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Functional measure for quantum field theory in curved spacetime

D. J. Toms
Department of Theoretical Physics, University of Newcastle upon Tyne, Newcastle upon Tyne, NE1 7RU England
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An examination of the functional measure for quantum field theory defined on a general curved background spacetime is presented. It is shown how to define the measure in field space to be invariant under general coordinate transformations based upon the simpler problem of defining an invariant inner product. The weight chosen for the variables of integration is seen not to matter in contrast with the claim of Fujikawa that they are uniquely specified. It is shown how the weight $-\frac{1}{2}$ variables advocated by Fujikawa are equivalent to working in a local orthonormal frame. In view of this, the interpretation of conformal anomalies as arising from the measure is reexamined. It is also shown how to define the invariant measure in phase space for a scalar field, which turns out not to be the naive generalization of the finite-dimensional result. The extension to complex and anticommuting fields is discussed. It is also shown how the choice of field variables does not alter the effective field equations.

I. INTRODUCTION

The use of functional-integral techniques in quantum field theory is now widespread. It is possible to derive many results which were established by other methods, often in a much more elegant way. In addition, functional integrals have led to a deeper understanding of many of the issues in quantum field theory. All of this is despite the fact that functional integration has not been established on as rigorous a footing as integration in finite-dimensional spaces, at least in the case of interest in quantum field theory. The major problem is that it has not been found possible to establish with any degree of rigor the notion of the functional measure. As a consequence, there has been some controversy in the literature over what it should be.[1] The fact that the nature of the functional measure is not without physical content is evident from the work of Fujikawa[2-7] which shows that anomalies are linked to the noninvariance of the measure, and also from Polyakov's[8,9] approach to the string in which the measure plays a key part in calculating the critical dimension. (A nice review is contained in Ref. 7.)

The purpose of the present paper is to try to elucidate the nature of the functional measure with the aim of correcting and clarifying some of the results in the literature. Because the problems discussed here are not present in flat Minkowski spacetime (with the exception of certain nonlinear models, such as the nonlinear σ model) attention will be focused on quantum fields on a fixed curved background spacetime. For the case of a scalar field $\phi_0(x)$ on a background with an arbitrary metric $g_{\mu\nu}(x)$, Fujikawa[3,5-7] has argued that the correct choice of variable is not $\phi_0(x)$ but rather $\phi(x)=[-g(x)]^{1/4}\phi_0(x)$ which transforms as a scalar field density[10] of weight $-\frac{1}{2}$. In this event, the functional measure should be $\prod_x d\phi(x)$ and not $\prod_x d\phi_0(x)$. As further evidence for the correctness of this viewpoint, Fujikawa shows that $\prod_x d\phi(x)$ is invariant under an arbitrary change of coordinates, whereas $\prod_x d\phi_0(x)$ is not. In Ref. 6, it is claimed that the variables for the functional integral are specified uniquely by the condition that the measure be anomaly-free under the Becchi-Rouet-Stora (BRS) transformation associated with general coordinate invariance. The fact that there is a preferred choice of variable might seem to be a bit strange if the variables are thought of as coordinates of points in function space.

In Sec. II it will be shown that the scalar field variables may be chosen to be densities of arbitrary weight provided that sufficient care is taken in defining the functional measure. There is nothing wrong with the choice made by Fujikawa, but it is no way compulsory. It will be shown that Fujikawa's choice of variables may be thought of as the function-space analogue of working in a local orthonormal frame. Just as in the finite-dimensional case, the choice of basis should not influence the final answer. Because the conformal anomaly as derived by Fujikawa[3,5] relied crucially on his choice of variable, this problem will be analyzed in Sec. III where it will be shown that the choice of weight made for the variables is irrelevant. It will also be shown how the conformal anomaly may be understood as arising from the fact that a conformal transformation does not preserve the orthonormality and completeness of the basis functions for the scalar field (i.e., conformal transformations are nonunitary).

Unz[11,12] has made the claim that the correct measure for a scalar field $\phi_0(x)$ on a curved background is

$$\prod_x \{(g^{00})^{1/2}[-g(x)]^{1/4}d\phi_0(x)\} .$$

An unsupported claim in his work is that even though the measure looks noncovariant due to the presence of the factor $(g^{00})^{1/2}$, it is actually in fact covariant. Unz's starting point is the functional integral defined in phase space, the factor of $(g^{00})^{1/2}[-g(x)]^{1/4}$ in the measure coming from integration over the momenta. This factor plays a key role in the analysis of five-dimensional

Kaluza-Klein theory presented in Ref. 12. The functional measure in phase space is discussed in Sec. IV where it is shown that it must be defined with a factor which involves $(g^{00})^{-1/2}$ if it is to be covariant. Integration over the canonical momenta then leads to a measure in configuration space without an explicit g^{00} dependence. The mistake made by Unz is that in quantum field theory in curved spacetime the phase-space measure is not just the naive generalization of that found in finite-dimensional quantum mechanics as given by Faddeev,[13] for example.[14] Unz starts out with a noncovariant expression in phase space, and it should therefore not be surprising that he ends up with a noncovariant expression in configuration space. Finally, we discuss the extension of our results to fields other than scalars.

II. CONFIGURATION-SPACE ANALYSIS

Consider the case of a real scalar field defined on an N-dimensional compact Riemannian manifold with metric tensor $g_{\mu\nu}(x)$. The aim of quantum calculations at the one-loop level is to compute the generating functional

$$Z=\int d\mu[\phi]\exp[-\tfrac{1}{2}(\phi,\Delta\phi)]\ . \tag{2.1}$$

Here $d\mu[\phi]$ is an as yet unspecified measure in function space, and (,) denotes an inner product which also needs to be specified. Δ represents a differential operator which is assumed to be self-adjoint with respect to the given inner product.

The possibility that the field $\phi(x)$ may be a scalar field density of arbitrary weight w is considered here.[10] [Recall that $g(x)=\det g_{\mu\nu}(x)$ transforms as a scalar density of weight -2.] If $\phi_0(x)$ denotes the scalar field (i.e., density of weight $w=0$), then

$$\phi(x)=[g(x)]^{-w/2}\phi_0(x) \tag{2.2}$$

is a scalar field density of weight w. The operator Δ in (2.1) will be taken to transform like a scalar, although this is not strictly necessary.

Given the variables $\phi(x)$, which have been chosen to be densities of weight w, the inner product in (2.1) may be defined. It must be generally coordinate invariant so that

$$(\phi,\psi)=\int d^Nx[g(x)]^{1/2}\phi_0(x)\psi_0(x)\ , \tag{2.3}$$

where $\phi_0(x)$ and $\psi_0(x)$ are scalar fields. From (2.2) it is observed that the inner product should be taken as

$$(\phi,\psi)=\int d^Nx\,[g(x)]^{w+1/2}\phi(x)\psi(x)\ . \tag{2.4}$$

Before defining the functional measure, it is helpful to consider the situation in a finite-dimensional, real vector space V. Assume that V is equipped with an inner product (,) and a positive-definite symmetric metric E such that

$$(u,v)=\sum_{i,j=1}^{n}u^iE_{ij}v^j \tag{2.5}$$

for u, $v\in V$, $\dim V=n$. (The restriction that E_{ij} be positive definite is not essential.) Here u^i and v^i denote the components of u and v with respect to some arbitrary basis in V. The definition of the inner product will be independent of this choice of basis provided that E_{ij} transforms like a second-rank, covariant tensor under a change of basis. The volume element in V which is invariant under a general linear change of basis is

$$d\mu(v)=(\det E_{ij})^{1/2}d^nv\ , \tag{2.6}$$

where

$$d^nv=\prod_{i=1}^{n}dv^i\ . \tag{2.7}$$

If a local, orthonormal basis is chosen in V, then $E_{ij}=\delta_{ij}$, and the invariant measure is $d\mu(v)=d^nv$, but of course this result is not true in general.

In the functional case, let $\phi^i(x)$ denote a real, bosonic field whose indices are specified by the index i. Let $\mathscr{F}$ denote the function space of all $\phi^i(x)$. The inner product between two elements ϕ and ψ of $\mathscr{F}$ may be written as

$$(\phi,\psi)=\int d^Nx\phi^i(x)\rho_{ij}(x)\psi^j(x)\ , \tag{2.8}$$

where $\rho_{ij}(x)$ is interpreted as the metric on $\mathscr{F}$, which is completely analogous to E_{ij} in the finite-dimensional case discussed above. $\rho_{ij}(x)$ is chosen to transform in such a way so as to make (ϕ,ψ) independent of the choice of coordinates. [See (2.4) for the special case of scalar field densities.] It then follows by direct analogy with the finite-dimensional case that

$$d\mu[\phi]=\prod_x[\det\rho_{ij}(x)]^{1/2}\prod_{i=1}^{n}d\phi^i(x) \tag{2.9}$$

should be chosen as the invariant measure on $\mathscr{F}$. Here $\det\rho_{ij}(x)$ denotes an ordinary determinant over the indices i and j. The analogy is furthered by thinking of the integration in (2.8) as a summation over a continuous index, and noting that the metric is diagonal in its continuous indices. The idea of using the invariant metric in function space to define the functional measure was suggested by DeWitt[15] and used in the context of quantum gravity. He also has emphasized the need for the measure to be invariant under coordinate transformations.[16]

In the scalar field case discussed above, the indices i and j in (2.8) are redundant, and we have

$$\rho(x)=[g(x)]^{w+1/2} \tag{2.10}$$

as the metric on $\mathscr{F}$, where $\mathscr{F}$ is chosen as the space of scalar field densities of weight w. The functional measure according to (2.9) should be

$$d\mu[\phi]=\prod_x[g(x)]^{w/2+(1/4)}d\phi(x)\ . \tag{2.11}$$

It is straightforward to now see that $d\mu[\phi]$ is independent of w and hence of the choice of scalar field variables. It is obvious from (2.2) that

$$d\mu[\phi]=\prod_x[g(x)]^{1/4}d\phi_0(x)\ , \tag{2.12}$$

since $g_{\mu\nu}(x)$ is the background metric which is kept fixed during functional integration. This is seen to be identical to what is obtained by setting $w=0$ in (2.11). Thus, if the ordinary scalar field is chosen as a variable, the functional measure must be taken to be (2.12) and not simply

$\prod_x d\phi_0(x)$. It is therefore not surprising that Fujikawa found that $\prod_x d\phi_0(x)$ was not invariant under general coordinate transformations. This is no more to be expected than that $d^n x$ would be the invariant volume element on a curved manifold.

Fujikawa advocated the choice of variable $\tilde{\phi}(x)=[g(x)]^{1/4}\phi_0(x)$ which is a scalar field density of weight $w=-\frac{1}{2}$. If $w=-\frac{1}{2}$ is chosen, then from (2.10) it is seen that $\rho(x)=1$. Thus, Fujikawa's choice of variables corresponds to choosing a local orthonormal basis in $\mathscr{F}$. However, it is seen that we are free to choose whatever variables we like, including the usual scalar field (with $w=0$) provided that we include the appropriate metric factor for the space $\mathscr{F}$ in the measure as given in (2.9). The fact that (2.9) [or (2.11)] really is invariant under a general change of coordinates may be seen following the analysis of Fujikawa. The simplest way is just to work out the Jacobian in the functional measure under a general change of coordinates and show that it is unity.

There is another way of thinking about the functional measure which is based on defining it so that Gaussian functional integration makes sense.[17] It is also a useful approach to understand the conformal anomaly which is discussed in Sec. III. Again, it is helpful to first consider a finite-dimensional example.

The finite-dimensional analogue of (2.1) is

$$I(A)=\int d\mu(x)\exp[-\tfrac{1}{2}(x,Ax)]\ , \tag{2.13}$$

where A is a nonsingular $n\times n$ matrix with positive eigenvalues, x is an n-dimensional vector, and $(x,Ax)=x^T Ax$ denotes the usual Euclidean inner product. Without loss of generality A may be taken to be symmetric. If $d\mu(x)$ is defined by

$$d\mu(x)=(2\pi)^{-n/2}d^n x \tag{2.14}$$

it then follows that

$$I(A)=(\det A)^{-1/2}\ . \tag{2.15}$$

Comparison of (2.13) with (2.1) leads to the conclusion that however the functional measure is defined, it should be such as to give

$$Z=[\mathrm{Det}(l^2\Delta)]^{-1/2}\ , \tag{2.16}$$

where the determinant is now a functional one. It is necessary to introduce the length scale l for dimensional reasons since Δ has dimensions of $(\text{length})^{-2}$. Another argument for demanding that the functional measure be defined so as to lead to (2.16) is that this result can be obtained without recourse to functional methods. We will now consider in more detail how the functional measure may be defined so as to lead to (2.16).

The biscalar Dirac distribution $\delta(x,x')$ is defined by

$$\phi_0(x)=\int d^N x'[g(x')]^{1/2}\delta(x,x')\phi_0(x') \tag{2.17}$$

for scalar field $\phi_0(x)$. For the scalar field density $\phi(x)$ defined in (2.2), define

$$\tilde{\delta}(x,x')=[g(x)]^{-w/2}[g(x')]^{-w/2}\delta(x,x') \tag{2.18}$$

which transforms like a scalar density of weight w in each argument under a general coordinate change. From (2.17) and (2.18) we have

$$\phi(x)=\int d^N x'[g(x')]^{w+1/2}\tilde{\delta}(x,x')\phi(x') \tag{2.19}$$

for $\phi(x)$ a scalar field density of weight w. $\tilde{\delta}(x,x')$ may be regarded as the matrix element of the identity operator I in the function space whose elements are the $\phi(x)$:

$$(I\phi)(x)=\int d^N x'[g(x')]^{w+1/2}\tilde{\delta}(x,x')\phi(x')\ . \tag{2.20}$$

Let $\{\lambda_n,f_n(x)\}$ be a spectral decomposition of Δ:

$$\Delta f_n(x)=\lambda_n f_n(x)\ . \tag{2.21}$$

Regardless of how $f_n(x)$ transforms under a general coordinate transformation (i.e., it can be a scalar density of arbitrary weight), the λ_n are invariant. The transformation properties and dimensions of $f_n(x)$ are not fixed by (2.21). A natural, although not essential choice is to take the $f_n(x)$ to have the same transformation properties and dimensions as $\phi(x)$. Assume that $f_n(x)$ forms a complete orthonormal set:

$$(f_n,f_{n'})=\int d^N x[g(x)]^{w+1/2}f_n^*(x)f_{n'}(x) \tag{2.22}$$

$$=l^2\delta_{nn'}\ . \tag{2.23}$$

The constant l^2 with dimensions of $(\text{length})^2$ is introduced so that the dimensions on both sides agree. The completeness relation reads

$$\sum_n f_n^*(x)f_n(x')=l^2\tilde{\delta}(x',x)\ . \tag{2.24}$$

Note that the dimensions on both sides of (2.24) agree, and that this result is consistent with (2.23).

Because $\{f_n(x)\}$ is complete, we may expand $\phi(x)$ as

$$\phi(x)=\sum_n \phi_n f_n(x)\ . \tag{2.25}$$

The expansion coefficients ϕ_n will be invariant under general coordinate transformations since $\phi(x)$ and $f_n(x)$ transform in an identical manner. A simple calculation leads to (for real ϕ_n)

$$(\phi,\Delta\phi)=\sum_n (l^2\lambda_n){\phi_n}^2\ . \tag{2.26}$$

If a functional measure $d\tilde{\mu}[\phi]$ is defined by

$$d\tilde{\mu}[\phi]=\prod_n \frac{d\phi_n}{(2\pi)^{1/2}}\ , \tag{2.27}$$

it then follows that

$$Z=\prod_n \int \frac{d\phi_n}{(2\pi)^{1/2}}\exp[-\tfrac{1}{2}(l^2\lambda_n){\phi_n}^2] \tag{2.28}$$

$$=\prod_n (l^2\lambda_n)^{-1/2} \tag{2.29}$$

$$=[\mathrm{Det}(l^2\Delta)]^{-1/2} \tag{2.30}$$

which is the required result. Note that (2.27) is directly analogous to the measure (2.14) which was used in the finite-dimensional case.

The above procedure is basically that used by Hawking.[18] One viewpoint which could be taken is that the functional measure could be defined by (2.27). This approach has the advantage that the functional measure is naturally coordinate invariant, is independent of the choice of weight made for the variables, and leads to an elegant way[18] of regularizing the infinite determinant in (2.30). It is easy to check that the definition (2.27) is independent of the choice of basis, since transformations between complete orthonormal sets of basis functions are unitary.

The formal equivalence between the measure defined in (2.27) and that defined in (2.11) may be established as follows. (Fujikawa[5] has given a similar analysis for his weight $-\frac{1}{2}$ variables, but again it must be emphasized that the functional measure is independent of weight.) Introduce Dirac notation by writing

$$\phi(x)=\langle x \mid \phi \rangle \; , \tag{2.31}$$

$$f_n(x)=\langle x \mid n \rangle \; . \tag{2.32}$$

Then (2.25) may be written as

$$\mid \phi \rangle = \sum_n \phi_n \mid n \rangle \; . \tag{2.33}$$

In this form it may be recognized that what is happening in (2.25) is just a change of basis in Hilbert space from $\mid \phi \rangle$ to $\mid n \rangle$. Associated with this change of basis will be a Jacobian, which formally is

$$J=\mathrm{Det}\langle x \mid n \rangle \; . \tag{2.34}$$

Now

$$\mid J \mid^2 = J^\dagger J = (\mathrm{Det}\langle n \mid x \rangle)(\mathrm{Det}\langle x' \mid n' \rangle) \tag{2.35}$$

$$= \mathrm{Det}\left[\int d^N x \, [g(x)]^{w+1/2} f_n^*(x) f_{n'}(x) \right] \tag{2.36}$$

$$= \mathrm{Det}(l^2 \delta_{nn'}) \; . \tag{2.37}$$

The result in (2.36) follows from (2.35) by combining the two determinants and using the metric (2.10) in the space of functions when multiplying $\langle n \mid x \rangle [=f_n^*(x)]$ by $\langle x' \mid n' \rangle [=f_{n'}(x')]$. Equation (2.37) shows that up to an irrelevant infinite constant factor, we have $\mid J \mid =1$, and therefore the measure defined in (2.27) is formally equivalent to that defined by (2.11).

III. THE FUNCTIONAL MEASURE AND THE CONFORMAL ANOMALY

As was discussed in the preceding section, the functional measure is independent of the weight chosen for the scalar field variables in the functional integral. The choice made by Fujikawa is allowed, but is not compulsory. Because the conformal anomaly may be understood as arising from the functional measure, it follows that it too must be independent of the weight chosen for the variables. Since Fujikawa's derivation[3] relied on the choice $w=-\frac{1}{2}$, it is instructive to see how the analysis proceeds for arbitrary w.

Let

$$\Delta_x = -\Box_x + \frac{1}{4}\left[\frac{N-2}{N-1}\right] R(x) \tag{3.1}$$

be the conformally invariant wave operator for a scalar field in N dimensions. A conformal transformation of the metric is given by

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \; . \tag{3.2}$$

The scalar field action which involves $(\phi, \Delta\phi)$ will be invariant[19] under (3.2) provided that

$$\phi(x) \to \bar{\phi}(x) = [\Omega(x)]^{1-N(w+1/2)} \phi(x) \; . \tag{3.3}$$

Note that the biscalar density Dirac distribution $\tilde{\delta}(x,x')$ defined in (2.18) transforms like

$$\tilde{\delta}(x,x') \to \bar{\tilde{\delta}}(x,x') = [\Omega(x)]^{-1-N(w+1/2)} [\Omega(x')]^{1-N(w+1/2)} \times \tilde{\delta}(x,x') \tag{3.4}$$

under (3.2).

Consider the partition function (2.1). Because the scalar field action is invariant under (3.2) and (3.3), if Z changes at all, any change must come from the functional measure. We also know that if the functional measure is written in the form (2.27) it will be invariant under a unitary transformation. Thus, any conformal anomaly must correspond to a nonunitary change of basis. This will now be demonstrated explicitly, and it will also be shown how the weight chosen for the variables is irrelevant.

The functions $f_n(x)$ have been chosen to transform like $\phi(x)$. Therefore, in the conformally related spacetime [cf. (2.25)],

$$\bar{\phi}(x) = \sum_n \phi_n \bar{f}_n(x) \; , \tag{3.5}$$

where

$$\bar{f}_n(x) = [\Omega(x)]^{1-N(w+1/2)} f_n(x) \; . \tag{3.6}$$

(The ϕ_n do not transform under the conformal transformation.) If $\{f_n(x)\}$ is a complete orthonormal set, then the conformally related basis $\{\bar{f}_n(x)\}$ is neither complete nor orthonormal in general. It is easy to see that

$$(\bar{f}_n, \bar{f}_{n'}) = \int d^N x [\bar{g}(x)]^{w+1/2} \bar{f}_n^*(x) \bar{f}_{n'}(x) \tag{3.7}$$

$$= \int d^N x \, [g(x)]^{w+1/2} \Omega^2(x) f_n^*(x) f_{n'}(x) \; , \tag{3.8}$$

$$\sum_n \bar{f}_n^*(x) \bar{f}_n(x') = \Omega(x)\Omega(x') l^2 \bar{\tilde{\delta}}(x',x) \; . \tag{3.9}$$

However, the derivation of (2.30) relied on a complete orthonormal set of basis functions which diagonalized Δ. It is therefore necessary to perform a further transformation

$$\bar{f}_n(x) = \Omega(x) g_n(x) \tag{3.10}$$

to a new basis $\{g_n(x)\}$. It is clear from (3.7)–(3.10) that

$$(g_n, g_{n'}) = \int d^N x [\bar{g}(x)]^{w+1/2} g_n^*(x) g_{n'}(x) \tag{3.11}$$

$$= l^2 \delta_{nn'} \tag{3.12}$$

and

$$\sum_n g_n^*(x) g_n(x') = l^2 \bar{\bar{\delta}}(x',x) \ . \tag{3.13}$$

The basis $\{g_n(x)\}$ is now complete and orthonormal in the conformally related spacetime.

As mentioned above, the transformation (3.10) from the basis $\{\bar{f}_n(x)\}$ to $\{g_n(x)\}$ will not be unitary, and hence a nontrivial Jacobian will arise. Write the scalar field density $\bar{\phi}(x)$ in the conformally related spacetime as

$$\bar{\phi}(x) = \sum_n \bar{\phi}_n g_n(x) \tag{3.14}$$

for some expansion coefficients $\bar{\phi}_n$. The original functional measure was $\prod_n d\phi_n/(2\pi)^{1/2}$ and the transformed one must involve $\prod_n d\bar{\phi}_n/(2\pi)^{1/2}$. Therefore we have

$$\prod_n \frac{d\phi_n}{(2\pi)^{1/2}} = J \prod_n \frac{d\bar{\phi}_n}{(2\pi)^{1/2}} \ , \tag{3.15}$$

where J is the Jacobian of the transformation from ϕ_n to $\bar{\phi}_n$. However, by comparing (3.5) with (3.14) it may be seen that

$$\phi_n = \sum_{n'} C_{nn'} \bar{\phi}_{n'} \ , \tag{3.16}$$

where

$$C_{nn'} = l^{-2} \int d^N x \, [\bar{g}(x)]^{w+1/2} \Omega^{-1}(x) g_n^*(x) g_{n'}(x) \tag{3.17}$$

$$= l^{-2} \int d^N x \, [g(x)]^{w+1/2} \Omega^{-1}(x) f_n^*(x) f_{n'}(x) \ . \tag{3.18}$$

The Jacobian appearing in (3.15) is therefore

$$J = \mathrm{Det} C_{nn'} \ . \tag{3.19}$$

From (3.17) and (3.18) it is clear that the Jacobian may be evaluated either in the original or conformally related spacetime. It is also clear that $C_{nn'}$ and hence J is independent of the choice of weight made for the field variables. A particular choice of $w = -\frac{1}{2}$ may be made if desired, but it is again worth emphasizing that this is completely arbitrary and is not essential. The Jacobian (3.19) may be evaluated as described by Fujikawa[3] where the usual scalar field conformal anomaly will be obtained.

IV. THE FUNCTIONAL MEASURE IN PHASE SPACE

Phase space for a finite-dimensional system is described by a set of coordinates $q^i(t)$ and their associated canonical momenta $p_i(t)$. The partition function is then[20]

$$Z = \int d\mu(p,q) \exp\left[i \int dt \, [p_i \dot{q}^i - H(q,p)] \right] \ , \tag{4.1}$$

where $H(q,p)$ is the Hamiltonian. The measure is just the usual one[13,14]

$$d\mu(p,q) = \prod_t \left[\prod_i dp_i(t) \right] \left[\prod_i dq^i(t) \right] \ . \tag{4.2}$$

As mentioned in the Introduction, this result is usually just transcribed directly without modification to quantum field theory. While this does not lead to any problems in flat Minkowski spacetime, it turns out not to be the correct procedure in curved spacetime.

Consider a scalar field theory on an N-dimensional curved manifold with a metric signature $(-++\cdots+)$. A careful discussion of canonical quantization in curved spacetime[21] is given by Fulling.[22] The canonical formalism requires that one of the coordinates, say x^0, be chosen as the time coordinate, and that for each t, the hypersurface $x^0 = t$ be a Cauchy surface for the spacetime region covered by the coordinates.[22] The behavior of quantities under a change of coordinates which involve only those on the Cauchy hypersurface or else a transformation of the local time scale with the coordinates on the Cauchy hypersurface held fixed may be considered.[23]

The action for a scalar field $\phi_0(x)$ is

$$S[\phi_0] = \int d^N x \, \mathscr{L}(\phi_0, \partial_\mu \phi_0, x) \ , \tag{4.3}$$

where

$$L(\phi_0, \partial_\mu \phi_0, x) = -\tfrac{1}{2} [-g(x)]^{1/2} g^{\mu\nu}(x) \partial_\mu \phi_0(x) \partial_\nu \phi_0(x) \tag{4.4}$$

is the Lagrangian density. (Only the simplest case of a free massless, minimally coupled scalar field is considered in this section.) The Lagrangian $L[\phi_0, \dot{\phi}_0, t]$ is defined by

$$L[\phi_0, \dot{\phi}_0, t] = \int d^{N-1} x \, \mathscr{L}(\phi_0, \partial_\mu \phi_0, x) \ , \tag{4.5}$$

where the integration is over the $(N-1)$-dimensional Cauchy hypersurface. Instead of using the scalar field $\phi_0(x)$, the choice of a scalar field density of weight w defined in (2.2) may be made. For this choice, the Lagrangian is

$$L[\phi, \dot{\phi}, t] = -\tfrac{1}{2} \int d^{N-1} x \, [-g(x)]^{1/2} g^{\mu\nu}(x) \partial_\mu [(-g)^{w/2} \phi] \partial_\nu [(-g)^{w/2} \phi] \ . \tag{4.6}$$

[Setting $w = 0$ recovers the scalar field result in (4.5).] The momentum which is canonically conjugate to ϕ is defined as usual by

$$\pi(x) = \frac{\partial L}{\partial \dot{\phi}(x)} \ . \tag{4.7}$$

It follows from (4.6) that[24]

$$\pi(x) = -[-g(x)]^{w+1/2} g^{0\nu}(x) \nabla_\nu \phi(x) \ . \tag{4.8}$$

The momentum which is canonically conjugate to the scalar field $\phi_0(x)$ is

$$\pi_0(x) = -[-g(x)]^{1/2} g^{0\nu}(x) \nabla_\nu \phi_0(x) \ . \tag{4.9}$$

Thus the canonical momentum for the scalar field density of weight w is related to that for the scalar field by

$$\pi(x)=[-g(x)]^{w/2}\pi_0(x) . \quad (4.10)$$

The Hamiltonian density is defined in general by

$$\mathscr{H}=\pi\dot{\phi}-\mathscr{L} \quad (4.11)$$

from which it follows that for the scalar field

$$\mathscr{H}=-\tfrac{1}{2}(g^{00})^{-1}(-g)^{-w-1/2}\pi^2-(g^{00})^{-1}g^{0i}\pi\nabla_i\phi$$
$$-\frac{w}{2}[\partial_0\ln(-g)]\pi\phi$$
$$-\tfrac{1}{2}(g^{00})^{-1}(-g)^{w+1/2}g^{0i}g^{0j}\nabla_i\phi\nabla_j\phi$$
$$+\tfrac{1}{2}(-g)^{w+1/2}g^{ij}\nabla_i\phi\nabla_j\phi . \quad (4.12)$$

As in Sec. II, focus first on the n-dimensional case of a system with a Lagrangian

$$L=\tfrac{1}{2}E_{ij}\dot{q}^i\dot{q}^j-V(q) , \quad (4.13)$$

where E_{ij} is a nonsingular matrix. The canonical momentum is

$$p_i=\frac{\partial L}{\partial \dot{q}^i}=E_{ij}\dot{q}^j \quad (4.14)$$

and the Hamiltonian is

$$H=\tfrac{1}{2}(E^{-1})^{ij}p_ip_j+V(q) , \quad (4.15)$$

where

$$(E^{-1})^{ik}E_{kj}=\delta^i_j . \quad (4.16)$$

The invariant volume element in momentum space is then

$$d\mu(p)=(\det E^{-1})^{1/2}d^np=(\det E)^{-1/2}d^np . \quad (4.17)$$

The invariant volume element in configuration space is

$$d\mu(q)=(\det E)^{1/2}d^nq . \quad (4.18)$$

Thus the invariant volume element in phase space is simply

$$d\mu(p,q)=d^np\, d^nq , \quad (4.19)$$

since the factors of $(\det E)^{1/2}$ cancel between (4.17) and (4.18). This justifies the choice (4.2) for quantum-mechanical examples based on this finite-dimensional system. Note, however, that there is no analogue of the g^{00} factor in this finite-dimensional example. Thus, caution must be used in translation of this result to the field theory case.

Based upon (4.15) and (4.12) and the discussion of Sec. II, it is clear that the momentum-space measure for the scalar field should be

$$d\mu[\pi]=\prod_x \{[-g^{00}(x)]^{-1/2}[-g(x)]^{-w/2-(1/4)}d\pi(x)\} . \quad (4.20)$$

The factor of

$$[-g^{00}(x)]^{-1/2}[-g(x)]^{-w/2-(1/4)}$$

is seen to be just the analogue of $(\det E^{-1})^{1/2}$ in the finite-dimensional case. When (4.20) is combined with the configuration space measure (2.11) it gives

$$d\mu[\pi,\phi]=\prod_x \{[-g^{00}(x)]^{-1/2}d\pi(x)d\phi(x)\} \quad (4.21)$$

as the invariant phase-space measure. It is easy to verify that this measure is invariant under arbitrary changes in the local time scale, as well as under changes in the coordinates on the spacelike Cauchy hypersurface. The invariant inner product in momentum space is easily read off from the Hamiltonian (4.12) to be

$$(\pi,\pi)=\int d^Nx[-g^{00}(x)]^{-1}[-g(x)]^{-w-1/2}\pi^2(x) . \quad (4.22)$$

Integration over the momentum variables in the functional integral will lead directly to the configuration-space functional integral of Sec. II. Without the factor of $(-g^{00})^{-1/2}$ present in (4.21) this will not be true, as integration over the momentum will lead to explicit factors of $(-g^{00})$ appearing in the configuration-space functional measure. This is the situation encountered by the Unz[11,12] who starts off with a noninvariant measure in phase space and ends up with a noninvariant measure in configuration space.

V. DISCUSSION AND CONCLUSIONS

The above discussion has focused on the nature of the functional measure in quantum field theory in curved spacetime, particularly in relation to Fujikawa's[2-7] work. It has been shown that contrary to the claims of Fujikawa, the variables for the functional integral are not uniquely specified by the demand that the functional measure (and hence the partition function) be invariant under general coordinate transformations. For example, scalar field densities of weight $-\frac{1}{2}$ are not any more unique than densities of any other weight, provided that the appropriate factors of $[-g(x)]^{1/2}$ occur in the functional measure to give one which is invariant in function space. It was seen how this problem was related to the simpler one of defining an invariant Hilbert space inner product for the field under consideration. The resulting modification of Fujikawa's[3] interpretation of conformal anomalies as arising from the behavior of the functional measure under conformal transformations was considered. Lastly, the passage from the functional integral defined in phase space to one defined in configuration space was described. The invariant measure in phase space was given for a scalar field, and we discussed why the results of Unz[11,12] were incorrect. The reason for this can be traced to a naive transcription of results valid for finite-dimensional systems to quantum field theory.

Although the preceding analysis has been restricted to real boson fields, there is no impediment to extending it to either complex or fermion fields. If $\phi^i(x)$ represents a set of complete boson fields with an invariant inner product

$$(\phi,\phi)=\int d^Nx\rho_{ij}(x)\phi^{i*}(x)\phi^j(x) , \quad (5.1)$$

then in place of (2.11) we have

$$d\mu[\phi^*,\phi]=\prod_x [\det\rho_{ij}(x)]\prod_i d\phi^{i*}(x)\prod_i d\phi^i(x)\ . \qquad (5.2)$$

In fact this may be seen to simply be a consequence of (2.11) by splitting up the complex fields into real and imaginary parts. If $\phi^i(x)$ is a set of real anticommuting fields, describing Majorana spinors for example, then the configuration space measure will be

$$d\mu[\phi]=\prod_x [\det\rho_{ij}(x)]^{-1/2}\prod_i d\phi^i(x) \qquad (5.3)$$

assuming that the invariant inner product is still given by (5.1). The reason why $\det\rho_{ij}(x)$ occurs to a negative power in this case is because of the rules for integrating over anticommuting fields.[25]

Because the choice of variables in the functional integral has been shown to be arbitrary, it is interesting to examine how the vacuum expectation value of the stress-energy tensor and the semiclassical Einstein equations are affected. In order to compute the vacuum expectation value of the stress-energy tensor, we vary the effective action Γ with respect to the background metric $g_{\mu\nu}(x)$ while holding the basic scalar field variables fixed. It is clear that for different choices of field variables, different expressions will result and we must examine how they are related.

It has been demonstrated above that the partition function, and hence the effective action, is independent of the weight chosen for the scalar field variables. We therefore have

$$\Gamma[\phi,g_{\mu\nu}]=\Gamma[\phi_0,g_{\mu\nu}]\ , \qquad (5.4)$$

where ϕ and ϕ_0 denote the background scalar fields which are related by (2.2). If we now vary $\Gamma[\phi,g_{\mu\nu}]$ with respect to $g_{\mu\nu}(x)$ while holding $\phi(x)$ fixed, it is clear that $\phi_0(x)$ cannot remain fixed. However, from (5.4) we see that

$$\left[\frac{\delta\Gamma[\phi,g_{\mu\nu}]}{\delta g_{\mu\nu}(x)}\right]_\phi=\left[\frac{\delta\Gamma[\phi_0,g_{\mu\nu}]}{\delta g_{\mu\nu}(x)}\right]_\phi . \qquad (5.5)$$

The right-hand side may now be evaluated to give

$$\left[\frac{\delta\Gamma[\phi,g_{\mu\nu}]}{\delta g_{\mu\nu}(x)}\right]_\phi=\left[\frac{\delta\Gamma[\phi_0,g_{\mu\nu}]}{\delta g_{\mu\nu}(x)}\right]_{\phi_0}+\int d^Nx'\left[\frac{\delta\Gamma[\phi_0,g_{\mu\nu}]}{\delta\phi_0(x')}\right]_{g_{\mu\nu}}\left[\frac{\delta\phi_0(x')}{\delta g_{\mu\nu}(x)}\right]_\phi . \qquad (5.6)$$

From (2.2) we have

$$\delta\phi_0(x)=\frac{w}{2}\phi_0(x)g^{\mu\nu}(x)\delta g_{\mu\nu}(x) \qquad (5.7)$$

if $\phi(x)$ is held fixed. Thus, (5.6) becomes

$$\left[\frac{\delta\Gamma[\phi,g_{\mu\nu}]}{\delta g_{\mu\nu}(x)}\right]_\phi=\left[\frac{\delta\Gamma[\phi_0,g_{\mu\nu}]}{\delta g_{\mu\nu}(x)}\right]_{\phi_0}+\frac{w}{2}\phi_0(x)g^{\mu\nu}(x)\left[\frac{\delta\Gamma[\phi_0,g_{\mu\nu}]}{\delta\phi_0(x)}\right] . \qquad (5.8)$$

The other effective field equation is obtained by varying $\Gamma[\phi,g]$ with respect to the basic field variable $\phi(x)$ while holding $g_{\mu\nu}(x)$ fixed. It leads to

$$\left[\frac{\delta\Gamma[\phi,g_{\mu\nu}]}{\delta\phi(x)}\right]_{g_{\mu\nu}}=[g(x)]^{w/2}\left[\frac{\delta\Gamma[\phi_0,g_{\mu\nu}]}{\delta\phi_0(x)}\right]_{g_{\mu\nu}} . \qquad (5.9)$$

It is seen clearly from (5.8) and (5.9) that the dynamics is the same regardless of which field variables are chosen. The difference between the expectation values of the two stress tensors vanishes when the effective field equations hold. It should be pointed out that the situation regarding different choices of field variables is different from that described by Duff[26] which relies on redefining the background metric in a way which involves the quantum fields.

Finally, I wish to make some comments concerning the relationship of this approach to Vilkovisky's[27] formulation of the effective action. Fields should be viewed as coordinates in function space. In Sec. II it was also assumed that they were sections of some vector bundle so that expressions such as (2.8) made sense. This will not be true in general; however, it still makes sense to define an inner product in function space between vectors. (It was done in the case of quantum gravity by DeWitt.[28]) Specifying an inner product gives a metric in function space. As Vilkovisky[27] has discussed, this metric can be used to construct a connection and give a formulation of the effective action which is invariant under arbitrary field redefinitions. All of this supposes that the functional measure is also invariant under field redefinitions. This is guaranteed to be the case if it is defined as described in Sec. II above. This gives the whole approach to quantization by the background-field method a very geometrical flavor.

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