Relativistic dynamics and Lorentz contraction

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Regarding the structure of a bound system of two particles in relativistic quantum mechanics we raise the following question: Does the shape of (i.e., the density distribution in) a bound system in motion always conform to Lorentz contraction? A negative possibility is illustrated by means of the Bakamjian-Thomas model which is relativistically covariant in the sense that the generators of the Poincaré algebra for the model can be constructed explicitly. The degree of deviation from Lorentz contraction will depend on the choice of the interaction. We examine another model of two interacting particles, which is based on field theory. The (approximate) results indicate that the deviation from Lorentz contraction in this model is, if not zero, very small.

I. INTRODUCTION

Consider an object which is spherical in shape when it is at rest. If the object is set in motion with a constant speed v in the z direction, one expects on the basis of Lorentz contraction that the shape of the object will become ellipsoidal; the radius in the z direction is shortened by the ratio of $1/\gamma=[1-(v/c)^2]^{1/2}$ while the radii in the x and y directions remain unchanged. This Lorentz contraction is based on pure kinematics, with no reference to the dynamical structure of the object. If one thinks of this problem as one of quantum dynamics, however, Lorentz contraction is not a very simple phenomenon. Let us consider a bound system of two spinless particles. Then the wave function squared determines the density distribution in the system. Lorentz contraction will imply that, when the system is boosted, the density and hence the wave function scale in a certain simple manner. On the other hand, the wave function is determined by the interaction between the two particles. How does the interaction conspire so that the resulting wave function exhibits such a scaling property? Does any interaction lead to exact Lorentz contraction provided that the system/model is relativistically covariant?

In order to answer the questions raised above, we examine a model of a two-body bound system in the Bakamjian-Thomas (BT) scheme.[1,2] This model belongs to what Dirac called the "instant form" of the formulation of the relativistic two-body problem.[3] The two particles are at equal times, and are interacting directly. By a direct interaction we mean that the interaction term in the Hamiltonian is expressed explicitly in terms of the dynamical variables of the particles (their positions and momenta). In the BT scheme one can set up an interaction in such a way that the ten generators of the Poincaré algebra can be constructed explicitly. These generators are the Hamiltonian H, the total momentum $\mathbf{P}$, the total angular momentum $\mathbf{J}$, and the Lorentz-boost operator $\mathbf{K}$. The model is thus relativistically covariant. Within the scheme there remains wide latitude for the choice of interaction. In fact one can start by assuming a wave function in a certain form and then determine an interaction which reproduces the assumed wave function. In this way the BT scheme allows one to construct a solvable, relativistically covariant model of interacting particles. Using such a model we examine how the wave function and the density distribution of a bound system depend on the velocity of the system. We find that the density distribution does not exactly conform to Lorentz contraction.

How well Lorentz contraction is observed will depend on the choice of the interaction. Despite its ingenious mechanism, we feel that the type of interaction introduced in the BT scheme is rather artificial. As a more natural and realistic alternative we consider a model of a two-body system based on field theory. Starting with interacting meson and nucleon fields, one can eliminate the meson field by means of Okubo's projection technique.[4] This can be done in successive orders in the meson-nucleon coupling constant g. We examine only the lowest order, i.e., g^2, in this paper. Up to the same order in g, one can construct all generators of the Poincaré algebra.[2,5] The model in this approximation does not exactly obey the relativistic energy-momentum relation and Lorentz contraction. However, we find the deviation unexpectedly small. This is unexpected because the model is supposed to be covariant only up to the order of g^2. Implications of this finding will be discussed.

II. BAKAMJIAN-THOMAS MODEL

Let us consider a system of two spinless particles of equal mass m. We work in momentum space until we examine the wave function in coordinate space. In the absence of interaction the energy of the system is given by

$$E_0=\sum_i E_i \ , \tag{2.1}$$

where i $(=1,2)$ refers to the particles, and

$$E_i=(m^2+p_i^2)^{1/2} \ , \tag{2.2}$$

where $\mathbf{p}_i$ is the momentum of particle i, and $p_i^2=|\mathbf{p}_i|^2$. We use units such that $c=\hbar=1$. We define the free mass operator by

$$M_0=(E_0^2-P^2)^{1/2} \ , \tag{2.3}$$

where

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2 \tag{2.4}$$

is the total momentum. For later convenience we also define the relative momentum by

$$\mathbf{p}=\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2) \ . \tag{2.5}$$

We now introduce energies ω_i and momenta $\mathbf{k}_i$, which are related to E_i and $\mathbf{p}_i$ via the Lorentz transformation associated with the velocity $\mathbf{P}/E_0$:

$$\omega_i=\frac{1}{M_0}(E_0E_i-\mathbf{p}_i\cdot\mathbf{P}) \ , \tag{2.6}$$

$$\mathbf{k}_i=\mathbf{p}_i-\frac{\mathbf{P}}{M_0}\left[E_i-\frac{\mathbf{p}_i\cdot\mathbf{P}}{M_0+E_0}\right] \ . \tag{2.7}$$

As expected, $\mathbf{k}_1+\mathbf{k}_2=0$ and $\omega_i=(m^2+k_i^2)^{1/2}$. We introduce $\mathbf{k}$ and ω such that $\mathbf{k}_1=-\mathbf{k}_2=\mathbf{k}$ and $\omega_1=\omega_2=\omega$. It is easy to confirm that

$$M_0=\sum_i \omega_i=2\omega \ . \tag{2.8}$$

The transformation inverse to Eqs. (2.6) and (2.7) is given by

$$E_i=\frac{1}{M_0}(E_0\omega_i+\mathbf{k}_i\cdot\mathbf{P}) \ , \tag{2.9}$$

$$\mathbf{p}_i=\mathbf{k}_i+\frac{\mathbf{P}}{M_0}\left[\omega_i+\frac{\mathbf{k}_i\cdot\mathbf{P}}{M_0+E_0}\right] \ . \tag{2.10}$$

Now the idea of Bakamjian and Thomas is to introduce an interaction through the mass operator

$$M=M_0+v \ , \tag{2.11}$$

where v depends only on the momentum $\mathbf{k}$ (Ref. 6). In the center-of-mass (c.m.) frame in which $\mathbf{P}=0$, the M of Eq. (2.11) is the Hamiltonian. In a general frame the Hamiltonian is given by

$$H=(M^2+P^2)^{1/2} \ . \tag{2.12}$$

Of course $[H,\mathbf{P}]=0$. One can go on to construct the total angular momentum $\mathbf{J}$ and the Lorentz-boost operator $\mathbf{K}$, which, together with H and $\mathbf{P}$, form the ten generators of the Poincaré algebra; for this we refer the reader to Refs. 1 and 2.

We are now in a position to examine the structure of a bound system. As mentioned above, M is the Hamiltonian of the two-body system in the c.m. frame. The bound-state wave function ψ in the c.m. frame is determined by the Schrödinger equation

$$M\psi=m_b\psi \ , \tag{2.13}$$

where the eigenvalue m_b is the rest mass of the bound system. Recalling Eqs. (2.8) and (2.11), one can write Eq. (2.13) more explicitly as

$$\begin{aligned}\int d\mathbf{k}'M(\mathbf{k},\mathbf{k}')\psi(\mathbf{k}')&=2(m^2+k^2)^{1/2}\psi(\mathbf{k})\\&\quad+\int d\mathbf{k}'v(\mathbf{k},\mathbf{k}')\psi(\mathbf{k}')\\&=m_b\psi(\mathbf{k}) \ .\end{aligned} \tag{2.14}$$

When $\mathbf{P}\neq 0$, Eq. (2.14) is replaced by

$$H\psi=(M^2+P^2)^{1/2}\psi=(m_b^2+P^2)^{1/2}\psi \ . \tag{2.15}$$

Since $\mathbf{P}$ commutes with H, $\mathbf{P}$ is a constant of the motion. Clearly the relativistic energy-momentum relation is satisfied. The eigenfunction ψ for Eq. (2.15) is the same as that for Eq. (2.14). Therefore ψ depends on $\mathbf{P}$ only through $\mathbf{k}$. In order to see the $\mathbf{P}$ dependence of ψ, one has to express $\mathbf{k}$ in terms of $\mathbf{p}$ and $\mathbf{P}$. Let us denote the wave function so obtained by $\phi(\mathbf{p},\mathbf{P})$. The probability must not be affected by the variable change from $\mathbf{k},\mathbf{P}$ to $\mathbf{p},\mathbf{P}$; hence

$$\int d\mathbf{k}\,|\psi(\mathbf{k})|^2=\int d\mathbf{p}\,|\phi(\mathbf{p},\mathbf{P})|^2 \ , \tag{2.16}$$

where $\mathbf{P}$ is a fixed constant vector. This is satisfied if ψ and ϕ are related by

$$\psi(\mathbf{k})=J^{1/2}\phi(\mathbf{p},\mathbf{P}) \ , \tag{2.17}$$

where J is the Jacobian for the variable transformation:

$$J=\frac{\partial(\mathbf{p},\mathbf{P})}{\partial(\mathbf{k},\mathbf{P})}=\frac{2E_1E_2}{\omega E_0} \ . \tag{2.18}$$

Equation (2.17) is a crucial formula in our model analysis.[7]

If the interaction is of the form of $v(|\mathbf{k}-\mathbf{k}'|)$, then it corresponds to a local central interaction in coordinate space, and Eq. (2.14) can be separated into partial waves. Since we are interested in the ground state, we take the s state; $\psi(\mathbf{k})$ then becomes spherically symmetric with respect to $\mathbf{k}$. Alternatively, one can assume that $v(\mathbf{k},\mathbf{k}')$ is separable, i.e.,

$$\int d\mathbf{k}'v(\mathbf{k},\mathbf{k}')\psi(\mathbf{k}')=v(k)\int d\mathbf{k}'v(k')\psi(\mathbf{k}') \ , \tag{2.19}$$

which represents a nonlocal separable interaction. This interaction acts only in the s state; again $\psi(\mathbf{k})=\psi(k)$ is spherical.

In the model analysis that follows, rather than starting with a given interaction, we assume an *ad hoc* $\psi(k)$ [Eq. (3.6)] and then determine $\phi(\mathbf{p},\mathbf{P})$ by Eq. (2.17). Once $\phi(\mathbf{p},\mathbf{P})$ is specified the structure of the bound state is completely determined. In case one wonders about the legitimacy of assuming $\psi(k)$ arbitrarily, let us point out that, when $\psi(k)$ and m_b are given, one can easily determine $v(\mathbf{k},\mathbf{k}')$ in the form of Eq. (2.19). If one chooses $v(k)$ such that

$$v^2(k)=\frac{(2\omega-m_b)^2\psi^2(k)}{\int d\mathbf{k}(2\omega-m_b)\psi^2(k)} \ , \tag{2.20}$$

then Eq. (2.14) is satisfied by the ψ and m_b that are in Eq. (2.20).

III. STRUCTURE OF THE BOUND SYSTEM

Let us denote the relative coordinate for the two particles by $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, which is conjugate to the relative momentum $\mathbf{p}$. We choose the z axis along the total momentum $\mathbf{P}$, and calculate $\langle x^2\rangle=\langle y^2\rangle$ and $\langle z^2\rangle$. We have done this calculation in two ways: methods I and II.

Method I starts with

$$\langle x^2\rangle = \int d\mathbf{p}\left|\frac{\partial\phi(\mathbf{p},\mathbf{P})}{\partial p_x}\right|^2$$
$$= \int d\mathbf{k}\,J\left|\frac{\partial}{\partial p_x}[J^{-1/2}\psi(k)]\right|^2 , \tag{3.1}$$

where J is given by Eq. (2.17). Equation (3.1) can be reduced to

$$\langle x^2\rangle = \int d\mathbf{k}\frac{p_x^2}{4}\left\{\frac{[E_1E_2(E_0^2+3M_0^2)-M_0^2E_0^2]\psi}{(M_0E_1E_2)^2} + \frac{E_0^2}{E_1E_2}\frac{\partial\psi}{\partial k^2}\right\}^2 . \tag{3.2}$$

Similarly one obtains

$$\langle z^2\rangle = \int d\mathbf{k}\frac{p_z^2}{4}\left\{\frac{[E_1E_2(3E_0^2+M_0^2)-M_0^2E_0^2]\psi}{(E_0E_1E_2)^2} + \frac{M_0^2}{E_1E_2}\frac{\partial\psi}{\partial k^2}\right\}^2 . \tag{3.3}$$

Equations (3.2) and (3.3) are valid for any value of P. If $P/m \ll 1$, one can expand the above into P, to obtain

$$\langle x^2\rangle_P = \langle x^2\rangle_0 + P^2\frac{2\pi}{3}\int_0^\infty dk\frac{k^4}{\omega^4}\frac{\partial\psi}{\partial k^2} \times\left[\psi+\frac{4k^2}{5}\left[\frac{\partial\psi}{\partial k^2}-\frac{\psi}{\omega^2}\right]\right] , \tag{3.4}$$

where the suffixes P and 0 refer to the total momentum, and

$$\langle z^2\rangle_P = \langle z^2\rangle_0 + P\frac{2\pi}{3}\int_0^\infty dk\frac{k^4}{\omega^4}\frac{\partial\psi}{\partial k^2}\left[3\left[1-\frac{4k^2}{5\omega^2}\right]\psi - 2\left[\omega^2-\frac{6k^2}{5}\right]\frac{\partial\psi}{\partial k^2}\right] . \tag{3.5}$$

As an example, let us assume

$$\psi(k) = N\omega^{-4} = N(m^2+k^2)^{-2} , \tag{3.6}$$

where $N=(8m^5)^{1/2}/\pi$ is the normalization factor. Substituting this ψ into Eqs. (3.4) and (3.5), we find

$$\langle x^2\rangle_P = \langle x^2\rangle_0\left[1-\frac{1}{192}\frac{P^2}{m^2}\right] , \tag{3.7}$$

$$\langle z^2\rangle_P = \langle z^2\rangle_0\left[1-\frac{31}{192}\frac{P^2}{m^2}\right] , \tag{3.8}$$

where $\langle x^2\rangle_0=\langle z^2\rangle_0=m^{-2}$.

On the basis of Lorentz contraction one expects that $\langle x^2\rangle_P=\langle x^2\rangle_0$ and $\langle z^2\rangle_P=\gamma^{-2}\langle z^2\rangle_0$, where $\gamma^{-2}=1-v^2$. Clearly Eqs. (3.7) and (3.8) do not conform to this expectation; $\langle x^2\rangle_P$ has been reduced. The contraction ratio for $\langle z^2\rangle_P$ given by Eq. (3.8) does not agree with γ^{-2}. The discrepancy is actually of a conceptual nature in the following sense. Note that $v=P/(m_b^2+P^2)^{1/2}$ or $P=\gamma m_b v$. Therefore if one wants to express the contraction factor given by Eq. (3.8) in terms of v, one has to know m_b. However, as discussed at the end of Sec. II, m_b and $\psi(k)$ can be chosen completely independently. In deriving Eqs. (3.7) and (3.8), it is sufficient to assume $\psi(k)$, and m_b does not enter at all. Unless m_b is specified, v is undetermined.

In method II we expand $\phi(\mathbf{p},\mathbf{P})$ into partial waves:

$$\phi(\mathbf{p},\mathbf{P}) = \sum_l (l+\tfrac{1}{2})P_l(s)\phi_l(p,P) , \tag{3.9}$$

where $s=\mathbf{p}\cdot\mathbf{P}/pP$, P_l with $l=0.2,\ldots$ being the Legendre polynomials, and

$$\phi_l(p,P) = \int_{-1}^{1} ds\, P_l(s)\phi(\mathbf{p},\mathbf{P}) . \tag{3.10}$$

The wave function in coordinate space is given by

$$\phi(\mathbf{r},\mathbf{P}) = (2\pi)^{-3/2}\int d\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{r}}\phi(\mathbf{p},\mathbf{P})$$
$$= \sum_l (l+\tfrac{1}{2})P_l(t)\phi_l(r,P) , \tag{3.11}$$

where $t=\mathbf{r}\cdot\mathbf{P}/rP$, and

$$\phi_l(r,P) = (2/\pi)^{1/2}i^l\int_0^\infty dp\, p^2 j_l(pr)\phi_l(p,P) . \tag{3.12}$$

For $P=0$, $\psi(k)=\phi_0(p,P)$. As P increases, components with $l=2,4,\ldots$ become important.

The mean-square radii can be calculated as

$$\langle z^2\rangle = \sum_{l,l'}(l+\tfrac{1}{2})(l'+\tfrac{1}{2})\int_0^\infty dr\, r^4\phi_l(r,P)\phi_{l'}(r,P) \times 2\pi\int_{-1}^{1} dt\, P_l(t)P_{l'}(t)t^2 . \tag{3.13}$$

Carrying out the t integration, we obtain

$$\langle z^2\rangle = \frac{\pi}{2}\int_0^\infty dr\, r^4\left(\tfrac{2}{3}\phi_0{}^2+\tfrac{8}{3}\phi_0\phi_2+\tfrac{110}{21}\phi_2{}^2 + \tfrac{144}{2}\phi_2\phi_4+\tfrac{702}{77}\phi_4{}^2+\cdots\right), \tag{3.14}$$

where $\phi_l=\phi_l(r,P)$. The $\langle x^2\rangle$ is obtained by

$$\langle x^2\rangle = \tfrac{1}{2}\langle r^2-z^2\rangle , \tag{3.15}$$

combined with

$$\langle r^2\rangle = \pi\int_0^\infty dr\, r^4(\phi_0{}^2+5\phi_2{}^2+9\phi_4{}^2+\cdots) . \tag{3.16}$$

For the example of Eq. (3.6) we calculated $\langle x^2\rangle_P$ and $\langle z^2\rangle_P$ by method II, and confirmed Eqs. (3.4)—(3.8). In all cases that we considered it was sufficient to keep $l=0$, 2, and 4.

IV. MODEL BASED ON FIELD THEORY

In the preceding section we examined a solvable, relativistically covariant model of two-body system in the BT scheme. We calculated $\langle x^2\rangle_P$ and $\langle z^2\rangle_P$, and found that their P dependence does not conform to the Lorentz contraction. In the BT scheme, the interaction v can be chosen arbitrarily as long as it is a function of the variable $\mathbf{k}$. We feel that, as a realistic model for, say, the nucleon-nucleon interaction, the BT model is too arbitrary and artificial. As a model which we think is more natural and realistic, let us examine one based on interacting meson and nucleon fields.[5]

We start with a local relativistic field theory of interacting fields of mesons and nucleons, for simplicity, both considered to be spinless. They interact through the usual Yukawa coupling with coupling constant g. Let η be the projection operator into the space of two nucleons. Then $\Lambda=1-\eta$ is the projection for the rest of the space. Following Okubo[4] one can decouple the η and Λ subspaces with respect to the Hamiltonian H. In other words, one can find a unitary transformation U such that $UHU^\dagger$ has no matrix element connecting the η and Λ subspaces. In practice one determines U in the form of a series in powers of g. When the η-Λ coupling is eliminated from H up to order g^2, one obtains an effective nucleon-nucleon potential to the same order.

Since we start with a relativistic field theory, we can easily write down the generators of the Poincaré algebra in terms of the field variables. It was shown in Ref. 5 that when the η-Λ coupling is eliminated up to order g^2 in $UHU^\dagger$, the same transformation U also eliminates the η-Λ coupling in all of the generators of the Poincaré algebra. In this way one obtains a model of the nucleon-nucleon interaction which is covariant to order g^2. The nucleon-nucleon interaction Hamiltonian is given by

$$H_{\rm int} = -\frac{g^2}{2(2\pi)^3}\int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_1' d\mathbf{p}_2' \frac{\delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')}{(E_1E_2E_2'E_2')^{1/2}}[\omega_q{}^2-(E_1-E_1')^2]^{-1}a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)a(\mathbf{p}_1')a(\mathbf{p}_2') , \tag{4.1}$$

where $a^\dagger$ and a are nucleon creation and annihilation operators, respectively, $E_1'=(m^2+p_1'{}^2)^{1/2}$, $\omega_q=(\mu^2+q^2)^{1/2}$, $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_1'$, and μ is the meson mass. The interaction term in the boost operator is given by

$$K_{{\rm int},m} = -\frac{ig^2}{2(2\pi)^3}\int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_1' d\mathbf{p}_2' \frac{\partial\delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')/\partial p_{1,m}'}{(E_1E_2E_1'E_2')^{1/2}}[\omega_q{}^2-(E_1-E_1')^2]^{-1}a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)a(\mathbf{p}_1')a(\mathbf{p}_2') , \tag{4.2}$$

where the suffix m $(=1,2,3)$ specifies the components of $\mathbf{K}$.

If we write the state vector of the two-nucleon system as

$$|\phi\rangle = \int d\mathbf{p}_1 d\mathbf{p}_2 \phi(\mathbf{p}_1,\mathbf{p}_2)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle , \tag{4.3}$$

we obtain the relativistic Schrödinger equation

$$(E_1+E_2)\phi(\mathbf{p}_1,\mathbf{p}_2)-\lambda\int d\mathbf{p}_1' d\mathbf{p}_2' \frac{\delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')}{(E_1E_2E_1'E_2')^{1/2}}\frac{\phi(\mathbf{p}_1',\mathbf{p}_2')}{\omega_q{}^2-(E_1-E_1')^2} = E\phi(\mathbf{p}_1,\mathbf{p}_2) , \tag{4.4}$$

where λ is essentially g^2. We now change the variables from $\mathbf{p}_1$ and $\mathbf{p}_2$ to $\mathbf{k}$ and $\mathbf{P}$ by means of Eqs. (2.6) and (2.7). Then Eq. (4.4) becomes

$$(4\omega^2+P^2)^{1/2}\psi(\mathbf{k},\mathbf{P})-\lambda\int d\mathbf{k}'\frac{[\Omega(k,P)\Omega(k',P)]^{1/2}\psi(\mathbf{k}'P)}{\omega_{\mathbf{k}-\mathbf{k}'}{}^2-(E_k-E_{k'})^2} = E\psi(\mathbf{k},\mathbf{P}) , \tag{4.5}$$

where $E_k=(m^2+k^2)^{1/2}$, and

$$\psi(\mathbf{k},\mathbf{P}) = J^{1/2}\phi(\mathbf{p}_1,\mathbf{p}_2) , \tag{4.6}$$

and

$$\Omega(k,P) = \frac{2}{\omega(E_1+E_2)} = \frac{2}{\omega(4\omega^2+P^2)^{1/2}} . \tag{4.7}$$

It is remarkable that the interaction in Eq. (4.5) does not depend on the direction of $\mathbf{P}$. Hence all partial waves are decoupled. The ground-state wave function $\psi(\mathbf{k},\mathbf{p})$ consists of an s component alone. Unless $P=0$, however, $\phi(\mathbf{p}_1,\mathbf{p}_2)$ which is related to $\psi(\mathbf{k},\mathbf{P})$ through Eq. (4.6) consists of many partial waves because $k=|\mathbf{k}|$ and hence the Jacobian J depends on the direction of $\mathbf{P}$.

At this point let us enumerate a few differences between Eq. (4.5) and Eqs. (2.14) and (2.15) of the BT model.

(1) As noted below Eq. (2.15), the ψ of the BT model depends on P only implicitly through **k**. On the contrary, $\psi(\mathbf{k},\mathbf{P})$ of the present model depends explicitly on **P**.

(2) When $P=0$, E of Eq. (4.5) is identified with the rest mass m_b of the bound system. The relativistic energy-momentum relation $E^2=m_b{}^2+P^2$ is not expected to be satisfied. However, if one keeps adding corrections of higher order in g, the relation will eventually be satisfied. In the BT model the relativistic energy-momentum relation is automatically satisfied by construction of the model. Equations (2.14) and (2.15) are exact equations for the BT model, whereas Eq. (4.5) is only approximate.

(3) The interaction $v(\mathbf{k},\mathbf{k}')$ in Eq. (2.14) can be chosen arbitrarily, whereas the interaction in Eq. (4.5) has been derived (approximately) from a field theory.

For numerical illustration, we took m to be the nucleon mass 938.9 MeV$=4.758$ fm^{-1}, and the meson mass $\mu=0.7$ fm^{-1}. For λ we tried with the following three values in fm^{-3}:

$$\lambda=\begin{cases}0.4135,\\1,\\2,\end{cases}\quad B=\begin{cases}2,\\58.96,\\272.85\,.\end{cases}\tag{4.8}$$

Here we also listed the binding energy $B=2m-m_b$ in MeV for the corresponding choice of λ. With $\lambda=0.4135$ fm^{-2} we mean to simulate the deuteron. For $\lambda=1$ and 2, the binding energy becomes enormous as compared with typical nuclear-binding energies.

We solved Eq. (4.5) for $P=0$, 5, and 10 fm^{-1}, and determined the energy E and the wave function $\psi(\mathbf{k},\mathbf{P})$. We then calculated $\langle x^2\rangle$ and $\langle z^2\rangle$ by means of method II as explained in Sec. III. For the energy we compare E and $(m_b{}^2+P^2)^{1/2}$, where $m_b=E_{P=0}$. For the mean-square radii, we compare $\langle x^2\rangle_P{}^{1/2}$ and $\langle z^2\rangle_P{}^{1/2}$ with $\gamma^{-1}\langle z^2\rangle_0{}^{1/2}$, where $\gamma^{-1}=[1-(P/E)^2]^{1/2}$. The exact Lorentz contraction means that $\langle z^2\rangle_P{}^{1/2}=\gamma^{-1}\langle z^2\rangle_0{}^{1/2}$.

The results are presented in Table I. Overall the model is remarkably relativistic. In the case of $\lambda=0.4135$ fm^{-3}, which simulates the deuteron, the relativistic energy-momentum relation $E=(m_b{}^2+P^2)^{1/2}$ is practically exactly satisfied. This feature was already emphasized in Ref. 2. For the mean-square radii, the deviation from Lorentz contraction is hardly discernible. This is surpising because the model is relativistic only up to the order of g^2. Even for $\lambda=1$ and 2 fm^{-3}, the energy-momentum relation is very well satisfied. The deviation from Lorentz contraction is noticeable, but still very small. As mentioned earlier, $\lambda=1$ or 2 fm^3 represents an enormous strength of the interaction from the nuclear structure standard. Likewise $P=10$ fm^{-1} is a tremendous momentum. It is therefore safe to say that our model is relativistic and that it conforms to Lorentz contraction as far as nuclear-physics applications are concerned.

Figure 1 shows contour plots for the wave function

$$\phi(\mathbf{r},\mathbf{P})=\tfrac{1}{2}\phi_0(r,P)+\tfrac{5}{2}\phi_2(r,P)P_2(t)\;,\tag{4.9}$$

for $P=0$, 5, and 10 fm^{-1} with $\lambda=0.4315$ fm^{-3} ($B=2$ MeV). The normalization is chosen (arbitrarily) such that $\phi_0(r,P)=1$ for $r=0.05$ fm. The contour numbers 1,2,3,... correspond to the values 0.40,0.35,0.30,... of $\phi(\mathbf{r},\mathbf{P})$, respectively. The horizontal axis is in the direction of **P**. Figure 2 is similar to Fig. 1, except that $\lambda=1$ fm^{-3} ($B=58.96$ MeV). Although only the $l=0$ and 2 terms have been included in Eq. (4.9), the effect of the $l=4$ term on the contour plots would be hardly visible.

V. DISCUSSION

The model of a two-body bound system that we constructed in Sec. II according to the BT scheme[1] is relativistically covariant in the sense that all the ten generators of the Poincaré algebra for the model can be constructed explicitly. Nevertheless, when the system is

TABLE I. The energy E in fm^{-1} and $\langle x^2\rangle^{1/2}$ and $\langle z^2\rangle^{1/2}$ in fm for the model of Sec. IV, versus the total momentum P. For the coupling constant λ, three values given in Eq. (4.8) are considered. Lorentz-contraction factor $\gamma^{-1}=[1-(P/E)^2]^{1/2}$ used in the table has been calculated using E listed in the column E [rather than those of column $(E_{P=0}{}^2+P^2)^{1/2}$]. The $\langle x^2\rangle$ and $\langle z^2\rangle$ have been estimated with $l=0$ and 2, except for the values given in parentheses which include the $l=4$ contribution.

λ	P	E	$(E_{P=0}{}^2+P^2)^{1/2}$	$\langle x^2\rangle^{1/2}$	$\langle z^2\rangle^{1/2}$	$\gamma^{-1}\langle z^2\rangle_{P=0}{}^{1/2}$
	0	9.51	9.51	2.395	2.395	2.395
	5	10.74	10.74	2.374	2.105	2.120
0.4135				(2.375)	(2.104)	
	10	13.80	13.80	2.330	1.645	1.650
				(2.342)	(1.619)	
	0	9.22	9.22	0.651	0.651	0.651
	5	10.48	10.49	0.643	0.574	0.572
1				(0.643)	(0.574)	
	10	13.58	13.60	0.625	0.449	0.441
				(0.629)	(0.443)	
	0	8.13	8.13	0.375	0.375	0.375
2	5	9.48	9.55	0.367	0.332	0.319
	10	12.75	12.89	0.352	0.263	0.233

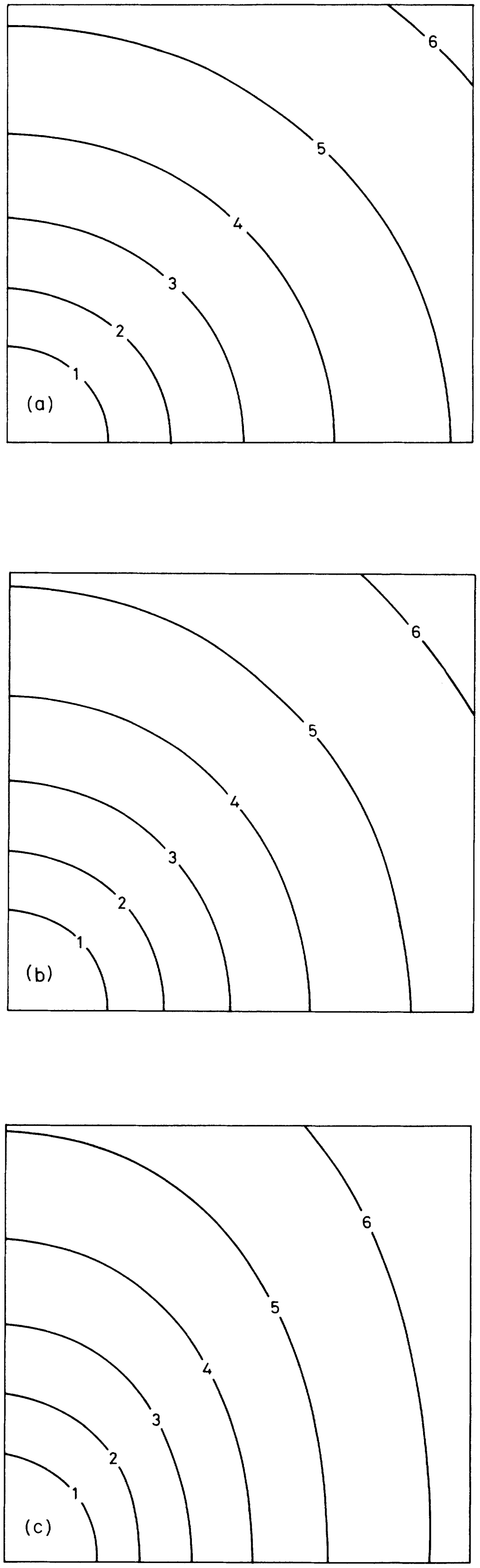

FIG. 1. Contour plots for $\psi(\mathbf{r},\mathbf{P})$ of Eq. (4.9) with $\lambda=0.4135$ fm^{-3}: (a) $P=0$, (b) $P=5\ \mathrm{fm}^{-1}$, (c) $P=10\ \mathrm{fm}^{-1}$. The contour indices 1,2,3,... are explained in Sec. IV.

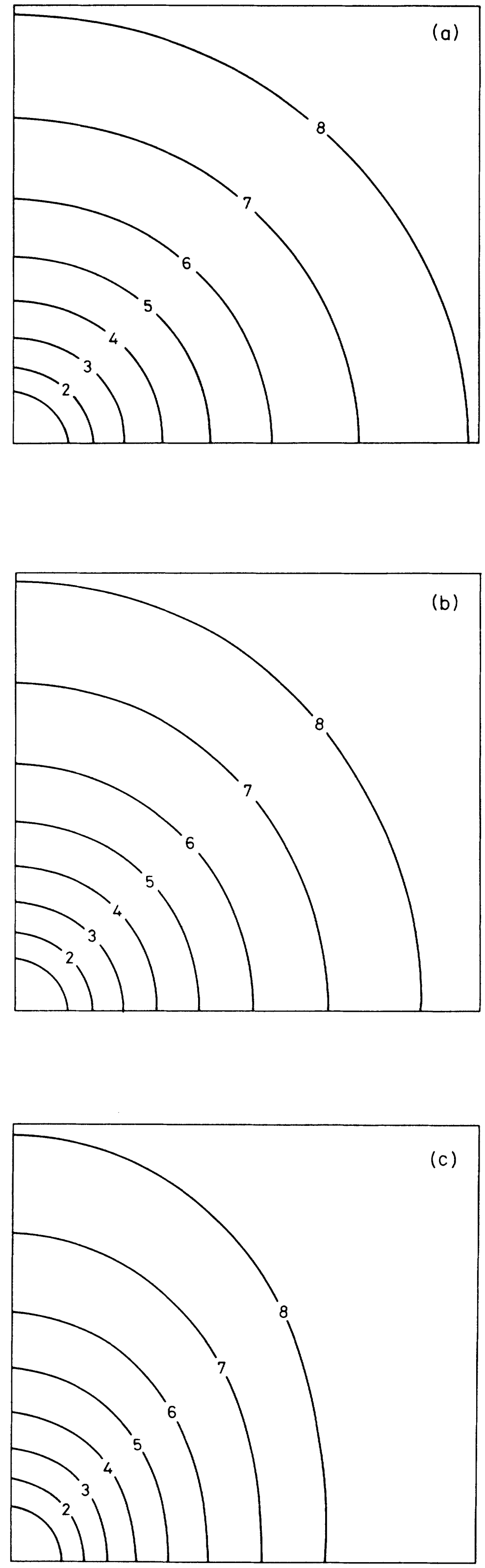

FIG. 2. The same as for Fig. 1, except that $\lambda=1\ \mathrm{fm}^{-3}$. In both figures, the horizontal axis is in the direction of $\mathbf{P}$.

boosted, its shape change does not follow the expected Lorentz contraction. The change in $\langle x^2 \rangle$ as shown in Eq. (3.7) was unexpected. Since the rest mass m_b of the bound system can be chosen arbitrarily in that model, the change in $\langle z^2 \rangle$ given by Eq. (3.8) can be made very different from what is expected on the basis of the Lorentz contraction. We feel, however, that the interaction in the BT scheme is too arbitrary and *ad hoc*.

In Sec. IV we examined a model two-body system based on field theory. We started with meson and nucleon fields interacting via the usual Yukawa coupling, and projected out the two-nucleon sector up to order g^2. The model is relativistic only to this approximation. To our pleasant surprise, however, we found the model practically relativistic. As we numerically illustrated, the relativistic energy-momentum relation for the moving bound system is satisfied to a very good approximation. The system also conforms to Lorentz contraction quite well.

This "success" of the model presented in Sec. IV leads us to conjecture the following. If the order of approximation $(g^2)^n$ is increased, the proximity to Lorentz contraction improves, and in the limit of $n \to \infty$ Lorentz contraction is exactly obeyed. This is a nontrivial conjecture in view of what we have found with the BT model.

There is another aspect of the model of Sec. IV which seems to suggest a novel avenue for relativistic phenomenological description of two-body systems. The interaction (4.1) is a covariant one-meson-exchange potential. Obviously this potential is too simple to be a realistic nucleon-nucleon interaction. Can one make the model somehow more realistic? In this connection, the model has an interesting flexibility. Note that $H_{\rm int}$ and $K_{\rm int}$ have a common factor $[\omega_q{}^2-(E_1-E_1')^2]^{-1}$, which corresponds to $[\omega_{\mathbf{k}-\mathbf{k}'}{}^2-(E_k-E_{k'})^2]^{-1}$ of Eq. (4.5). Even if this factor is replaced with an arbitrary function of $[\omega_q{}^2-(E_1-E_1')^2]$, the commutation relations among the generators of the Poincaré algebra remain valid up to the order of g^2. In this way one can replace the one-meson-exchange potential with a more realistic potential. By doing so, however, one loses contact with the field theory from which the model originates. It remains to be seen how relativistic such a phenomenological model can be, i.e., how well the relativistic energy-momentum relation and Lorentz contraction are obeyed.

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[6]The interaction v can depend also on the relative coordinate which is conjugate to $\mathbf{k}$. In this paper we assume that v is a function of $\mathbf{k}$ alone; see Eq. (2.14). The velocity of the system is also denoted by v, but the distinction should be obvious from the context.

[7]This is the same as Eq. (3.14) of Fong and Sucher (Ref. 1). We have to admit that ϕ might contain a factor $\exp[i\alpha(\mathbf{p},\mathbf{P})]$, which we do not consider.