Constrained Hamiltonian formalism for unconstrained classical mechanics

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The Hamiltonian formalism for the multidimensional equation $\ddot{x}=F(x,\dot{x},t)$ is presented. The link between the whole variety of Hamiltonians for a given F is established. It is shown that two equivalent Lagrangians are connected by a chain of successive transformations of the form $L \to KL+\dot{M}$ where both K and M are constants of the motion.

I. INTRODUCTION

Recently, an alternative approach for constructing a Lagrangian from its equation of motion (usually known as the inverse problem of the calculus of variations) has been presented.[1] It has been shown that any Lagrangian from which the regular n-dimensional second-order system of differential equations

$$\ddot{x}^i=F^i(x^j,\dot{x}^j,t) \tag{1.1}$$

can be derived, may be written (up to a total time derivative) as a linear combination of its own equations of motion (1.1).

As stated in Ref. 1 this universality is much wider than the usual $T-V$ construction which is valid only when forces are derivable from a potential.

On the other hand, it is sometimes impossible to find a (second-order) Lagrangian depending on $x^j,\dot{x}^j$, and t only, from which arbitrary equations of the type (1.1) could be obtained.[2,3]

These results are of fundamental importance not only in the understanding of classical systems, but for their quantizations. In fact, the whole set of equivalent Lagrangians (and Hamiltonians) of a given physical system can be treated on the same footing, the peculiarities distinguishing one from another being completely contained in the coefficients of the linear combination. There are, however, two important issues to be addressed before any quantization scheme is attempted: (a) the complete canonical treatment for acceleration-dependent Lagrangian systems; and (b) an interpretation for the ambiguities residing in the infinite classes of classical equivalent Lagrangians. In this paper we discuss the above-mentioned problems at the classical level. The treatment of the quantization problem will be presented elsewhere.[4]

In Sec. II the canonical formalism for Lagrangians $L(x,\dot{x},\ddot{x},t)$ is presented, following the method of Lanczos.[5] This method introduces a number of auxiliary variables and the resulting constraints are treated in Sec. III using Dirac's formalism for constrained Hamiltonian systems.[6] Section IV is devoted to the discussion of the transformations that link equivalent Lagrangians. Some open problems and prospects at the quantum level are presented in Sec. V.

II. *a* LAGRANGIANS AND THEIR HAMILTONIANS

The so-called inverse problem of the calculus of variations for classical mechanics has been largely studied[1–3,7] and a variety of approaches exists. For the present purpose, however, it is sufficient to recall that the whole set of s-equivalent Lagrangians[8] that originate the equations of motion

$$\ddot{x}^i-F^i(x^j,\dot{x}^j,t)=0 \tag{2.1}$$

can be written as[1]

$$L=L(x^j,\dot{x}^j,\ddot{x}^j,t)=\mu_i(x^j,\dot{x}^j,t)[\ddot{x}^i-F(x^j,\dot{x}^j,t)] \tag{2.2}$$

provided

$$\frac{\partial\mu_i}{\partial\dot{x}^j}=\frac{\partial\mu_j}{\partial\dot{x}^i}\ , \tag{2.3}$$

$$\frac{\bar{d}^2}{dt^2}\mu_i+\frac{\bar{d}}{dt}\left[\mu_j\frac{\partial F^j}{\partial\dot{x}^i}\right]-\mu_j\frac{\partial F^j}{\partial x^i}=0\ , \tag{2.4}$$

$$\det\left[\frac{\partial}{\partial\dot{x}^j}\left[\frac{\bar{d}}{dt}\mu_i+\mu_k\frac{\partial F^k}{\partial\dot{x}^i}\right]+\frac{\partial\mu_j}{\partial x^i}\right]\neq 0\ . \tag{2.5}$$

The symbol $\bar{d}/dt$ stands for on-shell time derivative; i.e.,

$$\frac{\bar{d}}{dt}=\frac{d}{dt}\bigg|_{\ddot{x}^i-F^i=0}=F^i\frac{\partial}{\partial\dot{x}^i}+\dot{x}^i\frac{\partial}{\partial x^i}+\frac{\partial}{\partial t}\ . \tag{2.6}$$

Note that Eq. (2.3) implies that there exists a function $\Lambda(x^i,\dot{x}^i,t)$ such that $\mu_i=-\partial\Lambda/\partial\dot{x}^i$ (see Ref. 1) and consequently $\tilde{L}=L+d\Lambda/dt$ depends only on x^i, $\dot{x}^i$, and t. This fact would suggest that it suffices to study acceleration-independent Lagrangians. However, because of the universality of the form (2.2) already mentioned in the Introduction, it deserves special attention. Moreover, it seems necessary to construct the associated Hamiltonian formalism—provided it exists—to compare the quantum behavior generated by L and $\tilde{L}$.

The complete set of solutions of (2.3)–(2.5) is given by

$$\mu_i=\left[C_1\frac{\partial C_2}{\partial\dot{x}^i}+C_3\frac{\partial C_4}{\partial\dot{x}^i}+\cdots+C_{2n-1}\frac{\partial C_{2n}}{\partial\dot{x}^i}\right]\ , \tag{2.7}$$

where the $C_i = C_i(x^j, \dot{x}^j, t)$ are $2n$ functionally independent constants of the motion of Eq. (2.1).

Care should be taken in using the correct Euler-Lagrange (EL) operator E_i to derive Eq. (2.1) from Lagrangian (2.2); i.e.,

$$E_i = -\frac{d^2}{dt^2}\frac{\partial}{\partial \ddot{x}^i} + \frac{d}{dt}\frac{\partial}{\partial \dot{x}^i} - \frac{\partial}{\partial x^i} . \quad (2.8)$$

We now want to emphasize that (2.2), together with (2.7), solves completely the problem of finding a Lagrangian from the equations of motion. Any other Lagrangian $\tilde{L}$ that gives rise to the same equations is necessarily of the same form as L modulo a total time derivative of a function of the coordinates, the velocities, and time:

$$\tilde{L} = \tilde{\mu}_i(\ddot{x}^i - F^i) + \dot{\Lambda}(x^j, \dot{x}^j, t) . \quad (2.9)$$

The Hamiltonian formalism for Lagrangian (2.9) will be constructed closely following Lanczos's procedure for Lagrangians containing higher than first derivatives.[5] By adding extra variables, the variational problem will be converted into another one which does not involve derivatives higher than the first. From now on, indices will be dropped and we will work formally in one dimension but all the results can be extended to an arbitrary number of degrees of freedom.

Let us define new variables v and a as

$$\dot{x} = v , \quad (2.10)$$

$$\dot{v} = a . \quad (2.11)$$

Then we can incorporate (2.10) and (2.11) into (2.9). In addition, to ensure that the above equations hold, we add them to the Lagrangian as subsidiary (constraint) conditions with appropriate Lagrange multipliers ω and α that should be considered as new variables as well.

Thus, our starting point will be of the modified Lagrangian

$$L' = \mu(x,v,t)[a - F(x,v,t)] + \omega(\dot{x} - v) + \alpha(\dot{v} - a) + \Lambda_{,v} a + \Lambda_{,x} v + \Lambda_{,t} \quad (2.12)$$

(the subscript ,x denotes differentiation with respect to x, etc.).

The equation of motion

$$\ddot{x} - F(x, \dot{x}, t) = 0 , \quad (2.13)$$

the defining equations (2.10) and (2.11), and the equations for the Lagrange multipliers ω and α are obtained from L' by independent variations of x, v, a, ω, and α, as can easily be verified.

To construct the associated Hamiltonian, the canonical momenta conjugated to x, v, a, ω, and α must be calculated. From (2.12),

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = \omega , \quad (2.14a)$$

$$p_v \equiv \frac{\partial L}{\partial \dot{v}} = \alpha , \quad (2.14b)$$

$$p_a \equiv \frac{\partial L}{\partial \dot{a}} = 0 , \quad (2.14c)$$

$$p_\omega \equiv \frac{\partial L}{\partial \dot{\omega}} = 0 , \quad (2.14d)$$

$$p_\alpha \equiv \frac{\partial L}{\partial \dot{\alpha}} = 0 . \quad (2.14e)$$

The Hamiltonian is then

$$H = p_x v + p_v a - \mu(a - F) - \Lambda_{,v} a - \Lambda_{,x} v - \Lambda_{,t} . \quad (2.15)$$

Equations (2.14) constitute a set of constraints in the sense that they do not allow us to solve the velocities in terms of the momenta and the coordinates. Consequently, to reach a consistent Hamiltonian picture use of Dirac's formalism for constrained systems[6] has to be made.

III. DIRAC'S METHOD FOR $\ddot{x} - F(x, \dot{x}, t) = 0$

Because of the fact that Eqs. (2.14) do not contain velocities at all, there are five primary constraints:

$$\phi_1 = p_x - \omega \approx 0 , \quad (3.1)$$

$$\phi_2 = p_v - \alpha \approx 0 , \quad (3.2)$$

$$\phi_3 = p_\omega \approx 0 , \quad (3.3)$$

$$\phi_4 = p_\alpha \approx 0 , \quad (3.4)$$

$$\phi_5 = p_a \approx 0 . \quad (3.5)$$

In order to get the correct equations of motion with the usual Poisson brackets (PB's) one must allow arbitrary linear combinations of these constraints to be added to the Hamiltonian (2.15):

$$H' = H + \lambda^i \phi_i = p_x v + p_v a - \mu(a - F) - \partial\Lambda + \lambda^i \phi_i . \quad (3.6)$$

Here, the λ^i are Lagrange multipliers and the ∂ denotes total time derivative of any function of x, $\dot{x}$, and t, where $\dot{x}$ and $\ddot{x}$ have been replaced by v and a, respectively.

The Poisson brackets are

$$[x^A, p_B] = \delta^A_B \quad (3.7)$$

with the identification $x^1 = x$, $x^2 = v$, $x^3 = a$, $x^4 = \omega$, $x^5 = \alpha$, etc.

Consistency requires that the constraints must be preserved in time, so their time derivatives should also be weakly zero:

$$\dot{\phi}_i = [\phi_i, H'] + \frac{\partial \phi_i}{\partial t} \approx 0 , \quad (3.8)$$

$$\dot{\phi}_1 = [\mu(a - F) + \Lambda_{,v} a + \Lambda_{,x} v + \Lambda_{,t}]_{,x} - \lambda_3 \approx 0 , \quad (3.9)$$

$$\dot{\phi}_2 = -p_x + [\mu(a - F) + \Lambda_{,v} a + \Lambda_{,x} v + \Lambda_{,t}]_{,v} - \lambda_4 \approx 0 , \quad (3.10)$$

$$\dot{\phi}_3 = \lambda_1 \approx 0 , \quad (3.11)$$

$$\dot{\phi}_4 = \lambda_2 \approx 0 , \quad (3.12)$$

$$\dot{\phi}_5 = -p_v + \mu + \Lambda_{,v} \approx 0 . \quad (3.13)$$

Equations (3.9)—(3.12) determine λ_1, λ_2, λ_3, λ_4, whereas Eq. (3.13) is a new (secondary) constraint.

Let us impose that this new constraint,

$$\phi_6 = -p_v + \mu + \Lambda_{,v} \approx 0 , \tag{3.14}$$

be maintained in time

$$\dot{\phi}_6 = \partial\mu + \partial(\Lambda_{,v}) - [p_v, H'] = \partial\mu + \partial(\Lambda_{,v}) + p_x - [\mu(a - F)]_{,v} - (\partial\Lambda)_{,v} \approx 0 . \tag{3.15}$$

Equations (3.15) is a new constraint that by virtue of the algebraic identities

$$\partial(\Lambda_{,v}) - (\partial\Lambda)_{,v} = -\Lambda_{,x} \tag{3.16}$$

and

$$\partial\mu - [\mu(a - F)]_{,v} = (\mu F)_{,v} + \mu_{,x} v + \mu_{,t} \tag{3.17}$$

can be written as

$$\phi_7 = p_x - \Lambda_{,x} + (\mu F)_{,v} + \mu_{,x} v + \mu_{,t} \approx 0 . \tag{3.18}$$

Imposing once again that the recently born constraint must be preserved in time, we get an equation for λ^6, and hence no new constraints appear in the theory.

The equations of motion are derived from the total Hamiltonian (where all the constraints are now incorporated)

$$H_T = p_x v + p_v a - \mu(a - F) - \partial\Lambda + \lambda^i \phi_i , \quad i = 1, \ldots, 7 , \tag{3.19}$$

in the usual way:

$$\dot{x} = [x, H_T] = v + \lambda^1 + \lambda^7 , \tag{3.20a}$$

$$\dot{v} = [v, H_T] = a + \lambda^2 - \lambda^6 , \tag{3.20b}$$

$$\dot{\omega} = [\omega, H_T] = \lambda^3 , \tag{3.20c}$$

$$\dot{\alpha} = [\alpha, H_T] = \lambda^4 , \tag{3.20d}$$

$$\dot{a} = [a, H_T] = \lambda^5 , \tag{3.20e}$$

$$\dot{p}_x = [p_x, H_T] = [\mu(a - F) + \partial\Lambda]_{,x} - \lambda^6(\mu + \Lambda_{,v})_{,x} - \lambda^7[(\mu F)_{,v} + \mu_{,x} v + \mu_{,t} - \Lambda_{,x}]_{,x} , \tag{3.20f}$$

$$\dot{p}_v = [p_v, H_T] = -p_x + [\mu(a - F) + \partial\Lambda]_{,v} - \lambda^6(\mu + \Lambda_{,v})_{,v} - \lambda^7[(\mu F)_{,v} + \mu_{,x} v + \mu_{,t} - \Lambda_{,x}]_{,v} , \tag{3.20g}$$

$$\dot{p}_\omega = [p_\omega, H_T] = \lambda^1 , \tag{3.20h}$$

$$\dot{p}_\alpha = [p_\alpha, H_T] = \lambda^2 , \tag{3.20i}$$

$$\dot{p}_a = [p_a, H_T] = -p_v + \mu + \Lambda_{,v} . \tag{3.20j}$$

Out of all the constraints $\phi_i, i = 1, \ldots, 7$, only ϕ_5 is first class (it has vanishing PB's with all the rest), it occurs because a has no dynamical equations and therefore p_a is an arbitrary parameter of the theory. The six remaining constraints are second class (have nonvanishing PB's among themselves) and arise from the introduction of the auxiliary variables v, a, ω, α, and their conjugate momenta. These variables are redundant in the sense that they do not correspond to genuine independent degrees of freedom and therefore should be eventually eliminated from the theory. This can be done in a systematic manner by defining the Dirac brackets.[9] We do this in two stages: (a) eliminate ϕ_1, ϕ_2, ϕ_3, and ϕ_4, defining the one-star brackets $[\ ,\ ]^*$ and (b) eliminates ϕ_6, ϕ_7 via the two-star brackets $[\ ,\ ]^{**}$.

(a) Define

$$C_{ij} = [\phi_i, \phi_j] , \quad i,j = 1, \ldots, 4 , \tag{3.21}$$

and the brackets

$$[y,z]^* = [y,z] - [y,\phi_i] C^{-1}{}_{ij} [\phi_j, z] , \tag{3.22}$$

where C^{-1} is the inverse matrix of C defined in (3.21):

$$C^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} . \tag{3.23}$$

The only nonvanishing brackets of any pair of variables of the set x, v, ω, α, a and their conjugate momenta are

$$[x,\omega]^* = [x,p_x]^* = [v,\alpha]^* = [v,p_v]^* = [a,p_a]^* = 1 . \tag{3.24}$$

Since the star brackets of any dynamical function with the constraints $\phi_1, \ldots, \phi_4$ vanishes identically, we can now set these constraints strongly to zero, solve the corresponding strong equations for redundant variables ω, α, p_ω, p_α and substitute them whenever they occur. This amounts to simply dropping $\phi_1, \ldots, \phi_4$ from H_T:

$$H^* = H_T \,|_{\phi_1, \ldots, \phi_4 = 0} = p_x v + p_v a - \mu(a - F) - \partial\Lambda + \lambda^i \phi_i , \quad i = 5,6,7 . \tag{3.25}$$

The only remaining variables are now x, v, a and their conjugate momenta.

In order to (b) eliminate ϕ_6 and ϕ_7 we define the 2×2 matrix

$$D_{ij} = [\eta_i, \eta_j] , \quad i,j = 1,2 , \tag{3.26}$$

and the (two-star) brackets

$$[y,z]^{**} = [y,z]^* - [y,\eta_i] D^{-1}{}_{ij} [\eta_j, z] , \tag{3.27}$$

where, by definition $DD^{-1} = I$ and $\eta_i = \phi_{5+i}$. Then

$$D = \begin{bmatrix} 0 & \Phi \\ -\Phi & 0 \end{bmatrix} \tag{3.28}$$

with

$$\Phi = [(\mu F)_{,v} + \mu_{,x} v + \mu_{,t}]_{,v} + \mu_{,x} . \tag{3.29}$$

The quantity Φ is different from zero as required by Eq. (2.5) so that

$$D^{-1} = \begin{bmatrix} 0 & -\Phi^{-1} \\ \Phi^{-1} & 0 \end{bmatrix} \tag{3.30}$$

is well defined.

Upon setting $\phi_6 = \phi_7 = 0$, the only subsisting independent dynamical variables are x and v. Hence, the only independent, nontrivial brackets left is

$$[x,v]^{**} = -\frac{1}{[(\mu F)_{,v} + \mu_{,x} v + \mu_{,t}]_{,v} + \mu_{,x}} . \tag{3.31}$$

The constraints η_1 and η_2 are now set strongly equal to zero so

$$p_v = \mu + \Lambda_{,v} \tag{3.32}$$

and

$$p_x = \Lambda_{,x} - [(\mu F)_{,v} + \mu_{,x} v + \mu_{,t}] \ . \tag{3.33}$$

The Hamiltonian is then

$$H^{**} = -[(\mu F)_{,v} + \mu_{,x} v + \mu_{,t}]v + \mu F - \Lambda_{,t} \ . \tag{3.34}$$

Since μ, Λ, and F are functions of x, v, and t, no further reduction can be carried out.

It should be stressed that although the dynamical variables are x and v, they are not in general the canonical conjugates of each other with respect to the two-star brackets. In fact, the canonical conjugate of x can be checked to be the quantity

$$\pi(x,v,t) = -\left[(\mu F)_{,v} + \mu_{,x} v + \mu_{,t} + \int \mu_{,x} dv + P(x,t)\right] , \tag{3.35}$$

where $P(x,t)$ is an arbitrary function of x and t.

In terms of it, the Hamiltonian (3.34) takes the form

$$H^{**}(x,\pi,t) = \left[\pi + p + \int \mu_{,x} dv\right] v + \mu F - \Lambda_{,t} \ , \tag{3.36}$$

where v is an implicit function of x, π, and t.

It is apparent from the form of the PB's $[x,v]^{**}$ [see Eq. (3.31)] that different choices of the functions μ might in general give rise to different realizations of the canonical algebra and, therefore, to different quantum theories, originating from the same classical equation (2.1).

In the multidimensional case, Eq. (3.31) reads

$$[x^i, v^j]^{**} = -W^{ij} \ , \tag{3.37}$$

where W is the inverse matrix of

$$M_{ij} = \frac{\partial}{\partial \dot{x}^j}\left[\frac{\bar{d}}{dt}\mu_i + \mu_k \frac{\partial F^k}{\partial \dot{x}^i}\right] + \frac{\partial \mu_j}{\partial x^i} \ , \tag{3.38}$$

M is a (symmetric) nonsingular matrix as ensured by Eq. (2.5). On the other hand, if π_j is defined through

$$[x^i, \pi_j] = \delta^i_j \tag{3.39}$$

then Eq. (3.37) implies that

$$\frac{\partial \pi_j}{\partial \dot{x}^k} = -M_{jk} \ . \tag{3.40}$$

These facts confirm that now there are no constraints left in the theory since all velocities can be expressed as functions of coordinate and momenta (π_j).

Moreover, by virtue of the symmetry of M,

$$\frac{\partial \pi_j}{\partial \dot{x}^k} = \frac{2\pi_k}{\partial \dot{x}^j} \ . \tag{3.41}$$

Thus, relation (3.35) can be written in the general case

$$\pi_i = -\left[\frac{\partial}{\partial \dot{x}^i}(\mu_k F^k) + \frac{\partial \mu_i}{\partial x^k} v^k + \frac{\partial \mu_i}{\partial t} + \int \frac{\partial \mu_k}{\partial x^i} d\dot{x}^k + p_i(x^l,t)\right] . \tag{3.42}$$

IV. LINK BETWEEN EQUIVALENT LAGRANGIANS (AND HAMILTONIANS)

As we already know, the most general solution of Eqs. (2.3)–(2.5) is given by

$$\mu_j(x^i,\dot{x}^i,t) = C_1 \frac{\partial C_2}{\partial \dot{x}^j} + C_3 \frac{\partial C_4}{\partial \dot{x}^j} + \cdots + C_{2n-1} \frac{\partial C_{2n}}{\partial \dot{x}^j} \ , \tag{4.1}$$

where the C_l are $2n$ suitably chosen functionally independent constants of motion of Eq. (2.1).

We will now show that any other choice for μ_j, say,

$$\tilde{\mu}_j(x^i,\dot{x}^i,t) = \tilde{C}_1 \frac{\partial \tilde{C}_2}{\partial \dot{x}^j} + \tilde{C}_3 \frac{\partial \tilde{C}_4}{\partial \dot{x}^j} + \cdots + \tilde{C}_{2n-1} \frac{\partial C_{2n}}{\partial \dot{x}^j} \tag{4.2}$$

can be obtained from (4.1) by iteration of the following transformation process:

$$\mu \to K\mu + M_{,v} \ , \tag{4.3}$$

where K and M are (matrix functions of) appropriately selected constants of motion. To see this it suffices to observe that multiplication of μ by any function K generates extra terms proportional to $(\bar{d}K/dt)$ or $(\bar{d}^2K/dt^2)$ in Eq. (2.4) for μ. Thus, if K is a constant of the motion (i.e., $\dot{K}=0$ on shell), $K\mu$ is a solution of the equation for μ as well. Furthermore, if we add a term of the form $M_{,v}$ [with $(\bar{d}M/dt)=0$] to any solution of (2.4) one obtains the extra term

$$\frac{\bar{d}}{dt}\left[\frac{\bar{d}M}{dt}\right]_{,v} - \left[\frac{\bar{d}M}{dt}\right]_{,x}$$

which vanishes identically.

As can easily be verified, μ_j can be turned into $\tilde{\mu}_j$ if it is acted upon $(2n+1)$ transformations of the type (4.3) defined by

$$K_s = C_{2s-3}(C_{2s-1})^{-1} \ \text{ with } C_{-1} \equiv 1 \ , \qquad M_s = -C_{2s} \tag{4.4}$$

for $s = 1,2,\ldots,n$, and

$$K_s = \tilde{C}_{2(s-n)-3}[\tilde{C}_{2(s-n)-1}]^{-1} \quad \text{with } \tilde{C}_{-1} \text{ arbitrary, } \tilde{C}_{2n+1} \equiv 1 \ , \tag{4.5}$$

$$M_s = \tilde{C}_{2(s-n)} \ \text{ with } \tilde{C}_{2n+2} \equiv 0$$

for $s = n+1, n+2, \ldots, 2n+1$. Consequently, any two Lagrangians L, $\tilde{L}$ which are constructed in the form (2.2) from μ and $\tilde{\mu}$, respectively, are related by successive transformations of the form

$$L \to KL + M_{,v}(\ddot{x} - F) \ . \tag{4.6}$$

Correspondingly, as it can be deduced from (3.34) and using the constancy of K and M, the associated Hamiltonians can be found by iteration of

$$H \to KH - M_{,t} \ . \tag{4.7}$$

V. QUANTIZATION: OPEN PROBLEMS AND PROSPECTS

In the preceding section we have outlined the canonical framework for the whole set of Lagrangians generating the same classical orbits. This can be done directly from the equations of motion of the system.

This could be the starting point for a quantization of such systems. One is immediately faced, however, with the usual problems and ambiguities connected with the inversion of the limiting process $\hbar\rightarrow 0$: given the classical limit ($\hbar=0$) of some quantum theory, how can we unambiguously reproduce the original theory ($\hbar\neq 0$)?

In the present case, over and above the usual ordering problem[10] for the operators in the quantum Hamiltonian, etc., we have a new source of ambiguity, namely, among the whole set of choices of the functions μ, which one should be chosen (if there is more than one) when we try to realize the algebra (3.31)? Are different choices of μ equivalent? If so, in what sense? In order to shed some light on these issues, which will be more extensively discussed elsewhere,[4] let us consider two Lagrangians connected by a single transformation of the type (4.6). This relation can be written in the more familiar form

$$\widetilde{L}=KL+\frac{d}{dt}M \ , \tag{5.1}$$

where use has been made of the property that M is a constant of the motion. Consider the particular case $K=1$. Now, Eq. (5.1) reduces to the standard result that two Lagrangians differing by a total time derivative give the same EL equations. This is true irrespective of the nature of M: the (appropriate) EL operator vanishes identically on any total time derivative. In the particular case where the function whose total derivative is added to L depends on x and t only, this transformation corresponds to a gauge transformation in the quantum theory. To illustrate this, consider the case of one degree of freedom with the simple Lagrangian

$$L=\tfrac{1}{2}\dot{x}^2-V(x) \tag{5.2}$$

and its equivalent

$$\widetilde{L}=L+\dot{\Lambda}(x,t) \ . \tag{5.3}$$

The corresponding canonical momenta are

$$p\equiv\frac{\partial L}{\partial\dot{x}}=\dot{x} \ , \tag{5.4}$$

$$\widetilde{p}\equiv\frac{\partial\widetilde{L}}{\partial\dot{x}}=\dot{x}+\Lambda_{,x}=p+\Lambda_{,x} \ , \tag{5.5}$$

and, therefore, the Hamiltonians are, respectively,

$$H=\frac{p^2}{2}+V \tag{5.6}$$

and

$$\widetilde{H}=\tfrac{1}{2}(\widetilde{p}-\Lambda_{,x})^2+V-\Lambda_{,t} \ . \tag{5.7}$$

Following the usual correspondence rule of substituting the canonical momenta conjugate to a variable z by $-i\hbar(\partial/\partial z)$ and H by $i\hbar(\partial/\partial t)$, one obtains the Schrödinger representations associated to (5.6) and (5.7) as

$$\hat{H}\psi=\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi(x,t)=i\frac{\partial}{\partial t}4(x,t) \tag{5.8}$$

and

$$\hat{\widetilde{H}}\psi=\left[\frac{1}{2}\left[i\hbar\frac{\partial}{\partial x}+\Lambda_{,x}\right]^2+V(x)-\Lambda_{,t}\right]\widetilde{\psi}(x,t)$$
$$=i\frac{\partial}{\partial t}\widetilde{\psi}(x,t) \ . \tag{5.9}$$

At first glance, Eqs. (5.8) and (5.9) may seem to describe completely different systems due to the arbitrariness of the function Λ. Their equivalence, however, can be manifestly exhibited if

$$\widetilde{\psi}=e^{i\Lambda(x,t)}\psi \tag{5.10}$$

is replaced in Eq. (5.9). Thus, the addition of a total time derivative into the Lagrangian can be compensated by the local gauge transformation (5.10). This result, nevertheless, breaks down if Λ is a function of $\dot{x}$ as well,[11] since in that case $\dot{x}$ is a much more complicated function of x and p_x and consequently $\widetilde{H}$ is not simply a quadratic expression in $(\widetilde{p}-\Lambda_{,x})$. As a result of this, the relation between the different pictures will not be of the form (5.10). It should be mentioned though that in particularly simple cases (e.g., Λ linear in $\dot{x}$), the complications are manageable and one can still prove the equivalence of the quantum theories derived from the classical Lagrangians related in this manner. This is the case of the supersymmetric point particle,[12] where under a supersymmetry transformation, the Lagrangian changes by a total time derivative of a function linear in the coordinates and velocities in the right combination.[4]

In the general case, when K is an arbitrary constant of the motion, the problem becomes extremely difficult and delicate. We only mention here, for the sake of completeness, that in general the transformation that links $\widetilde{\psi}$ with ψ will not be a simple change of basis in the Hilbert space, so that the classical equivalence will not be automatically extendible to the quantum theory.

VI. CONCLUDING REMARKS

We have carried out the Hamiltonian analysis for the whole class of Lagrangians that give rise to the same classical orbits. This is done via acceleration-dependent Lagrangians taking advantage of the fact that any Lagrangian can be written generically as a linear combination of its own equations of motion. After all the redundant variables of the theory are removed using Dirac's method, it is shown how the freedom in the choice of Lagrangians is mapped into the Poisson-brackets algebra.[13]

The quantum realizations are expected to be radically different depending on the choice of the Lagrangian:[14] in some cases (such as $L=T-V$, for instance), the right-hand side of the algebra of PB's consists of constants (c numbers in the quantum theory) whereas for another s-

equivalent Lagrangian they are functions of the dynamical variables (operators in the quantized version). The problem of the s-equivalent Lagrangians is reduced to the class of transformations involving multiplication by arbitrary constants of motion and addition of total time derivatives of them. This scheme makes apparent that the main obstacle for the quantization of an arbitrary classical system will be to understand how the classical equivalence is reflected at the quantum level.

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