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Massless minimally coupled scalar field in de Sitter space

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We quantize the massless minimally coupled scalar field in de Sitter space, and find a one-complex-parameter family of O(4)-invariant Hadamard Fock vacua which break de Sitter invariance. The different Fock spaces corresponding to the different choices of vacuum are subspaces of a single space of states. We make some remarks about the existence of E(3)- and O(1,3)-invariant Fock vacuum states.

I. INTRODUCTION

The quantum theory of the massless, minimally coupled scalar field propagating in de Sitter space is of fundamental importance in the cosmological context of the very early Universe. It plays a central role in the phenomenon of inflation[1] and it might imply the instability of de Sitter space and thus explain why the cosmological constant is so small today.[2]

It is well known[3] that the Green's functions for a scalar field in the Euclidean vacuum [which is O(1,4) invariant] are infrared divergent in the limits $\xi\rightarrow 0$ and $m\rightarrow 0$ (where ξ and m are, respectively, the coupling constant and the mass of the quanta). This infrared divergence corresponds to the fact that there is no de Sitter-invariant vacuum for the massless minimally coupled scalar field.[4] However, it is possible to look for vacua which break de Sitter invariance but which have as much symmetry as possible,[4] i.e., vacua which are invariant under the six-parameter subgroups of the de Sitter group: O(4), E(3), and O(1,3). [These subgroups are *not* maximal. There exists a seven-parameter subgroup $D\Box E(3)$ of O(1,4) containing an additional dilation D. However the orbit of this subgroup (acting on a point of the de Sitter spacetime) is the whole spacetime. Thus it is not as interesting as the other subgroups, whose orbits are three-dimensional foliations of the spacetime.]

In this paper we clarify some aspects of the quantization of a massless minimally coupled scalar field. In order to simplify the treatment we choose to work only with Hadamard states, i.e., with vacuum states in which the two-point functions have the Hadamard singularity.[5] After a description of the de Sitter manifold and some properties of the O(1,4), O(4), E(3), and O(1,3) groups in Sec. II, we review some aspects of the quantization of a massive scalar field propagating in de Sitter space in Sec. III. In Sec. IV we explain the existence of the infrared divergence in the Euclidean vacuum and we construct a two-real-parameter family of O(4)-invariant Hadamard vacua, which are well-defined Fock states. This family contains a one-parameter subfamily of states which are also time-reversal invariant. This construction clarifies the status of the E(3)-invariant Hadamard "vacuum" found by Allen[4] and of the O(1,3)-invariant one. In Sec. V we explain why the O(1,3)- and E(3)-invariant "states" are not vacuum Fock states but only idealizations.

Our spacetime conventions follow those of Hawking and Ellis[6] and we work in units $\hbar=c=1$.

II. DE SITTER SPACE

de Sitter space[6] is a maximally symmetric spacetime having a positive constant curvature R and topology $R^1\times S^3$. It is locally characterized by the relation $R_{abcd}=\frac{1}{12}R(g_{ac}g_{bd}-g_{ad}g_{bc})$ and it can be easily represented as the four-dimensional hyperboloid

$$\eta_{ab}x^a x^b=H^{-2}\quad (a,b=0,1,2,3,4) \tag{2.1}$$

embedded in a flat five-dimensional space R^5 with metric

$$\eta_{ab}=\mathrm{diag}(-1,1,1,1,1)\ . \tag{2.2}$$

The curvature is then

$$R=12H^2\ . \tag{2.3}$$

The symmetry group O(1,4) of de Sitter space is ten dimensional and has three subgroups O(4), E(3), and O(1,3) which are six dimensional.

Given two points x and x' on the hyperboloid, it is useful to introduce the real quadratic form

$$Z(x,x')=H^2\eta_{ab}x^a x'^b\ . \tag{2.4}$$

Z is invariant under the group O(1,4) and has the properties

$$\begin{aligned} &Z(x,x')>1 \ \text{ if } (x,x') \text{ are timelike related },\\ &Z(x,x')=1 \ \text{ if } (x,x') \text{ are null related },\\ &Z(x,x')<1 \ \text{ if } (x,x') \text{ are spacelike related }. \end{aligned} \tag{2.5}$$

The relationship between Z and the geodesic distance is

$$Z(x,x')=\cos\left[\frac{R}{6}\sigma(x,x')\right]^{1/2}, \tag{2.6}$$

where $\sigma(x,x')$ is one-half the square of the geodesic distance between x and x'.

As is well known, different coordinate systems can be

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used to parametrize de Sitter space.[6,7] In particular, there are three coordinate systems which correspond to three different ways of slicing the manifold into space and time. These correspond to foliations with closed, flat, or open spatial sections that are, respectively, O(4), E(3), and O(1,3) invariant. We will frequently use the subscript $+1$, 0, and -1, respectively, to refer to the time coordinates in these three cases.

The closed coordinates $(t_1,\chi,\theta,\varphi)$ (see Figs. 1 and 2) are defined by the following transformation of coordinates:

$$\begin{aligned} x^0 &= H^{-1}\sinh(Ht_1)\,, \\ x^1 &= H^{-1}\cosh(Ht_1)\cos\chi, \quad -\infty < t_1 < +\infty\,, \\ x^2 &= H^{-1}\cosh(Ht_1)\sin\chi\,\cos\theta, \quad 0\le\chi,\ \theta\le\pi\,, \\ x^3 &= H^{-1}\cosh(Ht_1)\sin\chi\,\sin\theta\cos\varphi, \quad 0\le\varphi<2\pi\,, \\ x^4 &= H^{-1}\cosh(Ht_1)\sin\chi\,\sin\theta\sin\varphi\,. \end{aligned} \tag{2.7}$$

The line element ds^2 is

$$ds^2 = -dt_1{}^2 + H^{-2}\cosh^2(Ht_1)[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\,d\varphi^2)] \tag{2.8}$$

and the quadratic form $Z(x,x')$ can be written as

$$Z(x,x') = -\sinh(Ht_1)\sinh(Ht_1') + \cosh(Ht_1)\cosh(Ht_1')\cos\Omega\,, \tag{2.9}$$

where Ω is the angle between the spacelike components of x and x' on S^3:

$$\cos\Omega = \cos\chi\,\cos\chi' + \sin\chi\,\sin\chi'[\cos\theta\cos\theta' + \sin\theta\,\sin\theta'\cos(\varphi-\varphi')]\,.$$

The coordinates $(t_1,\chi,\theta,\varphi)$ cover the whole de Sitter manifold. The spatial sections $t_1=$const are spheres S^3 of positive curvature and are Cauchy surfaces. O(4) is the set of transformations of de Sitter space which leaves invariant these hypersurfaces. It is also useful to define the conformal time

$$\eta_1 = 2\arctan(e^{Ht_1}), \quad 0\le\eta_1\le\pi \tag{2.10}$$

and the coordinate system $(t_1,\chi,\theta,\varphi)$ for which

$$ds^2 = H^{-2}(\sin\eta_1)^{-2}[-d\eta_1{}^2 + d\chi^2 + \sin^2\chi(d\theta^2+\sin^2\theta\,d\varphi^2)]\,. \tag{2.11}$$

The flat coordinates $(\eta_0,\rho,\theta,\varphi)$ and the open coordinates $(t_{-1},\lambda,\theta,\varphi)$ do not cover all the de Sitter manifold. In the flat coordinates the line element is given by

$$ds^2 = -dt_0{}^2 + H^{-2}e^{2Ht_0}[d\rho^2+\rho^2(d\theta^2+\sin^2\theta\,d\varphi^2)] \tag{2.12}$$

with $-\infty < t_0 < +\infty$ and $0\le\rho<+\infty$. Using the conformal time $\eta_0 = -H^{-1}e^{-Ht_0}$ and allowing η_0 to range over all real numbers, we can cover the whole manifold with the system $(\eta_0,\rho,\theta,\varphi)$. In this coordinate system the metric takes the form

$$ds^2 = H^{-2}\eta_0{}^{-2}[-d\eta_0{}^2 + d\rho^2 + \rho^2(d\theta^2+\sin^2\theta\,d\varphi^2)]\,. \tag{2.13}$$

The spatial sections $\eta_0=$const (or $t_0=$const) are flat. E(3) is the set of transformations of de Sitter space which leave invariant these sections.

In the open coordinates $(t_{-1},\lambda,\theta,\varphi)$ (with $-\infty < t_{-1} < +\infty$ and $0\le\lambda<+\infty$) the metric becomes

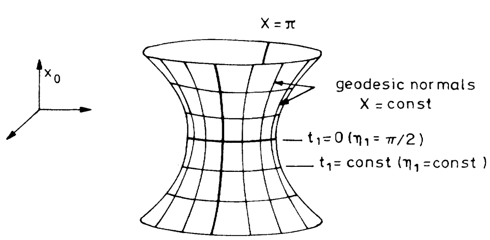

FIG. 1. de Sitter space is a hyperboloid embedded in a five-dimensional flat space (two dimensions are suppressed in the figure). Coordinates $(t_1,\chi,\theta,\varphi)$ or $(\eta_1,\chi,\theta,\varphi)$ cover the whole manifold. The spatial sections $t_1=$const ($\eta_1=$const) are spheres S^3 of positive curvature and are Cauchy surfaces.

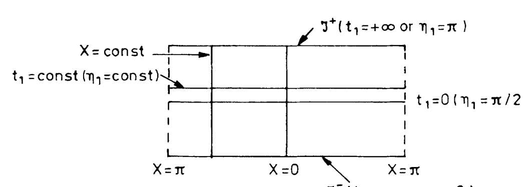

FIG. 2. Penrose diagram of de Sitter space. Null lines are at 45°. The left and right edges of this diagram must be identified along the dashed lines. de Sitter space has a past and future spacelike infinity, for timelike and null geodesics. The surfaces $t_1=$const ($\eta_1=$const) are O(4) invariant.

$$ds^2 = -dt_{-1}{}^2 + H^{-2}\sinh^2(Ht_{-1})[d\lambda^2 + \sinh^2\lambda(d\theta^2 + \sin^2\theta\, d\varphi^2)] \tag{2.14}$$

or, using the conformal time $\eta_{-1} = 2\text{arccoth}^{-1}[\exp(Ht_{-1})]$,

$$ds^2 = H^{-2}(\sinh\eta_{-1})^{-2}[-d\eta_{-1}{}^2 + d\lambda^2 + \sinh^2\lambda(d\theta^2 + \sin^2\theta\, d\varphi^2)] . \tag{2.15}$$

The spatial sections $\eta_{-1} = \text{const}$ (or $t_{-1} = \text{const}$) are O(1,3) invariant. (For more details about the flat and open coordinate systems see Ref. 7.)

III. QUANTIZATION OF THE MASSIVE SCALAR FIELD

In this section we review the canonical quantization of a real scalar field $\phi(x)$ propagating on de Sitter space, previously studied by Chernikov and Tagirov, Schomblond and Spindel, Allen, Mottola, and other authors.[4,7–10]

In order to quantize the scalar field obeying the wave equation

$$(\Box - m^2 - \xi R)\phi = 0 \tag{3.1}$$

we must obtain a complete set of mode solutions of (3.1), which are orthogonal under the inner product

$$(\psi_1, \psi_2) = -i \int_\Sigma d\Sigma^\mu (\psi_1 \overleftrightarrow{\nabla}_\mu \psi_2^*) \tag{3.2}$$

(where Σ is a Cauchy surface), i.e., satisfying

$$(u_n, u_{n'}) = \delta_{nn'}, \quad (u_n, u_{n'}^*) = 0 . \tag{3.3}$$

We then expand the field $\phi(x)$ in the form

$$\phi = \sum_n (a_n u_n + a_n^\dagger u_n^*) \tag{3.4}$$

and we quantize the theory be adopting commutation relations for the operators a_n and $a_n^\dagger$:

$$[a_n, a_{n'}^\dagger] = \delta_{nn'} , \quad [a_n, a_{n'}] = [a_n^\dagger, a_{n'}^\dagger] = 0 . \tag{3.5}$$

We then define a vacuum state $|0\rangle$ by

$$a_n |0\rangle = 0 \quad \forall n \tag{3.6}$$

and by operating on $|0\rangle$ with the creation operators $a_n^\dagger$ one can construct a Fock space. A particularly interesting object which contains all the information about the Fock-space structure is the vacuum expectation value of the anticommutator function:

$$G^{(1)}(x,x') = \langle 0 | \phi(x)\phi(x') + \phi(x')\phi(x) | 0 \rangle = \sum_n [u_n(x)u_n^*(x') + u_n(x')u_n^*(x)] . \tag{3.7}$$

In order to obtain a complete set of orthonormal modes of (3.1) we must specify a coordinate system. In the closed coordinates $(\eta_1, \chi, \theta, \varphi)$ the wave equation can be solved by separation of variables. We look for modes having the form

$$u_{klm}(x) = H \sin\eta_1 X_k(\eta_1) Y_{klm}(\chi, \theta, \varphi) , \tag{3.8}$$

where the Y_{klm} are S^3 spherical harmonics obeying

$$\Delta^3 Y_{klm} = -k(k+2) Y_{klm} \tag{3.9}$$

and where $m = -l, -l+1, \ldots, +l$; $l = 0, 1, \ldots, k$ and $k = 0, 1, \ldots$. (The properties of the S^3 spherical harmonics can be found in the Appendix of Ref. 7.) It should be noted that we have the orthogonality relation

$$\int d\chi\, d\theta\, d\varphi \sin^2\chi \sin\theta\, Y_{klm}(\chi,\theta,\varphi) Y_{k'l'm'}^*(\chi,\theta,\varphi) = \delta_{kk'}\delta_{ll'}\delta_{mm'} \tag{3.10}$$

and that the $\{Y_{klm}\}_{lm}$ form a basis for the $(k+1)^2$-dimensional representation of SO(4). Substituting (3.8) in (3.1) we obtain

$$\frac{d^2 X_k}{d\eta_1{}^2} + \{(k+1)^2 + H^{-2}\sin\eta_1{}^{-2}[m^2 + (\xi - \tfrac{1}{6})R]\} X_k = 0 \tag{3.11}$$

while the relations (3.3) become (taking for Σ a surface $\eta_1 = \text{const}$)

$$X_k \frac{\partial}{\partial \eta_1} X_k^* - X_k^* \frac{\partial}{\partial \eta_1} X_k = i . \tag{3.12}$$

The general solution of (3.11) can be written

$$X_k(\eta_1) = (\sin\eta_1)^{1/2}[A_k P_{k+1/2}^{\nu}(-\cos\eta_1) + B_k Q_{k+1/2}^{\nu}(-\cos\eta_1)] , \tag{3.13}$$

where

$$\nu \equiv \left[\frac{9}{4} - \frac{12}{R}(m^2 + \xi R)\right]^{1/2} \tag{3.14}$$

and the normalization condition (3.12) reduces to

$$A_k B_k^* - B_k A_k^* = i \frac{\Gamma(k + \frac{3}{2} - \nu)}{\Gamma(k + \frac{3}{2} + \nu)} . \tag{3.15}$$

There is an infinity of A_k and B_k satisfying (3.15) and therefore an infinity of vacua. As is well known,[4,7,8] if we only require the de Sitter invariance for the vacuum states, we find there is a one-complex-parameter family of such states, the Euclidean vacuum being a particular member of this family. In this paper we choose to work only with Hadamard vacua. If we require the vacuum state to be O(1,4) invariant and of Hadamard type, i.e., the $G^{(1)}$ function in this state has the Hadamard form for small σ (i.e., $|\sigma R| \ll 1$ and $|\sigma m^2| \ll 1$),

$$G^{(1)}=\frac{1}{(2\pi)^2}\left[\frac{1}{\sigma}+\tfrac{1}{2}[m^2+(\xi-\tfrac{1}{6})R]\ln\sigma+O(\sigma^0)\right] \tag{3.16}$$

then the A_k and B_k are uniquely determined. We have[8,10]

$$A_k=\left[\frac{\pi}{4}\frac{\Gamma(k+\frac{3}{2}-v)}{\Gamma(k+\frac{3}{2}+v)}\right]^{1/2}e^{i(\pi/2)v}\,,$$
$$B_k=-\frac{2i}{\pi}A_k\,, \tag{3.17}$$

and the corresponding vacuum is the Euclidean vacuum.

From (3.7), (3.8), (3.13), and (3.17) $G^{(1)}(x,x')$ can be calculated. When x and x' are spacelike separated $G^{(1)}(x,x')$ is given by[10]

$$G^{(1)}(x,x')=G^{(1)}(Z(x,x'))$$
$$=\frac{R}{96\pi}\frac{\frac{1}{4}-v^2}{\cos\pi v}F\left[\tfrac{3}{2}+v,\tfrac{3}{2}-v;2;\frac{1+Z(x,x')}{2}\right], \tag{3.18}$$

where F is a hypergeometric function. When x and x' are timelike separated $[Z(x,x')>1]$, $G^{(1)}(x,x')$ is obtained from (3.18) by taking the real part of the function analytically continued around the branch cut of F from $Z=+1$ to $+\infty$. The expectation values $\langle\phi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$ in the Euclidean vacuum are given by[11]

$$\langle\phi^2\rangle_{\rm ren}=-\frac{1}{16\pi^2}\left[-\frac{R}{18}+[m^2+(\xi-\tfrac{1}{6})R]\left[\psi(\tfrac{3}{2}+v)+\psi(\tfrac{3}{2}-v)+\ln\frac{R}{12m^2}\right]\right], \tag{3.19}$$

$$\langle T_{\mu\nu}\rangle_{\rm ren}=-\frac{g_{\mu\nu}}{64\pi^2}\left[m^2[m^2+(\xi-\tfrac{1}{6})R]\left[\psi(\tfrac{3}{2}+v)+\psi(\tfrac{3}{2}-v)+\ln\frac{R}{12m^2}\right]\right.$$
$$\left.-m^2(\xi-\tfrac{1}{6})R-\tfrac{1}{18}m^2R-\tfrac{1}{2}(\xi-\tfrac{1}{6})^2R^2+\frac{R^2}{2160}\right]. \tag{3.20}$$

(The arbitrary renormalization mass has been removed by requiring that $\langle\phi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$ vanish in the flat-space limit $R\to 0$).

In this section we have described the quantization of the massive scalar field using the closed coordinates. In fact it is also possible to work with the flat[9] or open coordinates and to construct in these systems the modes giving rise to the Euclidean vacuum.

IV. THE MASSLESS MINIMALLY COUPLED SCALAR FIELD: O(4) HADAMARD VACUA

In this section we study the breakdown of de Sitter invariance for the vacuum state of a massless minimally coupled scalar field, and we construct well-defined Hadamard Fock vacua which are O(4) invariant.

Let us take $\xi=0$ and $m^2\to 0$ in the formulas (3.18), (3.19), and (3.20). One finds

$$G^{(1)}(x,x')=\frac{R^2}{192\pi^2m^2}+\frac{R}{48\pi^2}\left[\frac{1}{1-Z}-\ln(1-Z)\right]$$
$$+O(m^2)\,, \tag{4.1}$$

$$\langle\phi^2\rangle_{\rm ren}=\frac{R^2}{384\pi^2m^2}+O(m^2)\,, \tag{4.2}$$

$$\langle T_{\mu\nu}\rangle_{\rm ren}=-\frac{61R^2}{138\,240\pi^2}g_{\mu\nu}+O(m^2R)\,. \tag{4.3}$$

(In the previous relations we have dropped the constant terms which do not depend on m^2 and on the coordinates of x and x'.) The infrared divergence which appears for $G^{(1)}(x,x')$ and $\langle\phi^2\rangle_{\rm ren}$ defined in the O(1,4)-invariant Hadamard vacuum is well known[3] and corresponds to the breakdown of de Sitter invariance for the vacuum states of the massless minimally coupled scalar field: Allen[4] has shown that no de Sitter-invariant vacuum state exists for this field while other authors[1,12,13] have shown that the vacuum expectation value of ϕ^2, for the massless minimally coupled field, must be time dependent. The time dependence of $\langle\phi^2\rangle$ necessarily breaks O(1,4) invariance because if the vacuum state was de Sitter invariant, it would require $\langle\phi^2\rangle=$const [note that for $m^2=0,\xi=0$, $\langle T_{\mu\nu}\rangle_{\rm ren}$ remains O(1,4) invariant].

The origin of the infrared divergence for the O(1,4)-invariant Hadamard vacuum can be easily understood. For $m^2=0,\xi=0$ the modes $u_{klm}(x)$, $u^*_{klm}(x)$ given by (3.8), (3.13), and (3.17) are well defined if $k\neq 0$ while the "zero modes" $u_{000}(x)$ and $u^*_{000}(x)$ diverge as m^{-1}. More explicitly, using relations between the Legendre functions $P^v_{1/2},Q^v_{1/2}$ and the hypergeometric functions,[14] we find after a tedious calculation that, for $\xi=0$ and $m^2\to 0$, the zero-mode term in the two-point function is

$$u_{000}(x)u^*_{000}(x')+u_{000}(x')u^*_{000}(x)=\frac{R^2}{192\pi^2m^2}+\frac{R}{48\pi^2}[\ln(2\sin\eta_1)+\ln(2\sin\eta'_1)+\sin^2\eta_1+\sin^2\eta'_1]+O(m^2)\ . \quad (4.4)$$

Here again we have dropped the constant term. It will be useful to consider the mode sum, identical in form to the sum (3.7) which defines the two-point function, but *without* the zero modes. We denote this by $G^{(1)}_{\mathrm{NZM}}$ (the subscript means "no zero modes"):

$$G^{(1)}_{\mathrm{NZM}}(x,x')\equiv\sum_{\substack{k,l,m\\k\neq0}}[u_{klm}(x)u^*_{klm}(x')+u_{klm}(x')u^*_{klm}(x)]$$
$$=\frac{R}{48\pi^2}\left[\frac{1}{1-Z}-\ln(1-Z)-\ln(2\sin\eta_1)-\ln(2\sin\eta'_1)-\sin^2\eta_1-\sin^2\eta'_1\right]\ . \quad (4.5)$$

It should be noted that $G^{(1)}_{\mathrm{NZM}}(x,x')$ is not the expectation value of the anticommutator function in some vacuum state because the modes $\{u_{klm}(x)\}_{k\neq0}$ do not form a complete set of modes solutions of the wave equation.

However the wave equation (3.11) with $k=0$, $m^2=0$, and $\xi=0$ allows us to obtain nondivergent zero-mode solutions of the form

$$v_{000}=H[A(\eta_1-\tfrac{1}{2}\sin2\eta_1-\pi/2)+B]Y_{000}(\chi,\theta,\varphi)\ . \quad (4.6)$$

The constants A and B are normalized by the relation

$$A^*B-AB^*=\frac{i}{2}\ . \quad (4.7)$$

Let us now consider the complete set of modes $\{v_{klm}(x)\}_{klm}$ which are solutions of the wave equation $\Box\phi=0$, and are defined by

$$v_{000}(x)=H[A(\eta_1-\tfrac{1}{2}\sin2\eta_1-\pi/2)+B]\times Y_{000}(\chi,\theta,\varphi)\ ,$$
$$v_{klm}(x)=\lim_{\substack{\xi=0\\m^2\to0}}u_{klm}(x)\ \text{ if } k\neq0\ . \quad (4.8)$$

Because this set now forms a complete set of orthonormal modes, one can expand the field ϕ as

$$\phi(x)=\sum_{klm}[a_{klm}v_{klm}(x)+a^\dagger_{klm}v^*_{klm}(x)] \quad (4.9)$$

and define a vacuum state $|0\rangle_{A,B}$ by

$$a_{klm}|0\rangle_{A,B}=0\ \ \forall k,l,m\ . \quad (4.10)$$

Thus, we find that there is a family of Hadamard vacua, depending on the parameters A,B. The vacuum expectation value of $\phi(x)\phi(x')+\phi(x')\phi(x)$ in these vacua is

$$G^{(1)}_{A,B}(x,x')=\sum_{klm}[v_{klm}(x)v^*_{klm}(x')+v_{klm}(x')v^*_{klm}(x)]$$
$$=G^{(1)}_{\mathrm{NZM}}(x,x')+v_{000}(x)v^*_{000}(x')+v_{000}(x')v^*_{000}(x) \quad (4.11)$$

or more explicitly

$$G^{(1)}_{A,B}(x,x')=\frac{R}{48\pi^2}\left[\frac{1}{1-Z}-\ln(1-Z)-\ln(2\sin\eta_1)-\ln(2\sin\eta'_1)-\sin^2\eta_1-\sin^2\eta'_1\right]$$
$$+\frac{R}{24\pi^2}[2AA^*(\eta_1-\tfrac{1}{2}\sin2\eta_1-\pi/2)(\eta'_1-\tfrac{1}{2}\sin2\eta'_1-\pi/2)$$
$$+(AB^*+BA^*)(\eta_1-\tfrac{1}{2}\sin2\eta_1+\eta'_1-\tfrac{1}{2}\sin2\eta'_1-\pi)+2BB^*]+\text{const}\ . \quad (4.12)$$

Obviously, the vacua $|0\rangle_{A,B}$ break O(1,4) invariance but are O(4) invariant. Because a change of phase $A,B\to e^{i\theta}A,e^{i\theta}B$ is unobservable, there exists a two-real-parameter set of vacua, for A and B given by

$$A=-i\alpha\ ,$$
$$B=\alpha^{-1}(\tfrac{1}{4}+i\beta)\ . \quad (4.13)$$

Here the real numbers α and β lie in the range $\alpha\in(0,+\infty)$, $\beta\in(-\infty,+\infty)$.

Now consider the effects of time reversal on the state $|0\rangle_{A,B}$. The antiunitary operator of time reversal will be denoted by T, and changes the time coordinate of a space-time point from η_1 to $\pi-\eta_1$:

$$T^\dagger\phi(\eta_1,\mathbf{\Omega})T=\phi(\pi-\eta_1,\mathbf{\Omega})\ , \quad (4.14)$$

where $\mathbf{\Omega}=(\chi,\theta,\varphi)$ denotes the angular coordinates on S^3.

If the state $|0\rangle_{A,B}$ is invariant under the transformation T then its two-point function $G^{(1)}$ transforms in the following way:

$$G^{(1)}(\pi-\eta_1,\mathbf{\Omega};\pi-\eta_1',\mathbf{\Omega}')=G^{(1)}(\eta_1,\mathbf{\Omega};\eta_1',\mathbf{\Omega}') . \quad (4.15)$$

Condition (4.15) implies that time-reversal-invariant states obey

$$AB^*+BA^*=0 . \quad (4.16)$$

Together with the normalization condition (4.7) this gives a one-real-parameter family of time-reversal- and O(4)-invariant vacuum states:

$$A=-i\alpha, \quad B=(4\alpha)^{-1} , \quad (4.17)$$

where α is a real number $\alpha>0$. In fact it can be easily shown that this is because the time reversal of the vacuum state $|0\rangle_{A,B}$ is

$$T|0\rangle_{A,B}=|0\rangle_{-A^*,B^*} .$$

Thus in the two-parameter family of O(4)-invariant vacua labeled by (α,β) it is only those states with $\beta=0$ which are time-reversal invariant.

In summary, we see that although there is no de Sitter-invariant Hadamard vacuum for the massless minimally coupled scalar field, there does exist an infinity of O(4)-invariant Hadamard vacua, which are well defined and whose correlation functions are free of infrared divergences.

A calculation of the renormalized vacuum expectation value of the stress tensor in the vacuum state $|0\rangle_{A,B}$ can be easily done using the Hadamard formalism.[15] We find that

$$\langle T_{\mu\nu}\rangle_{A,B\,\mathrm{ren}}=\frac{119R^2}{138\,240\pi^2}g_{\mu\nu} . \quad (4.18)$$

$\langle T_{\mu\nu}\rangle_{A,B\,\mathrm{ren}}$ (1) is de Sitter invariant, (2) does not depend on A and B and (3) is different than the value given by considering the limit $\xi\to 0$, $m^2\to 0$ of (3.20), which is given in (4.3). Thus one can see that the stress tensor in a O(1,4)-invariant Hadamard vacuum state does not see the infrared divergence but the breakdown of O(1,4) invariance changes the value of the stress tensor.

The vacuum expectation value of ϕ^2 can also be easily found:

$$\langle\phi^2\rangle_{A,B\,\mathrm{ren}}=\mathrm{const}-\frac{R}{48\pi^2}(\ln 2\sin\eta_1+\sin^2\eta_1)+\frac{R}{24\pi^2}|A(\eta_1-\tfrac{1}{2}\sin 2\eta_1-\pi/2)+B|^2 . \quad (4.19)$$

In the early and late-time limits $|t_1|\to 0,\pm\infty$ we find that this is asymptotic to

$$\langle\phi^2\rangle_{A,B\,\mathrm{ren}}\simeq\mathrm{const}+\frac{H^2}{2\pi^2}|B|^2+\frac{H^3}{\pi^2}(AB^*+BA^*)t_1+\frac{H^4}{8\pi^2}(3+16|A|^2)t_1^2+O(t_1^3), \quad |Ht_1|\ll 1 , \quad (4.20)$$

$$\langle\phi^2\rangle_{A,B\,\mathrm{ren}}\simeq\frac{H^3}{4\pi^2}|t_1|, \quad |Ht_1|\gg 1 . \quad (4.21)$$

The quantization of the massless minimally coupled scalar field done previously and the existence of an infinity of vacua linked to an infinity of possible choices for the zero modes can be understood in another way. We follow DeWitt's treatment of a massless field in a compact and stationary universe.[16] Although de Sitter space is not stationary a construction similar to that developed in Ref. 16 can be done here. We define the Hermitian operators P and Q by

$$P\equiv Aa_0+A^*a_0^\dagger , \quad (4.22)$$

$$Q\equiv 2(Ba_0+B^*a_0^\dagger) , \quad (4.23)$$

and we write the field operator (4.9) as

$$\phi(x)=\sum_{\substack{klm\\k\neq 0}}[a_{klm}v_{klm}(x)+a_{klm}^\dagger v_{klm}^*(x)]+\frac{H}{2\sqrt{2}\pi}Q+\frac{H}{\sqrt{2}\pi}P(\eta_1-\tfrac{1}{2}\sin 2\eta_1-\pi/2) \quad (4.24)$$

with the operators $a_{klm},a_{klm}^\dagger(k\neq 0)$ and Q,P which satisfy

$$[a_{klm},a_{k'l'm'}]=[a_{klm},Q]=[a_{klm},P]=0 ,$$
$$[a_{klm},a_{k'l'm'}^\dagger]=\delta_{kk'}\delta_{ll'}\delta_{mm'} , \quad (4.25)$$
$$[Q,P]=i .$$

All reference to the constants A and B [introduced in the zero modes (4.8)] has now disappeared but the construction of the space of states does not proceed as usual. The operators of the theory are represented in a space spanned by the $a_{klm}^\dagger(k\neq 0)$ and e^{ipQ} acting on an unnormalizable "ground state" denoted $|p=0\rangle$ and defined by

$$a_{klm}|p=0\rangle=0 \quad \forall k,l,m,\ k\neq 0 ,$$
$$P|p=0\rangle=0 . \quad (4.26)$$

The "state" $|p=0\rangle$ can be considered as carrying "zero momentum" and an eigenstate $|p\rangle$ of "momentum" p is defined by

$$|p\rangle\equiv e^{ipQ}|p=0\rangle , \quad (4.27)$$

and satisfies

$$P|p\rangle=p|p\rangle ,$$
$$a_{klm}|p\rangle=0 \quad \forall k,l,m,\ k\neq 0 , \quad (4.28)$$
$$\langle p|p'\rangle=\delta(p-p') .$$

A basis of the space of states is given by the vectors

$$(a_{k_1l_1m_1}^\dagger)^{i_1}\cdots(a_{k_nl_nm_n}^\dagger)^{i_n}e^{ipQ}|p=0\rangle . \quad (4.29)$$

The connection between the two different ways of constructing the space of states can be easily made by remarking that up to a phase

$$|0\rangle_{A,B}=\left[\frac{\pi}{2BB^*}\right]^{1/4}\exp\left[i\frac{A^*}{4B^*}Q^2\right]|p=0\rangle . \quad (4.30)$$

Using the states $\{|q\rangle\}_q$ which satisfy $Q|q\rangle = q|q\rangle$, $\langle q'|q\rangle = \delta(q-q')$, and $\langle q|p\rangle = (2\pi)^{-1/2}e^{ipq}$, one can also express the Fock vacuum state in the form

$$|0\rangle_{A,B} = \left[\frac{1}{8\pi BB^*}\right]^{1/4} \int |q\rangle \exp\left[i\frac{A^*}{4B^*}q^2\right] dq \quad (4.31)$$

and

$$|0\rangle_{A,B} = \left[\frac{1}{2\pi AA^*}\right]^{1/4} \int |p\rangle \exp\left[-i\frac{B^*}{A^*}p^2\right] dp \; . \quad (4.32)$$

Note that these integrals converge because the normalization condition (4.7) implies that $\mathrm{Re}(iA^*/B^*) < 0$ and hence that the integrals (4.31) and (4.32) contain a Gaussian damping factor.

Obviously the new space of states contains an infinity of Fock spaces constructed from the vacua $|0\rangle_{A,B}$. It may be that these different Fock spaces are related to the invariance of the Lagrangian under the global transformations $\phi \rightarrow \phi + \mathrm{const}$.

V. MASSLESS MINIMALLY COUPLED SCALAR FIELD: E(3) AND O(1,3) HADAMARD "VACUA"

Here we make some remarks about the E(3)-invariant "vacuum" found by Allen[4] and about the analogous O(1,3)-invariant "vacuum."

Working in the flat coordinates $(\eta_0,\rho,\theta,\varphi)$, Allen has found a family of E(3)-invariant "vacua." Let us consider the Hadamard "vacuum" of this family. In flat coordinates the Green's functions are defined by continuous sums of the type $\int_m^\infty dk\, f(k)$, where m is an infrared cutoff, or regulator. These integrals are not finite as $m \rightarrow 0$. The expectation value of the anticommutator in the Hadamard "vacuum" is[4]

$$G^{(1)}(x,x') = \frac{R}{48\pi^2}\left[\frac{1}{1-Z} - \ln(1-Z) - \ln(2\eta_0\eta_0')\right] + \frac{R}{48\pi^2}\ln\frac{m^2}{R} + \mathrm{const} \; . \quad (5.1)$$

In light of Sec. IV we can understand the regularization process of Allen as follows: the introduction of a regulator m drops the contribution of the zero modes, i.e., it excludes the range $k \in [0,m]$ from the mode sum.

But this means that one does not have a complete set of modes. Thus the "state" giving rise to (5.1) is not in fact a Fock vacuum state. Here, because the spatial sections are not compact, it is impossible to save the situation by introducing new normalized zero modes, as we have done in Sec. IV. In fact, Ford and Vilenkin[17] have shown that Allen's E(3) "vacuum" can be considered as the unrealizable limit of a continuous family of Fock vacuum states. Thus the E(3)-invariant "vacua" of Allen are an idealization.

Similar considerations apply to the open coordinates $(\eta_{-1},\lambda,\theta,\varphi)$ and the corresponding O(1,3) vacua. In the same way one should be able to show that there exists an O(1,3)-invariant Hadamard "vacuum" which is not a physical vacuum state but is also an idealization and whose symmetric function is given by

$$G^{(1)}(x,x') = \frac{R}{48\pi^2}\left[\frac{1}{1-Z} - \ln(1-Z) - \ln(2\sinh\eta_{-1}) - \ln(2\sinh\eta'_{-1}) + \sinh^2\eta_{-1} + \sinh^2\eta'_{-1}\right] + \frac{R}{48\pi^2}\ln\frac{m^2}{R} + \mathrm{const} \; . \quad (5.2)$$

VI. CONCLUSION

The quantization of the massless minimally coupled scalar field in de Sitter space necessarily breaks de Sitter invariance. For this reason, it is interesting to look for vacua which break de Sitter invariance as little as possible.

We have constructed a two-real-parameter family of Fock vacuum states which are invariant under the O(4) subgroup of the de Sitter group. This family contains a one-parameter subfamily which is also invariant under the time-symmetry reflection T. It is noteworthy that the stress tensor in these states is independent of the state, and that it is not equal to the naive $\xi \rightarrow 0, m^2 \rightarrow 0$ limit of the massive de Sitter-invariant one. There are no infrared "problems": the two-point functions of these vacua approach zero at large spacelike separations. We have given a construction of the complete Fock space of states, for each choice of a vacuum, and have shown that they are embedded in a larger "overcomplete" space of states.

Finally we have shown that the E(3)-invariant states constructed by Allen are not Fock vacuum states because they correspond to a regularization procedure which leaves out the zero modes.

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[1]A. Linde, Phys. Lett. **116B**, 335 (1982).
[2]L. H. Ford, Phys. Rev. D **31**, 710 (1985).
[3]L. H. Ford and L. Parker, Phys. Rev. D **16**, 245 (1977).
[4]B. Allen, Phys. Rev. D 32, 3136 (1985).
[5]J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, CT, 1923).
[6]S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Spacetime* (Cambridge University Press, London, 1973).
[7]E. Mottola, Phys. Rev. D **31**, 754 (1985).

[8]N. A. Chernikov and E. A. Tagirov, Ann. Inst. Henri Poincaré **A9**, 109 (1968); E. A. Tagirov, Ann. Phys. (N.Y.) **76**, 561 (1973).

[9]C. Schomblond and P. Spindel, Ann. Inst. Henri Poincaré **A25**, 67 (1976).

[10]N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, London, 1982).

[11]T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London **A360**, 117 (1978).

[12]A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).

[13]A. Vilenkin and L. H. Ford, Phys. Rev. D **26**, 1231 (1982).

[14]*Higher Transcendental Functions* (Bateman manuscript project) edited by A. Erdelyi *et al.* (McGraw-Hill, New York, 1953), Vol. 1, Eqs. 3.4.9 and 3.2.44.

[15]M. R. Brown and A. C. Ottewill, Phys. Rev. D **34**, 1776 (1986); D. Bernard and A. Folacci, *ibid.* **34**, 2286 (1986).

[16]B. S. DeWitt, in *Les Houches 1983,* edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1983).

[17]L. H. Ford and A. Vilenkin, Phys. Rev. D **33**, 2833 (1986).