and only then made a perturbative expansion in order to obtain one-loop 1PI amplitudes.

The second place where our approach differs from the usual Feynman perturbation series is in the regularization techniques used. The usual approach is to render divergent Feynman integrals finite through the introduction of a regulating parameter into the initial Lagrangian (e.g., n, the number of dimensions of space-time, or m^2, the mass of a Pauli-Villars regulating field). Such a method has two disadvantages. First of all, it is not always possible to introduce the regulating parameter in such a way that all symmetries of the initial theory are respected. Second, when the regulating parameter approaches its physical value divergences appear which must be removed through a renormalization procedure. Neither of these shortcomings are present in our approach. By evaluating the functional integral over quantum fields prior to making our perturbation expansion, we are left with determinants of operators. It is these *operators* that we regulate, as in Eq. (2.14a); we do not have to alter the initial Lagrangian in order to render finite the momentum integrals in the perturbation expansion. Regulating operators, and not the Lagrangian, ensures that symmetries of the initial Lagrangian are undisturbed. It is a remarkable bonus in our approach that after having performed all momentum integrations in the perturbation expansion of Eq. (2.21) we find no divergences as the regulating parameter s approaches its limiting value of zero.

We would like to speculate that both of the advantages of our approach persist when we apply our methods to so-called nonrenormalizable interactions such as quantum gravity. It is worth investigating whether operator regularization provides a means of computing finite, symmetry-preserving quantum corrections to the classical theory of gravity. This is an issue we shall address in a future publication.

Operator regularization can also be applied beyond one-loop order. Although the details will be presented elsewhere[1] we would like to indicate how operator regularization can be applied to two-loop order and beyond.

If, in Eq. (2.2), we include the terms of order h^3 and h^4 in Eq. (2.3), then for the generating functional we have

$$Z[f,j]=\left\{\sum_{n=0}^{\infty}\frac{1}{n!}\left[\int dx\left[\frac{1}{3!}a_{ijk}(f)\frac{\delta^3}{\delta J_i\delta J_j\delta J_k}+\frac{1}{4!}b_{ijkl}\frac{\delta^4}{\delta J_i\delta J_j\delta J_k\delta J_l}\right]\right]^n \right.$$
$$\left.\times\int dh_k\exp\left[\int dx\left[\tfrac{1}{2}h_iM_{ij}(f)h_j+J_ih_i\right]\right]\right\} . \tag{7.2}$$

Beyond one-loop order Eq. (7.2) gives rise to contributions to Z that involve inverses of supermatrices rather than superdeterminants [as in Eq. (2.12)]. It is interesting to note that the inverse of a supermatrix has two representations:[13] namely,

$$\begin{bmatrix}M_{bb} & M_{bf}\\ M_{fb} & M_{ff}\end{bmatrix}^{-1}=\begin{bmatrix}(M_{bb}-M_{bf}M_{ff}{}^{-1}M_{fb})^{-1} & 0\\ -M_{ff}{}^{-1}M_{fb}(M_{bb}-M_{bf}M_{ff}{}^{-1}M_{fb})^{-1} & 1\end{bmatrix}\begin{bmatrix}1 & -M_{bf}M_{ff}{}^{-1}\\ 0 & M_{ff}{}^{-1}\end{bmatrix} \tag{7.3a}$$

and

$$\begin{bmatrix}M_{bb} & M_{bf}\\ M_{fb} & M_{ff}\end{bmatrix}^{-1}=\begin{bmatrix}1 & -M_{bb}{}^{-1}M_{bf}(M_{ff}-M_{fb}M_{bb}{}^{-1}M_{bf})^{-1}\\ 0 & (M_{ff}-M_{fb}M_{bb}{}^{-1}M_{bf})\end{bmatrix}\begin{bmatrix}M_{bb}{}^{-1} & 0\\ -M_{fb}M_{bb}{}^{-1} & 1\end{bmatrix} . \tag{7.3b}$$

It is only in theories with both external bosons and external fermions that these two representations are distinct much as was seen in Sec. IV for the superdeterminant. To illustrate the use of operator regularization to two-loop order we consider the simple $(\phi^3)_6$ model of Sec. II for which both representations of M^{-1} reduce to $M_{bb}{}^{-1}$.

The two loop 1PI Green's functions for this model come from the generating functional

$$Z_2=\frac{\lambda^2}{2!3!}\int d^6x\,d^6y[\langle x\,|\,(p^2+\lambda f)^{-1}\,|\,y\rangle]^3 \tag{7.4}$$

as can be seen from Eq. (7.2). To regulate strings of inverses of operators of the form $A_1{}^{-1}A_2{}^{-1}\cdots A_p{}^{-1}$ at m-loop order, such as occur in Eq. (7.4), we use Eq. (2.14b) so that

$$A_1{}^{-1}A_2{}^{-1}\cdots A_p{}^{-1}=\lim_{s\to 0}\left[\frac{d^m}{ds^m}\left[\frac{s^m}{m!}A_1{}^{-s-1}A_2{}^{-s-1}\cdots A_p{}^{-s-1}\right]\right]. \tag{7.5}$$

Thus, for example, the regulated form of Eq. (7.4) is

$$Z_2=\lim_{s\to 0}\frac{d^2}{ds^2}\left[\frac{s^2}{2!}\frac{\lambda^2}{2!3!}\int d^6x\,d^6y\langle x\,|\,(p^2+\lambda f)^{-s-1}\,|\,y\rangle\langle x\,|\,(p^2+\lambda f)^{-s-1}\,|\,y\rangle\langle x\,|\,(p^2+\lambda f)^{-s-1}\,|\,y\rangle\right]. \tag{7.6}$$

For any operator we can write

$$A^{-s-1}=\frac{1}{\Gamma(s+1)}\int_0^\infty dt\,t^s e^{-At} \tag{7.7}$$

and employ a second perturbative expansion due to Schwinger[9] to write

$$\begin{aligned} e^{-(A_0+A_I)t}&=e^{-A_0t}+(-t)\int_0^1 du\,e^{-(1-u)A_0t}A_I e^{-uA_0t}\\ &\quad+(-t)^2\int_0^1 du\,u\int_0^1 dv\,e^{-(1-u)A_0t}A_I e^{-u(1-v)A_0t}A_I e^{-uvA_0t}+\cdots . \end{aligned} \tag{7.8}$$

Equations (7.5), (7.7), and (7.8) can together be used to compute n-point 1PI Green's functions for any theory to two-loop order and beyond. We evaluate matrix elements of the perturbation expansion of the regulated operators in momentum space, exactly as is explained in Sec. II for one-loop Green's functions.

Operator regularization leads to results that are finite and symmetry preserving. As all the integrals are finite there are no overlapping divergences in our approach. A key ingredient in obtaining finiteness is to have m in Eq. (7.5) equal to the number of (loop) momentum integrals to be evaluated; this ensures that no poles occur in the limit $s\to 0$ performing all loop momentum integrals. For completeness, we give the expression for the two-point, two-loop 1PI amplitude arising from Eq. (7.6). It is

$$\begin{aligned} Z_{2ff}=\lim_{s\to 0}\frac{d^2}{ds^2}\Bigg\{&\frac{s^2}{8}\frac{\lambda^4}{(2\pi)^{12}}\int d^6p\,f(p)f(-p)\\ &\times\int d^6k\,d^6q\left[\frac{\Gamma(s+2)}{\Gamma(s+1)}\int_0^1 du\,u\int_0^1 dv\left[\frac{1}{q^2}\frac{1}{(q+k+p)^2}\right]^{s+1}\frac{1}{[(1-u)k^2+u(k+p)^2]^{s+3}}\right.\\ &\quad+\frac{\Gamma^2(s+2)}{\Gamma^2(s+1)}\int_0^1 du_1du_2\left[\frac{1}{(q+k+p)^2}\right]^{s+1}\frac{1}{[(1-u_1)k^2+u_1(k+p)^2]^{s+2}}\\ &\quad\left.\times\frac{1}{[(1-u_2)q^2+u_2(q+p)^2]^{s+2}}\right]\Bigg\}. \end{aligned} \tag{7.9}$$

No explicit divergences occur in Eq. (7.9), either in the integrals over k and q, or in the limit $s\to 0$. We postulate that finiteness is a feature of operator regularization at two-loop order and beyond, even for nonrenormalizable theories such as gravity.

Finding a suitable regularization scheme for supersymmetric theories has proved to be a particularly troublesome problem. For example, dimensional regularization adequately preserves any gauge symmetry that may be present but, since supersymmetry is intrinsically linked to four dimensions, it has been necessary to adopt a variant of dimensional regularization, known as dimensional reduction,[27] in order to handle supersymmetric theories; even this proposed scheme may fail at higher-loop order.[28] Operator regularization does not appear to have such shortcomings. Indeed, we have shown[1] how our approach can be used to handle the Wess-Zumino model written in terms of superfields. We postulate that operator regularization is a general method for computing 1PI Green's functions to arbitrary order, without destroying gauge invariance or supersymmetry, in such a way that explicit divergences are never encountered.

We would now like to mention some problems that we feel should be addressed within the context of operator regularization.

First of all, we have already indicated that quantum gravity may be rendered finite in our approach; this conjecture must be tested by explicit computations. Similarly, we must see if operator regularization is a suitable regulating technique for supersymmetric theories.

Another problem we plan to address is how to deal with noncovariant gauge in non-Abelian gauge theories. In Sec. III and IV we have indicated problems that arise when we try to use operator regularization in conjunction with such gauges as the Coulomb gauge, the axial gauge, the planar gauge, or the light-cone gauge. The light-cone gauge in the conventional Feynman perturbative approach leads to nonlocal counterterms;[29] it would be interesting to see what this means in the context of operator regularization.

Operator regularization may provide some additional insight on how anomalies arise in quantum field theory. In particular, the anomalous four- and five-point functions in theories with non-Abelian axial-vector currents might be investigated using our technique.

We would also like to apply operator regularization to see how the anomaly in the divergence of the superfield supercurrent[30] arises. Discussions on this subject have to data been obscured by the fact that no suitable regularization procedure for supersymmetry has been devised.[31] This particular problem may be overcome by operator regularization.

It is also important that we apply our method to problems of direct physical interest, such as computing the electromagnetic moment of the muon in the standard Weinberg-Salam model. The finite results that we obtain differ by (at most) a finite renormalization from the results obtained using, say, the minimal subtraction (MS), modified minimal subtractions ($\overline{\text{MS}}$), or momentum-space (MOM) subtraction scheme in conventional perturbation theory.[32] It would be of interest to see explicitly the differences between results obtained in these different approaches.

Finally, we note that the arbitrary dimensionful parameter μ^2 that arises in the Feynman perturbation series also arises in our method. We simply have to rescale the parameter t in Eq. (2.21) so that $\tau=\mu^2 t$ is dimensionless.[33] This approach to the renormalization group in the context of operator regularization will be dealt with elsewhere.[1]

In conclusion, we hope that we have provided an easily applicable calculational procedure for evaluating Green's functions in quantum field theory that can be used in a wide variety of problems.

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APPENDIX A

When both external fermion fields and external boson fields occur in a given model the one-loop 1PI generating functional is a superdeterminant.[13–15,20] In the formalism of Sec. II we wrote the Lagrangian in terms of a supermatrix to show how this superdeterminant arises

$$\mathscr{L}=\tfrac{1}{2}(b^T M_{bb}b+f^T M_{fb}b+b^T M_{bf}f+f^T M_{ff}f) \tag{A1}$$

$$=\tfrac{1}{2}(b^T,f^T)\begin{bmatrix} M_{bb} & M_{bf} \\ M_{fb} & M_{ff}\end{bmatrix}\begin{bmatrix} b \\ f\end{bmatrix} \tag{A2}$$

$$\equiv\tfrac{1}{2}h_i M_{ij} h_j \ . \tag{A3}$$

As explained in detail in Sec. II, by completing the square in the associated Gaussian path integral in either (a) the fermionic variable f or (b) the bosonic variable b we are led to two representations of the superdeterminant

$$\text{sdet}M=\det(M_{bb}-M_{bf}M_{ff}{}^{-1}M_{fb})\det{}^{-1}M_{ff} \ , \tag{A4a}$$

$$\text{sdet}M=\det M_{bb}\det{}^{-1}(M_{ff}-M_{fb}M_{bb}{}^{-1}M_{bf}) \ . \tag{A4b}$$

However, it is also possible to derive the one-loop 1PI generating functional without writing $\mathscr{L}$ in terms of a supermatrix. Indeed, as we saw in Sec. IV, when Dirac spinors are used the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}b^T M_{bb}b+\bar{F}M_{\bar{F}b}b+b^T M_{bF}F+\bar{F}M_{\bar{F}F}F \ , \tag{A5}$$

where $\bar{F}$ and F are independent quantum Fermi fields, is not of the form (A1). To see how the same result for the one-loop generating functional can be obtained directly from this form of the Lagrangian we consider the general integral

$$J=\int d\bar{F}\,dF\,db\,\exp(\tfrac{1}{2}b^T M_{bb}b+\bar{F}M_{\bar{F}b}b+b^T M_{bF}F+\bar{F}M_{\bar{F}F}F) \ . \tag{A6}$$

To rewrite the integrals in (A6) in the form of the standard Gaussian integrals

$$\int db\,\exp(\tfrac{1}{2}b^T A b)=\det{}^{-1/2}A \ , \tag{A7a}$$

$$\int d\bar{F}\,dF\,\exp(\bar{F}BF)=\det B \ , \tag{A7b}$$

$$\int dF\,\exp(\tfrac{1}{2}F^T C F)=\det{}^{1/2}C \ . \tag{A7c}$$

We can proceed in two ways: either (a) to complete the "square" in the fermionic variables $\bar{F}$ and F or (b) to complete the square in the bosonic variable b.

Consider (a) first. We find

$$J=\int d\bar{F}\,dF\,dB\,\exp[\tfrac{1}{2}b^T M_{bb}b-b^T M_{bF}M_{FF}{}^{-1}M_{Fb}b+(\bar{F}+bM_{bF}M_{FF}{}^{-1})\times M_{\bar{F}F}(F+M_{\bar{F}F}{}^{-1}M_{\bar{F}b}b)] \ . \tag{A8}$$

The change of variables

$$F'=F+M_{\bar{F}F}{}^{-1}M_{\bar{F}b}b, \quad \bar{F}'=\bar{F}+bM_{bF}M_{\bar{F}F}{}^{-1} \tag{A9}$$

gives

$$J=\int d\bar{F}'dF'db\,\exp[\tfrac{1}{2}b^T(M_{bb}-2M_{bF}M_{\bar{F}F}{}^{-1}M_{Fb})b+\bar{F}'M_{\bar{F}F}F']$$
$$=\frac{\det M_{\bar{F}F}}{\det^{1/2}(M_{bb}-2M_{bF}M_{\bar{F}F}{}^{-1}M_{Fb})}\ . \qquad (A10)$$

To demonstrate the equivalence of the result (A10) and the generating functional obtained using (A4a) we supply the link between the two Lagrangians (A1) and (A5). This is

$$f=\begin{bmatrix}F\\ \bar{F}^T\end{bmatrix},\quad f^T=[F^T,\bar{F}]\ , \qquad (A11a)$$

$$M_{ff}=\begin{bmatrix}0 & -(M_{\bar{F}F})^T\\ M_{\bar{F}F} & 0\end{bmatrix}\ , \qquad (A11b)$$

$$M_{fb}=\begin{bmatrix}-(M_{bF})^T\\ M_{\bar{F}b}\end{bmatrix},\quad M_{bf}=[M_{bF},-(M_{\bar{F}b})^T]\ . \qquad (A11c)$$

It is now straightforward to show that

$$\det{}^{1/2}M_{ff}=\det M_{\bar{F}F}\ , \qquad (A12a)$$

$$\det(M_{bb}-M_{bf}M_{ff}{}^{-1}M_{fb})=\det(M_{bb}-2M_{bF}M_{\bar{F}F}{}^{-1}M_{Fb})\ , \qquad (A12b)$$

and, consequently, the equivalence of the two forms of the generating functional has been established. It was this latter approach that was used by Lee and Rim in Ref. 15 to derive the one-loop generating functional.

Let us now proceed to complete the square in the variable b in the integral (A6). We find

$$J=\int d\bar{F}\,dF\,db\,\exp(\tfrac{1}{2}\{b+M_{bb}{}^{-1}[M_{bF}F+(\bar{F}M_{\bar{F}b})^T]\}^TM_{bb}\{b+M_{bb}{}^{-1}[M_{bF}F+(\bar{F}M_{\bar{F}b})^T]\}+\bar{F}M_{\bar{F}F}F-\tfrac{1}{2}[\bar{F}M_{\bar{F}b}+(M_{bF}F)^T]M_{bb}{}^{-1}[M_{bF}F+(\bar{F}M_{\bar{F}b})^T])\ . \qquad (A13)$$

The change of variable

$$b'=b+M_{bb}{}^{-1}[M_{bF}F+(\bar{F}M_{\bar{F}b})^T] \qquad (A14)$$

gives the somewhat simpler integral

$$J=\int d\bar{F}\,dF\,db'\times\exp[\tfrac{1}{2}b'^TM_{bb}b'+\bar{F}(M_{\bar{F}F}-M_{\bar{F}b}M_{bb}{}^{-1}M_{bF})F+\tfrac{1}{2}\bar{F}M_{\bar{F}b}M_{bb}{}^{-1}M_{\bar{F}b}{}^T\bar{F}^T+\tfrac{1}{2}F^TM_{bF}{}^TM_{bb}{}^{-1}M_{bF}F]\ . \qquad (A15)$$

However, J is not yet in a form in which the standard integrals (A7) can be used due to the terms in the exponent of (A15) quadratic in $\bar{F}$ and F. To proceed we define

$$\Delta\equiv M_{\bar{F}F}-M_{\bar{F}b}M_{bb}{}^{-1}M_{bF}\ , \qquad (A16a)$$

$$\Omega\equiv M_{\bar{F}b}M_{bb}{}^{-1}M_{\bar{F}b}{}^T,\quad \Omega^T=-\Omega\ , \qquad (A16b)$$

$$\theta\equiv M_{bF}{}^TM_{bb}{}^{-1}M_{bF},\quad \theta^T=-\theta \qquad (A16c)$$

and complete the square in $\bar{F}$ to find

$$J=\int d\bar{F}\,dF\,db'\times\exp[\tfrac{1}{2}b'^TM_{bb}b'+\tfrac{1}{2}(\bar{F}-F^T\Delta^T\Omega^{-1})\Omega(\bar{F}-F^T\Delta^T\Omega^{-1})^T+\tfrac{1}{2}F^T(\theta+\Delta^T\Omega^{-1}\Delta)F]\ . \qquad (A17)$$

The change of variables

$$\bar{F}'=\bar{F}-F^T\Delta^T\Omega^{-1} \qquad (A18)$$

allows us to write

$$J=\int d\bar{F}\,dF\,db'\exp[\tfrac{1}{2}b'^TM_{bb}b'+\tfrac{1}{2}\bar{F}'\Omega\bar{F}'^T+\tfrac{1}{2}F^T(\theta+\Delta^T\Omega^{-1}\Delta)F]\ . \qquad (A19)$$

Using the standard integrals (A7a) and (A7c) we find

$$J=\frac{\det^{1/2}\Omega\,\det^{1/2}(\theta+\Delta^T\Omega^{-1}\Delta)}{\det^{1/2}M_{bb}}=\frac{\det^{1/2}\Delta\,\det^{1/2}(\Delta+\Omega\Delta^{T^{-1}}\theta)}{\det^{1/2}M_{bb}}\ . \qquad (A20)$$

To demonstrate the equivalence of the forms of the one-loop functional obtained from Eqs. (A4b) and (A20) we use Eqs. (A11) in (A4b). We find

$$M_{ff}-M_{fb}M_{bb}{}^{-1}M_{bf}=\begin{bmatrix}\theta & -\Delta^T\\ \Delta & \Omega\end{bmatrix}\ . \qquad (A21)$$

The determinant of this antisymmetric matrix can be written in terms of θ, Ω, and Δ, using Eq. (4.17), as

$$\det(M_{ff}-M_{fb}M_{bb}{}^{-1}M_{bf})=\det(-\Delta^T)\det(-\Delta^T-\theta\Delta^{-1}\Omega)=\det\Delta\,\det(\Delta+\Omega\Delta^{T^{-1}}\theta)\ . \qquad (A22)$$

This result clearly demonstrates the equality of the two forms of the generating functional.

APPENDIX B

In this appendix we write explicitly those terms omitted in Eq. (4.18). With P, Q, and R defined in Eq. (4.16) we find that

$$R+PR^{-1^T}Q=(\not p-e\not V-e^2\gamma_\mu\eta\Pi_{\mu\nu}^{\alpha^{-1}}\bar\eta\gamma_\nu)$$
$$-e^4\gamma_\rho\eta\Pi_{\rho\lambda}^{\alpha^{-1}}\bar\eta\gamma_\lambda(\not p-e\not V-e^2\gamma_\mu\eta\Pi_{\mu\nu}^{\alpha^{-1}}\bar\eta\gamma_\nu)^{-1}\gamma_\tau\eta\Pi_{\tau\sigma}^{\alpha^{-1}}\bar\eta\gamma_\sigma\,. \qquad (B1)$$

Upon expanding

$$(\not p-e\not V-e^2\gamma_\mu\eta\Pi_{\mu\nu}^{\alpha^{-1}}\bar\eta\gamma_\nu)^{-1}=(\not p)^{-1}+(\not p)^{-1}(-e\not V-e^2\gamma_\mu\eta\Pi_{\mu\nu}^{\alpha^{-1}}\bar\eta\gamma_\nu)(\not p)^{-1}$$
$$+(\not p)^{-1}(-e\not V-e^2\gamma_\mu\eta\Pi_{\mu\nu}^{\alpha^{-1}}\bar\eta\gamma_\nu)(\not p)^{-1}(-e\not V-e^2\gamma_\lambda\eta\Pi_{\lambda\rho}^{\alpha^{-1}}\bar\eta\gamma_\rho)(\not p)^{-1}+\cdots\,. \qquad (B2)$$

We find that, by Eq. (4.17),

$$Z_1=\det{}^{1/2}[R(R+R^{-1^T}Q)]$$
$$=\det{}^{1/2}\Bigg\{p^2-e\{\not p,\not V\}+e^2V^2-e^2\{\not p,\gamma_\mu\eta\Pi_{\mu\nu}^{\alpha^{-1}}\bar\eta\gamma_\nu\}+e^3\{\not V,\gamma_\mu\eta\Pi_{\mu\nu}^{\alpha^{-1}}\bar\eta\gamma_\nu\}+e^4\gamma_\mu\eta\Pi_{\mu\nu}^{\alpha^{-1}}\bar\eta\gamma_\nu\gamma_\lambda\eta\Pi_{\lambda\sigma}^{\alpha}\bar\eta\gamma_\sigma$$
$$-e^4\left[\not p-e\not V-e^2\gamma_\mu\eta\Pi_{\mu\nu}^{\alpha^{-1}}\bar\eta\gamma_\nu,\gamma_\rho\eta\Pi_{\rho\lambda}^{\alpha^{-1}}\bar\eta\gamma_\lambda\left[\frac{1}{\not p}+\frac{1}{\not p}(-e\not V-e^2\gamma_\beta\eta\Pi_{\beta\delta}^{\alpha^{-1}}\bar\eta\gamma_\delta)\frac{1}{\not p}+\cdots\right]\gamma_\sigma\eta\Pi_{\sigma\tau}^{\alpha^{-1}}\bar\eta\gamma_\tau\right]$$
$$+e^8\left[\gamma_\rho\eta\Pi_{\rho\lambda}^{\alpha^{-1}}\bar\eta\gamma_\lambda\left[\frac{1}{\not p}+\frac{1}{\not p}(-e\not V-e^2\gamma_\beta\eta\Pi_{\beta\delta}^{\alpha^{-1}}\bar\eta\gamma_\delta)\frac{1}{\not p}+\cdots\right]\gamma_\sigma\eta\Pi_{\sigma\tau}^{\alpha^{-1}}\bar\eta\gamma_\tau\right]\Bigg\}\,. \qquad (B3)$$

It is evident that Eq. (B3) is not as convenient as Eq. (4.8) for computing one-loop 1PI Green's functions with more than three external lines. However, if one wants to employ a noncovariant gauge in QED it seems necessary to use approach B and, hence, Eq. (B3) as the Schwinger expansion cannot easily be applied in approach A when using a noncovariant gauge.

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