Test-particle motion in the nonsymmetric gravitation theory

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(Received 21 May 1986)

A derivation of the motion of test particles in the nonsymmetric gravitational theory (NGT) is given using the field equations in the presence of matter. The motion of the particle is governed by the Christoffel symbols, which are formed from the symmetric part of the fundamental tensor $g_{\mu\nu}$, as well as by a tensorial piece determined by the skew part of the contracted curvature tensor $R_{\mu\nu}$. Given the energy-momentum tensor for a perfect fluid and the definition of a test particle in the NGT, the equations of motion follow from the conservation laws. The tensorial piece in the equations of motion describes a new force in nature that acts on the conserved charge in a body. Particles that carry this new charge do not follow geodesic world lines in the NGT, whereas photons do satisfy geodesic equations of motion and the equivalence principle of general relativity. Astronomical predictions, based on the exact static, spherically symmetric solution of the field equations in a vacuum and the test-particle equations of motion, are derived in detail. The maximally extended coordinates that remove the event-horizon singularities in the static, spherically symmetric solution are presented. It is shown how an inward radially falling test particle can be prevented from forming an event horizon for a value greater than a specified critical value of the source charge. If a test particle does fall through an event horizon, then it must continue to fall until it reaches the singularity at $r=0$.

I. THE NONSYMMETRIC FIELD EQUATIONS

The motion of a particle in Einstein's general relativity (GR) is determined by the nonlinear field equations of the gravitational field. This was recognized early for the restricted problem of the motion of a test particle represented by a local concentration of the energy-momentum tensor $T^{\mu\nu}$. The electrically neutral particle is pictured as a narrow tube of timelike direction, $T^{\mu\nu}$ being nonzero inside the tube and zero outside. A test particle is obtained from the limiting procedure in which the tube shrinks to a world line, while the mass of the particle, represented by an integral of $T^{\mu\nu}$, tends to zero. This world line describes a test particle. A consequence of the conservation equations is that the world line of the test particle is not arbitrary but must be a geodesic of the continuous metric field.[1–4] In the nonsymmetric gravitation theory[5–11] (NGT), the motion of a test particle can also be determined by the generalized conservation equations, although now the definition of a test particle must be extended to include a new source S^μ, which corresponds to a conserved current density in a body.[6,7] The particle has a concentration of the current vector S^μ in the timelike tube, and as the tube shrinks to the world line, the charge described by an integral of S^μ tends to zero. The ratio of the charge to the mass of the test particle does not vanish as the world line is approached, and the test particle does not follow a geodesic unless the test-particle charge is identically zero. A photon in the NGT will have zero charge and move along a geodesic and satisfy locally the (weak) principle of equivalence.

The mathematical formulation of the NGT is based on a nonsymmetric field structure with a nonsymmetric fundamental tensor $g_{\mu\nu}$, which can be decomposed according to

$$g_{\mu\nu}=g_{(\mu\nu)}+g_{[\mu\nu]}\ , \tag{1.1}$$

where $g_{(\mu\nu)}=\frac{1}{2}(g_{\mu\nu}+g_{\nu\mu})$ and $g_{[\mu\nu]}=\frac{1}{2}(g_{\mu\nu}-g_{\nu\mu})$. A contravariant tensor $g^{\mu\nu}$ satisfies the relationship

$$g^{\mu\nu}g_{\sigma\nu}=g^{\nu\mu}g_{\nu\sigma}=\delta^\mu_\sigma\ . \tag{1.2}$$

The fundamental geometrical object in the NGT is the nonsymmetric connection defined by

$$W^\lambda_{\mu\nu}=\Gamma^\lambda_{\mu\nu}-\tfrac{2}{3}\delta^\lambda_\mu W_\nu\ , \tag{1.3}$$

where

$$W_\nu\equiv W^\alpha_{[\nu\alpha]}=\tfrac{1}{2}(W^\alpha_{\nu\alpha}-W^\alpha_{\alpha\nu})\ . \tag{1.4}$$

From (1.3) it can be shown that

$$\Gamma_\mu\equiv\Gamma^\alpha_{[\mu\alpha]}=0\ . \tag{1.5}$$

A curvature tensor can be formed from the connection $W^\lambda_{\mu\nu}$:

$$B^\sigma_{\mu\nu\rho}=W^\sigma_{\mu\nu,\rho}-W^\sigma_{\mu\rho,\nu}-W^\sigma_{\alpha\nu}W^\alpha_{\mu\rho}+W^\sigma_{\alpha\rho}W^\alpha_{\mu\nu} \tag{1.6}$$

and a contracted curvature tensor

$$B_{\mu\nu}=W^\beta_{\mu\nu,\beta}-W^\beta_{\mu\beta,\nu}-W^\beta_{\alpha\nu}W^\alpha_{\mu\beta}+W^\beta_{\alpha\beta}W^\alpha_{\mu\nu}\ , \tag{1.7}$$

where we have used the notation $X_{,\nu}=\partial X/\partial x^\nu$. By symmetrizing $B_{\mu\nu}$ in the second term, we get

$$R_{\mu\nu}(W)=W^\beta_{\mu\nu,\beta}-\tfrac{1}{2}(W^\beta_{\mu\beta,\nu}+W^\beta_{\nu\beta,\mu})-W^\beta_{\alpha\nu}W^\alpha_{\mu\beta}+W^\beta_{\alpha\beta}W^\alpha_{\mu\nu}\ . \tag{1.8}$$

Substitution of (1.3) into (1.8) gives

$$R_{\mu\nu}(W)=R_{\mu\nu}(\Gamma)+\tfrac{2}{3}W_{[\mu,\nu]}\ , \tag{1.9}$$

where

$$R_{\mu\nu}(\Gamma)=\Gamma^{\beta}_{\mu\nu,\beta}-\tfrac{1}{2}(\Gamma^{\beta}_{(\mu\beta),\nu}+\Gamma^{\beta}_{(\nu\beta),\mu}) -\Gamma^{\beta}_{\alpha\nu}\Gamma^{\alpha}_{\mu\beta}+\Gamma^{\beta}_{(\alpha\beta)}\Gamma^{\alpha}_{\mu\nu}\ . \tag{1.10}$$

The variational principle, based on a Lagrangian density, plays a fundamental role in the development of GR. Likewise, in the NGT the choice of a Lagrangian density determines the structure of the theory. The principle of transposition symmetry plays an important role in removing the arbitrariness in the choice of a Lagrangian density.[12] A quantity $T^{\cdots}_{\cdots\mu\nu\cdots}(\Gamma)$ will be called transposition symmetric in the indices μ and ν if $T^{\cdots}_{\cdots\mu\nu\cdots}(\widetilde{\Gamma})=T^{\cdots}_{\cdots\nu\mu\cdots}(\Gamma)$, where $\widetilde{\Gamma}^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\nu\mu}$ defines the operation of *transposition* for the affine connection. If a tensor $A_{\mu\nu}(\widetilde{\Gamma})$ is transposition symmetric, then a field equation of the form $A_{\mu\nu}(\Gamma)=0$ implies the field equation $A_{\mu\nu}(\widetilde{\Gamma})=0$ and we say that this system of equations is transposition invariant. The operation of raising the suffixes α and β for a tensor $A_{\alpha\beta}$, while simultaneously retaining the transposition symmetry of $A_{\alpha\beta}$, can be implemented by using the expression $A^{\mu\nu}=g^{\mu\alpha}g^{\beta\nu}A_{\alpha\beta}$, where the tensor $A_{\alpha\beta}$ satisfies $\widetilde{A}_{\alpha\beta}=A_{\beta\alpha}$.

The property of transposition symmetry replaces the symmetry in Riemannian geometry; it severely restricts the arbitrariness of the NGT. We shall require that all the equations derived from a variational principle be transposition invariant. Once the principle of transposition invariance is adopted, then the field equations of the NGT follow uniquely from the variational principle, using the Palatini method, up to a normalization of the nonsymmetric connection, if we assume that the Lagrangian density is linear in the contracted curvature tensor $R_{\mu\nu}$. The physical interpretation of transposition invariance is that the same field equations are satisfied by a particle and an antiparticle. The operation of transposition conjugation corresponds to passing from a particle to an antiparticle. All observable quantities, such as the flux of radiation from a massive body, must be derived in the NGT from a transposition symmetric formula. The mathematical basis for the transposition conjugation operation has been described by a hyperbolic complex tangent space that leads to the local group GL(4,R) (Ref. 9).

The Lagrangian density has the form[6]

$$\mathcal{L}=\mathfrak{g}^{\mu\nu}R_{\mu\nu}(W)+\mathcal{L}_m\ , \tag{1.11}$$

where $\mathfrak{g}^{\mu\nu}=(-g)^{1/2}g^{\mu\nu}$. The matter Lagrangian density $\mathcal{L}_m$ is given by

$$\mathcal{L}_m=-8\pi g^{\mu\nu}\mathcal{T}_{\mu\nu}+\frac{8\pi}{3}W_{\mu}\mathcal{S}^{\mu}\ . \tag{1.12}$$

By means of the variational principle

$$\delta\int\mathcal{L}\,d^4x=0\ , \tag{1.13}$$

varying $g_{\mu\nu}$ and $W^{\lambda}_{\mu\nu}$ as independent field variables, we get the field equations

$$G_{\mu\nu}(W)=8\pi T_{\mu\nu}\ , \tag{1.14}$$

$$\mathfrak{g}^{[\mu\nu]}{}_{,\nu}=4\pi\mathcal{S}^{\mu}\ , \tag{1.15}$$

$$g_{\mu\nu,\sigma}-g_{\rho\nu}\Lambda^{\rho}_{\mu\sigma}-g_{\mu\rho}\Lambda^{\rho}_{\sigma\nu}=0\ . \tag{1.16}$$

The tensor $G_{\mu\nu}$ is defined by

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R\ , \tag{1.17}$$

where $R_{\mu\nu}=g_{\mu\alpha}g_{\beta\nu}R^{\alpha\beta}$ and $R=g^{\mu\nu}R_{\mu\nu}$. The connection $\Lambda^{\lambda}_{\mu\nu}$ is defined by the equation

$$\Lambda^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu}+D^{\lambda}_{\mu\nu}\ , \tag{1.18}$$

where the tensor $D^{\lambda}_{\mu\nu}$ is given by

$$g_{\rho\nu}D^{\rho}_{\mu\sigma}+g_{\mu\rho}D^{\rho}_{\sigma\nu}=-\left[\frac{4\pi}{3}\right]S^{\rho}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\rho}g_{\sigma\nu}+g_{\mu\nu}g_{[\sigma\rho]})\ . \tag{1.19}$$

In the NGT the phenomenological sources $\mathcal{T}^{\mu\nu}$ and $\mathcal{S}^{\mu}$ are interpreted as the generalized energy-momentum tensor and the conserved NGT current density, respectively. The conservation equation

$$\mathcal{S}^{\mu}{}_{,\mu}=0 \tag{1.20}$$

follows from the field equation (1.15). This conservation law is a consequence of the invariance of the Lagrangian density $\mathcal{L}$ with respect to the Abelian transformation

$$W'_{\mu}=W_{\mu}+\lambda_{,\mu}\ , \tag{1.21}$$

where λ is an arbitrary scalar field. This invariance follows immediately from an inspection of Eq. (1.9). The group of transformations associated with general coordinate transformations in the four-dimensional manifold is extended in the NGT to include the R_+ [or U(1)] transformation (1.21).

The NGT charge of a body is given by the equation[6]

$$l^2=\int\mathcal{S}^0d^3x\ . \tag{1.22}$$

The energy-momentum tensor $T^{\mu\nu}$ for a perfect fluid in the NGT can be derived from a variational principle.[13] The matter Lagrangian density $\mathcal{L}_m$ now takes the form

$$\mathcal{L}_m=\frac{8\pi}{3}W_{\mu}\mathcal{S}^{\mu}+16\pi(-g)^{1/2}\rho_0[\epsilon(\rho_0,s)+1]+16\pi(-g)^{1/2}\lambda_1(g_{(\mu\nu)}u^{\mu}u^{\nu}-1)+16\pi\lambda_2[(-g)^{1/2}\rho_0u^{\mu}]_{,\mu} +\lambda_3(-g)^{1/2}X_{,\mu}u^{\mu}+\lambda_4(-g)^{1/2}s_{,\mu}u^{\mu}\ , \tag{1.23}$$

where ρ_0 is the rest mass density, $\epsilon(\rho_0,s)$ is the rest specific internal energy of the fluid, s is the entropy, and u^{μ} is the four-velocity vector of the fluid element following the world line $x^{\mu}(\tau)$. The Lagrange multipliers λ_1, λ_2, λ_3, and λ_4 force the following constraints to be satisfied.

(a) The normalization of the velocity vectors

$$g_{\mu\nu}u^\mu u^\nu = g_{(\mu\nu)}u^\mu u^\nu = 1 \ . \tag{1.24}$$

(b) Conservation of the rest mass of the system

$$[(-g)^{1/2}\rho_0 u^\mu]_{,\mu} = 0 \ . \tag{1.25}$$

(c) Conservation of particle number for the fluid

$$X_{,\mu}u^\mu = 0 \ , \tag{1.26}$$

where X is a number assigned to each particle element of the fluid. A variation of $\mathscr{L}_m$ in (1.23) leads to the following form for the energy-momentum tensor of a perfect fluid[13]:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \ , \tag{1.27}$$

where ρ denotes the total energy density. This energy-momentum tensor has the same form as in GR.

II. BIANCHI IDENTITIES AND THE CONSERVATION LAWS

From the coordinate invariance of the Lagrangian density (1.11), we can obtain the four Bianchi identities in the NGT:

$$[\mathfrak{g}^{\alpha\nu}G_{\rho\nu}(\Gamma) + \mathfrak{g}^{\nu\alpha}G_{\nu\rho}(\Gamma)]_{,\alpha} + g^{\mu\nu}{}_{,\rho}\mathscr{G}_{\mu\nu} = 0 \ . \tag{2.1}$$

If we form the scalar curvature $R(W)$ from (1.9), we get

$$R(W) \equiv g^{\mu\nu}R_{\mu\nu}(W) = R(\Gamma) + \tfrac{2}{3}g^{[\mu\nu]}W_{[\mu,\nu]} \tag{2.2}$$

and using (1.9) and (1.17), we find that

$$\begin{aligned} G_{\mu\nu}(W) &= R_{\mu\nu}(\Gamma) + \tfrac{2}{3}W_{[\mu,\nu]} \\ &\quad - \tfrac{1}{2}g_{\mu\nu}R(\Gamma) - \tfrac{1}{3}g_{\mu\nu}g^{[\alpha\beta]}W_{[\alpha,\beta]} \\ &= G_{\mu\nu}(\Gamma) + \tfrac{2}{3}W_{[\mu,\nu]} - \tfrac{1}{3}g_{\mu\nu}g^{[\alpha\beta]}W_{[\alpha,\beta]} \ . \end{aligned} \tag{2.3}$$

The field equation (1.14) together with (2.3) yields

$$G_{\mu\nu}(\Gamma) = 8\pi T_{\mu\nu} - \tfrac{2}{3}W_{[\mu,\nu]} + \tfrac{1}{3}g_{\mu\nu}g^{[\alpha\beta]}W_{[\alpha,\beta]} \ . \tag{2.4}$$

Substituting (2.4) into the Bianchi identities (2.1) results in the equation

$$\begin{aligned} &8\pi(g^{\alpha\nu}\mathscr{T}_{\rho\nu} + g^{\nu\alpha}\mathscr{T}_{\nu\rho})_{,\alpha} - \tfrac{2}{3}(\mathfrak{g}^{\alpha\nu}W_{[\rho,\nu]} + \mathfrak{g}^{\nu\alpha}W_{[\nu,\rho]})_{,\alpha} + \tfrac{1}{3}(\mathfrak{g}^{\alpha\nu}g_{\rho\nu}g_{[\sigma\beta]}W_{[\sigma,\beta]} + g^{\nu\alpha}g_{\nu\rho}g^{[\sigma\beta]}W_{[\sigma,\beta]})_{,\alpha} + 8\pi g^{\mu\nu}{}_{,\rho}\mathscr{T}_{\mu\nu} \\ &\qquad - \tfrac{2}{3}g^{\sigma\beta}{}_{,\rho}(-g)^{1/2}W_{[\sigma,\beta]} + \tfrac{1}{3}(-g)^{1/2}g^{\mu\nu}{}_{,\rho}g_{\mu\nu}g^{[\sigma\beta]}W_{[\sigma,\beta]} = 0 \ . \end{aligned} \tag{2.5}$$

The second term in (2.5) can be calculated to give

$$\tfrac{2}{3}(\mathfrak{g}^{\alpha\nu}{}_{,\alpha}W_{[\rho,\nu]} + \mathfrak{g}^{\nu\alpha}{}_{,\alpha}W_{[\nu,\rho]} + \mathfrak{g}^{\alpha\nu}W_{[\rho,\nu]_{,\alpha}} + \mathfrak{g}^{\nu\alpha}W_{[\nu,\rho]_{,\alpha}}) = -\frac{16\pi}{3}W_{[\rho,\nu]}\mathscr{S}^\nu - \tfrac{4}{3}\mathfrak{g}^{[\nu\alpha]}W_{[\rho,\nu]_{,\alpha}} \ . \tag{2.6}$$

From the field equations (1.16), we can derive the relation

$$g_{\mu\nu}g^{\mu\nu}{}_{,\rho} = 2\Lambda^\sigma_{(\rho\sigma)} \tag{2.7}$$

which yields

$$(-g)^{1/2}{}_{,\rho} = (-g)^{1/2}\Lambda^\sigma_{(\rho\sigma)} \ . \tag{2.8}$$

By using (2.6) and (2.8) in (2.5), we get

$$\tfrac{1}{2}(g^{\nu\alpha}\mathscr{T}_{\nu\rho} + g^{\alpha\nu}\mathscr{T}_{\rho\nu})_{,\alpha} + \tfrac{1}{2}g^{\alpha\beta}{}_{,\rho}\mathscr{T}_{\alpha\beta} + \tfrac{1}{3}W_{[\rho,\nu]}\mathscr{S}^\nu + \tfrac{2}{3}\mathfrak{g}^{[\nu\alpha]}W_{[\rho,\nu]_{,\alpha}} + \tfrac{1}{3}\mathfrak{g}^{[\sigma\beta]}W_{[\sigma,\beta]_{,\rho}} = 0 \ . \tag{2.9}$$

The last two terms in (2.9) give

$$\begin{aligned} \tfrac{2}{3}\mathfrak{g}^{[\nu\alpha]}W_{[\rho,\nu]_{,\alpha}} + \tfrac{1}{3}\mathfrak{g}^{[\sigma\beta]}W_{[\sigma,\beta]_{,\rho}} &= \tfrac{1}{3}(\mathfrak{g}^{[\nu\alpha]}W_{[\rho,\nu]_{,\alpha}} + \mathfrak{g}^{[\alpha\nu]}W_{[\rho,\alpha]_{,\nu}} + \mathfrak{g}^{[\sigma\beta]}W_{[\sigma,\beta]_{,\rho}}) \\ &= \tfrac{1}{3}\mathfrak{g}^{[\nu\alpha]}(W_{[\rho,\nu]_{,\alpha}} + W_{[\nu,\alpha]_{,\rho}} + W_{[\alpha,\rho]_{,\nu}}) = 0 \ . \end{aligned} \tag{2.10}$$

We now obtain the result

$$\tfrac{1}{2}(g^{\nu\alpha}\mathscr{T}_{\nu\rho} + g^{\alpha\nu}\mathscr{T}_{\rho\nu})_{,\alpha} + \tfrac{1}{2}g^{\alpha\beta}{}_{,\rho}\mathscr{T}_{\alpha\beta} + \tfrac{1}{3}W_{[\rho,\nu]}\mathscr{S}^\nu = 0 \ . \tag{2.11}$$

By using the expressions $\mathscr{T}^{\sigma\nu} = g^{\alpha\nu}g^{\sigma\beta}\mathscr{T}_{\alpha\beta}$ and $g^{\alpha\beta}{}_{,\rho} = -g^{\alpha\nu}g^{\sigma\beta}g_{\sigma\nu_{,\rho}}$ we can write (2.11) in the form

$$\tfrac{1}{2}(g_{\sigma\rho}\mathscr{T}^{\sigma\alpha} + g_{\rho\sigma}\mathscr{T}^{\alpha\sigma})_{,\alpha} - \tfrac{1}{2}g_{\mu\beta_{,\rho}}\mathscr{T}^{\mu\beta} + \tfrac{1}{3}W_{[\rho,\nu]}\mathscr{S}^\nu = 0 \ . \tag{2.12}$$

Carrying out the differentiation in the first term of (2.12), we obtain the four conservation laws in the NGT:

$$\tfrac{1}{2}(g_{\mu\rho}\mathscr{T}^{\mu\nu}{}_{,\nu} + g_{\rho\mu}\mathscr{T}^{\nu\mu}{}_{,\nu}) + [\mu\nu,\rho]\mathscr{T}^{\mu\nu} + \tfrac{1}{3}W_{[\rho,\nu]}\mathscr{S}^\nu = 0 \ , \tag{2.13}$$

where $[\mu\nu,\rho]$ is

$$[\mu\nu,\rho] = \tfrac{1}{2}(g_{\mu\rho,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) \ . \tag{2.14}$$

Equation (2.13) represents the rigorous conservation laws of the NGT, which will be used to determine the equations of motion of particles.[14]

III. EQUATIONS OF MOTION OF TEST PARTICLES

Since we are presently only interested in test-particle motion, we shall assume that the particle is confined to a tube. The linear dimensions of the cross section of the tube Σ are small compared with the length R characterizing the gradient of the background metric. Let us describe the fundamental tensor $g_{\mu\nu}$ as consisting of two parts, the piece $g^{(0)}_{\mu\nu}$ corresponding to the continuous field at points along the world line of the test particle, and the part $\delta g_{\mu\nu}$ that describes the correction to the background $g^{(0)}_{\mu\nu}$ field due to the field of the test particle.[15] We can then write

$$g_{\mu\nu}=g^{(0)}_{\mu\nu}+\delta g_{\mu\nu} . \tag{3.1}$$

To the first-order of approximation, it is sufficient to keep in the field equations only the terms which are linear in $\delta g_{\mu\nu}$. The energy-momentum tensor associated with the test particle is $\delta T^{\mu\nu}$ and the current density belonging to the particle is δS^{μ}. We shall assume that the test particle is moving outside massive bodies, so that the matter tensor $T^{(0)\mu\nu}$ and the current $S^{(0)\mu}$, associated with the background metric $g^{(0)}_{\mu\nu}$, vanish inside as well as near the test particle. Let us adopt the convenient notation

$$(-g)^{1/2}\delta T^{\mu\nu}=\mathcal{T}^{\mu\nu}, \quad (-g)^{1/2}\delta S^{\mu}=\mathcal{S}^{\mu} . \tag{3.2}$$

From (1.9) and (1.14), we obtain the relation

$$W_{[\rho,\nu]}=\tfrac{3}{2}[R_{[\rho\nu]}(W)-R_{[\rho\nu]}(\Gamma)] \tag{3.3}$$

and

$$R_{[\mu\nu]}(W)=\tfrac{1}{2}g_{[\mu\nu]}R(W)+8\pi T_{[\mu\nu]} = -4\pi g_{[\mu\nu]}T+8\pi T_{[\mu\nu]} . \tag{3.4}$$

By using these relations in (2.13), it follows that

$$\tfrac{1}{2}(g^{(0)}_{\mu\rho}\mathcal{T}^{\mu\nu}{}_{,\nu}+g^{(0)}_{\rho\mu}\mathcal{T}^{\nu\mu}{}_{,\nu})+[\mu\nu,\rho]^{(0)}\mathcal{T}^{\mu\nu} -2\pi\left[g^{(0)}_{[\rho\nu]}\delta T-2\delta T_{[\rho\nu]}+\frac{1}{4\pi}R^{(0)}_{[\rho\nu]}(\Gamma)\right]\mathcal{S}^{\nu}=0 , \tag{3.5}$$

where

$$T_{[\nu\rho]}=(g_{(\nu\tau)}g_{[\sigma\rho]}+g_{(\sigma\rho)}g_{[\nu\tau]})T^{\sigma\tau} . \tag{3.6}$$

Let us denote by X^{ρ} the coordinates of a point on the world line, which is some smooth curve inside the tube Σ, and by x^{ρ} the coordinates of a point exterior to the world line of the test particle. Because $g^{(0)}_{\mu\nu}$ varies very little inside the section of the tube, we can expand the symbols $[\mu\nu,\rho]^{(0)}$ in a Taylor series about the point X^{ρ}:

$$[\mu\nu,\rho]^{(0)}(x^{\rho})=[\mu\nu,\rho]^{(0)}(X^{\rho}) +(x^{\sigma}-X^{\sigma})\partial[\mu\nu,\rho]^{(0)}(X^{\rho})/\partial X^{\sigma}+\cdots . \tag{3.7}$$

We shall assume that the test particle has the simple structure of a monopole, so that the dipole and higher moments of $\mathcal{T}^{\mu\nu}$ and $\mathcal{S}^{\mu}$ vanish:

$$\begin{aligned}&\int(x^{\rho}-X^{\rho})\mathcal{T}^{\lambda\mu}d^3x=0 ,\\&\int(x^{\rho}-X^{\rho})(x^{\sigma}-X^{\sigma})\mathcal{T}^{\lambda\mu}d^3x=0 ,\\&\int(x^{\rho}-X^{\rho})\mathcal{S}^{\nu}d^3x=0 ,\\&\int(x^{\rho}-X^{\rho})(x^{\sigma}-X^{\sigma})\mathcal{S}^{\nu}d^3x=0 .\end{aligned} \tag{3.8}$$

Separating (3.5) into symmetric and skew-symmetric parts, we get

$$g^{(0)}_{(\mu\rho)}\mathcal{T}^{(\mu\nu)}{}_{,\nu}+g^{(0)}_{[\mu\rho]}\mathcal{T}^{[\mu\nu]}{}_{,\nu}+[(\mu\nu),\rho]^{(0)}\mathcal{T}^{(\mu\nu)}+[[\mu\nu],\rho]^{(0)}\mathcal{T}^{[\mu\nu]}-2\pi\left[g^{(0)}_{[\rho\nu]}\delta T-2\delta T_{[\rho\nu]}+\frac{1}{4\pi}R^{(0)}_{[\rho\nu]}(\Gamma)\right]\mathcal{S}^{\nu}=0 . \tag{3.9}$$

Because we are presently interested only in test-particle motion, we shall neglect all self-coupling terms and set $\delta T\delta S=0$. The skew part of $\delta T^{\mu\nu}$ is obtained from the perfect fluid energy-momentum tensor (1.27):

$$\delta T^{[\mu\nu]}=-pg^{(0)[\mu\nu]} . \tag{3.10}$$

In the test-particle limit $p\rightarrow 0$ and, therefore, from Eq. (3.10), it follows that $\delta T^{[\mu\nu]}\rightarrow 0$. The equations of motion for the test particle are then given by

$$\mathcal{T}^{(\mu\nu)}{}_{,\nu}+\begin{Bmatrix}\mu\\ \alpha\beta\end{Bmatrix}^{(0)}\mathcal{T}^{\alpha\beta}=\tfrac{1}{2}\gamma^{(0)(\mu\rho)}R^{(0)}_{[\rho\nu]}(\Gamma)\mathcal{S}^{\nu} , \tag{3.11}$$

where the NGT Christoffel symbols are defined by

$$\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}=\tfrac{1}{2}\gamma^{(\lambda\rho)}(g_{(\mu\rho),\nu}+g_{(\rho\nu),\mu}-g_{(\mu\nu),\rho}) \tag{3.12}$$

which follow from the equations of compatibility

$$g_{(\mu\nu)|\sigma}\equiv g_{(\mu\nu),\sigma}-g_{(\rho\nu)}\begin{Bmatrix}\rho\\ \mu\sigma\end{Bmatrix}-g_{(\mu\rho)}\begin{Bmatrix}\rho\\ \nu\sigma\end{Bmatrix}=0 . \tag{3.13}$$

Also, the inverse tensor $\gamma^{(\mu\nu)}$ is defined by the relation

$$\gamma^{(\lambda\sigma)}g_{(\lambda\nu)}=\delta^{\sigma}_{\nu} . \tag{3.14}$$

When the test particle carries zero charge, e.g., as in the case of a photon, then the tensorial term on the right-hand side of (3.11) vanishes, and the test particle moves along a geodesic of the NGT background metric $g^{(0)}_{(\mu\nu)}$. The tensor force in (3.11) corresponds to a new long-range force (henceforth, we shall call it the NGT force), which violates the weak equivalence principle. Charge carrying matter will fall in a gravitational field in a way that depends on the composition of the matter.

Let us introduce (3.7) into (3.11) and integrate the resulting expression over the hypersurface $t=\text{const}$. Since $\mathcal{T}^{(\mu\nu)}$ vanishes outside Σ, we obtain, using the single-pole particle conditions (3.8),

$$\frac{d}{dt}\int \mathcal{T}^{(\mu 0)}d^3x+\begin{Bmatrix}\mu\\ \alpha\beta\end{Bmatrix}^{(0)}(X^\rho)\int \mathcal{T}^{(\alpha\beta)}d^3x =\tfrac{1}{2}\gamma^{(0)(\mu\rho)}R^{(0)}_{[\rho\nu]}(\Gamma)\int \mathcal{S}^{\nu}d^3x \ . \quad (3.15)$$

As a consequence of (3.11), it follows that

$$\mathcal{T}^{(\mu\nu)}=(x^\mu\mathcal{T}^{(\rho\nu)})_{,\rho}+x^\mu\begin{Bmatrix}\nu\\ \alpha\beta\end{Bmatrix}^{(0)}\mathcal{T}^{(\alpha\beta)}-x^\mu H^\nu_\beta\mathcal{S}^\beta \ , \quad (3.16)$$

where

$$H^\nu_\beta=\tfrac{1}{2}\gamma^{(0)(\nu\rho)}R^{(0)}_{[\rho\beta]}(\Gamma) \ . \quad (3.17)$$

Integration of (3.16) results in the equation

$$\int \mathcal{T}^{(\mu\nu)}d^3x=\frac{d}{dt}\int x^\mu\mathcal{T}^{(\nu 0)}d^3x +\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}^{(0)}(X^\alpha)\int x^\mu\mathcal{T}^{(\rho\sigma)}d^3x -H^\nu_\beta(X^\alpha)\int x^\mu\mathcal{S}^\beta d^3x \ . \quad (3.18)$$

In view of (3.8) we get

$$\int x^\mu\mathcal{T}^{(\nu 0)}d^3x=X^\mu(t)\int \mathcal{T}^{(\nu 0)}d^3x \ , \qquad \int x^\mu\mathcal{T}^{(\rho\sigma)}d^3x=X^\mu(t)\int \mathcal{T}^{(\rho\sigma)}d^3x \ , \quad (3.19)$$

and

$$\int x^\mu\mathcal{S}^\nu d^3x=X^\mu(t)\int \mathcal{S}^\nu d^3x \ . \quad (3.20)$$

Substituting (3.19) and (3.20) into (3.18) and taking into account (3.15), we obtain

$$\int \mathcal{T}^{(\mu\nu)}d^3x=\frac{dX^\mu}{dt}\int \mathcal{T}^{(\nu 0)}d^3x \quad (3.21)$$

and

$$\int \mathcal{T}^{(\mu 0)}d^3x=\frac{dX^\mu}{dt}\int \mathcal{T}^{00}d^3x \ . \quad (3.22)$$

Substituting (3.22) into (3.21) gives

$$\int \mathcal{T}^{(\mu\nu)}d^3x=\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}\int \mathcal{T}^{00}d^3x \ . \quad (3.23)$$

Introducing (3.22) and (3.23) into (3.15) yields

$$\frac{d}{dt}\left[\frac{dX^\mu}{dt}\int \mathcal{T}^{00}d^3x\right]+\begin{Bmatrix}\mu\\ \alpha\beta\end{Bmatrix}^{(0)}\frac{dX^\alpha}{dt}\frac{dX^\beta}{dt}\int \mathcal{T}^{00}d^3x =H^\mu_\nu\int \mathcal{S}^\nu d^3x \ . \quad (3.24)$$

We are now in a position to derive the final form of the equations of motion for the test particle. Let us adopt the line element

$$ds^2=g^{(0)}_{(\mu\nu)}dX^\mu dX^\nu \quad (3.25)$$

and the four-velocity vector $u^\lambda=dX^\lambda/d\tau$, where τ is the proper time along the world line of the particle. We define the proper mass of the test particle by the equation

$$\int \mathcal{T}^{00}d^3x=u^0\int \rho_0(-g)^{1/2}d^3x=m_p\frac{dt}{d\tau} \ , \quad (3.26)$$

where ρ_0 is the proper mass density and m_p denotes the test-particle mass. As a consequence of (1.20), we can write

$$\mathcal{S}^\nu=(x^\nu\mathcal{S}^\rho)_{,\rho} \ . \quad (3.27)$$

Integrating (3.27) and using (3.20) gives

$$\int \mathcal{S}^\nu d^3x=\frac{d}{dt}\int x^\nu\mathcal{S}^0d^3x=\frac{dX^\nu}{dt}\int \mathcal{S}^0d^3x=l_p{}^2\frac{dX^\nu}{dt} \ , \quad (3.28)$$

where l_p denotes the l value for the test particle. The equation of motion now takes the form

$$\frac{d}{d\tau}(m_pu^\mu)+m_p\begin{Bmatrix}\mu\\ \alpha\beta\end{Bmatrix}^{(0)}u^\alpha u^\beta=l_p{}^2H^\mu_\nu u^\nu \ . \quad (3.29)$$

The first term in (3.29) gives

$$\frac{d}{d\tau}(m_pu^\mu)=m_p\frac{du^\mu}{d\tau}+\frac{dm_p}{d\tau}u^\mu \ . \quad (3.30)$$

From the condition

$$g^{(0)}_{(\mu\nu)}u^\mu u^\nu=1 \ , \quad (3.31)$$

where $u^\rho=\gamma^{(0)(\mu\rho)}u_\mu$, we obtain

$$\tfrac{1}{2}(g_{(\mu\nu)}u^\mu u^\nu)_{|\sigma}=g_{(\mu\nu)}u^\nu u^\mu{}_{|\sigma}=u_\mu u^\mu{}_{|\sigma}=0 \quad (3.32)$$

and

$$u_\lambda\left[\frac{du^\lambda}{d\tau}+\begin{Bmatrix}\lambda\\ \alpha\beta\end{Bmatrix}^{(0)}u^\alpha u^\beta\right]=u_\lambda u^\lambda{}_{|\mu}u^\mu=0 \ . \quad (3.33)$$

Here the vertical bar denotes the covariant derivative defined in (3.13). Multiplying (3.29) by u_μ and using the result

$$\gamma^{(0)(\mu\rho)}R^{(0)}_{[\rho\nu]}u^\nu u_\mu=R^{(0)}_{[\rho\nu]}u^\nu u^\rho=0 \ , \quad (3.34)$$

we obtain

$$\frac{dm_p}{d\tau}=0 \ . \quad (3.35)$$

Equation (3.29) reduces to

$$\frac{du^\mu}{d\tau}+\begin{Bmatrix}\mu\\ \alpha\beta\end{Bmatrix}^{(0)}u^\alpha u^\beta=\frac{l_p{}^2}{m_p}H^\mu_\nu u^\nu \ . \quad (3.36)$$

We observe that for particles with $l^2=0$, Eq. (3.36) reduces to the geodesic equation for the motion of a single-pole test particle as in GR, although the orbits of these test particles will, of course, be quantitatively different from those of GR, because of the use of the NGT Christoffel symbols (3.12).

IV. MOTION OF TEST PARTICLES IN THE NGT IN A STATIC SPHERICALLY SYMMETRIC FIELD

A static, spherically symmetric solution of the NGT field equations in a vacuum, obtained by using the spherical polar coordinates $x^1=r$, $x^2=\theta$, $x^3=\phi$, $x^0=t$, yields the nonvanishing components of the tensor $g_{\mu\nu}$ (Ref. 16) (see the Appendix):

$$\begin{aligned} g_{11} &\equiv -\alpha = -(1-2m/r)^{-1}, \\ g_{22} &= -r^2, \quad g_{33} = -r^2\sin^2\theta, \\ g_{00} &\equiv \gamma = (1+l^4/r^4)(1-2m/r), \end{aligned} \tag{4.1}$$

and

$$w \equiv g_{[10]} = \pm l^2/r^2 . \tag{4.2}$$

All the other components of $g_{\mu\nu}$ are zero, and the solution satisfies the boundary conditions that $g_{(\mu\nu)} \to \eta_{\mu\nu}$ and $g_{[\mu\nu]} \to 0$ as $r \to \infty$, where $\eta_{\mu\nu}$ is the Minkowski flat-space metric: $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

The nonvanishing components of the Christoffel symbols that we need are obtained using (3.12) and (3.14):

$$\begin{aligned} &\left\{ {1 \atop 11} \right\} = \frac{\alpha'}{2\alpha}, \quad \left\{ {1 \atop 22} \right\} = -\frac{r}{\alpha}, \\ &\left\{ {1 \atop 33} \right\} = -\frac{r\sin^2\theta}{\alpha}, \quad \left\{ {1 \atop 00} \right\} = \frac{\gamma'}{2\alpha}, \\ &\left\{ {2 \atop 12} \right\} = \frac{1}{r}, \quad \left\{ {2 \atop 33} \right\} = -\sin\theta\cos\theta, \\ &\left\{ {3 \atop 31} \right\} = \frac{1}{r}, \quad \left\{ {3 \atop 32} \right\} = \cot\theta, \quad \left\{ {0 \atop 01} \right\} = \frac{\gamma'}{2\gamma}, \end{aligned} \tag{4.3}$$

where we use the notation $f' = df/dr$. A calculation of $R_{[10]}$ gives (see the Appendix)

$$R_{[10]} = \frac{4l_s^2 m_s}{r^5}, \tag{4.4}$$

where l_s denotes the l value of the point-particle source. By using the inverse relations $\gamma^{11} = -1/\alpha$ and $\gamma^{00} = 1/\gamma$, it follows that

$$\tfrac{1}{2}\gamma^{11} R_{[10]} = -\frac{2l_s^2 m_s}{\alpha r^5}, \tag{4.5}$$

$$\tfrac{1}{2}\gamma^{00} R_{[01]} = -\frac{2l_s^2 m_s}{\gamma r^5}. \tag{4.6}$$

Substituting the Christoffel symbols (4.3) and the results (4.5) and (4.6) into the equations of motion (3.36), we obtain

$$\begin{aligned} \frac{d^2r}{d\tau^2} + \frac{\alpha'}{2\alpha}\left(\frac{dr}{d\tau}\right)^2 - \frac{r}{\alpha}\left(\frac{d\theta}{d\tau}\right)^2 \\ - r\left(\frac{\sin^2\theta}{\alpha}\right)\left(\frac{d\phi}{d\tau}\right)^2 + \frac{\gamma'}{2\alpha}\left(\frac{dt}{d\tau}\right)^2 \\ + \left(\frac{2A}{\alpha r^5}\right)\left(\frac{dt}{d\tau}\right) = 0, \end{aligned} \tag{4.7}$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\left(\frac{d\theta}{d\tau}\right)\left(\frac{dr}{d\tau}\right) - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0, \tag{4.8}$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\left(\frac{d\phi}{d\tau}\right)\left(\frac{dr}{d\tau}\right) + 2\cot\theta\left(\frac{d\phi}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right) = 0, \tag{4.9}$$

$$\frac{d^2t}{d\tau^2} + \frac{\gamma'}{\gamma}\left(\frac{dt}{d\tau}\right)\left(\frac{dr}{d\tau}\right) + \left(\frac{2A}{\gamma r^5}\right)\left(\frac{dr}{d\tau}\right) = 0, \tag{4.10}$$

where $A = m_s l_s^2 l_p^2 / m_p$. The orbit of a test particle can be shown to lie in a plane and by an appropriate choice of axes, we can make $\theta = \pi/2$. Then integration of Eq. (4.9) gives

$$r^2\frac{d\phi}{d\tau} = J. \tag{4.11}$$

Equation (4.10) can be expressed in the form

$$\frac{1}{\gamma}\left(\gamma\frac{d^2t}{d\tau^2} + \frac{d\gamma}{d\tau}\frac{dt}{d\tau}\right) - \frac{1}{\gamma}\frac{d}{dr}\left(\frac{A}{2r^4}\right)\left(\frac{dr}{d\tau}\right) = 0. \tag{4.12}$$

We can now write (4.12) as

$$\frac{d}{d\tau}\left(\gamma\frac{dt}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{A}{2r^4}\right) \tag{4.13}$$

which integrates to

$$\frac{dt}{d\tau} = \frac{1}{\gamma}\left(1 + \frac{A}{2r^4}\right). \tag{4.14}$$

By substituting (4.11) and (4.14) into (4.7), we get

$$\begin{aligned} \frac{d^2r}{d\tau^2} + \frac{\alpha'}{\alpha}\left(\frac{dr}{d\tau}\right)^2 - \frac{J^2}{\alpha r^3} + \frac{\gamma'}{2\alpha}\frac{1}{\gamma^2}\left(1 + \frac{A}{2r^4}\right)^2 \\ + \left(\frac{2A}{\alpha r^5\gamma}\right)\left(1 + \frac{A}{2r^4}\right) = 0. \end{aligned} \tag{4.15}$$

Multiplying this last equation by $2\alpha\, dr/d\tau$, we obtain

$$\frac{d}{d\tau}\left[\alpha\left(\frac{dr}{d\tau}\right)^2 + \frac{J^2}{r^2} - \frac{1}{\gamma}\left(1 + \frac{A}{2r^4}\right)^2\right] = 0 \tag{4.16}$$

and integrating this expression gives

$$\alpha\left(\frac{dr}{d\tau}\right)^2 + \frac{J^2}{r^2} - \frac{1}{\gamma}\left(1 + \frac{A}{2r^4}\right)^2 = -E, \tag{4.17}$$

where E is a constant of integration. If we write the line element as

$$ds^2 = \gamma\, dt^2 - \alpha\, dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2 \tag{4.18}$$

set $\theta = \pi/2$, divide the resulting expression by $d\tau^2$, and use Eq. (4.14), we obtain (4.17) as we might have anticipated. We can then show that $ds^2 = E\, d\tau^2$, so that $ds/d\tau$ is a constant. For material particles $E > 0$ and for photons $E = 0$.

If we substitute $dr/d\tau = dr/d\phi\, d\phi/d\tau$ into (4.17), use (4.11), and integrate, we get the exact solution to the orbital problem:

$$\phi = \pm \int \frac{dr\,\alpha^{1/2}}{r^2\left[\frac{1}{J^2\gamma}(1+A/2r^4)^2 - E/J^2 - 1/r^2\right]^{1/2}} . \tag{4.19}$$

Let us now calculate the acceleration experienced by a test particle starting from rest in the static, spherically symmetric field given by (4.1) and (4.2). We choose $\theta=\pi/2$, $J=0$, and $dr/d\tau=0$ and obtain from (4.7)–(4.10), to leading orders in $1/r$,

$$\frac{d^2r}{dt^2}+\frac{m_s}{r^2}-\frac{2m_s^2}{r^3}-\frac{2l_s^4}{r^5}+\frac{2l_p^2 l_s^2 m_s}{m_p r^5}=0 . \tag{4.20}$$

The difference in the acceleration of two test particles falling in a gravitational field can be obtained from (4.20):

$$\Delta a = \frac{2m_s l_s^2 c^2}{R_s^5}\Delta\left[\frac{l_p^2}{m_p}\right] , \tag{4.21}$$

where we write $\Delta x = x_1 - x_2$ and we choose a to be positive. Note that the repulsive NGT term in (4.20) does not appear in this formula. The NGT gradient above the surface of a body is given to leading order by ($g>0$)

$$\frac{dg}{dh}=\frac{10m_s l_s^2 c^2}{R_s^6}\left[\frac{l_s^2}{m_s}-\frac{l_p^2}{m_p}\right] . \tag{4.22}$$

For the case of the motion of two massive point particles, the equations of motion can be derived using a post-Newtonian expansion of the field equations and the conservation laws in terms of v/c. The lowest order yields the Newtonian equations of motion, while in the post-Newtonian order, we get, in relative coordinates,[17]

$$\mathbf{a} = -\frac{m\mathbf{r}}{r^3}+\frac{m\mathbf{r}}{r^3}\left[\frac{4m}{r}-v^2+\frac{2\mu}{r}-\frac{3\mu v^2}{m}+\frac{3\mu}{2mr^2}(\mathbf{v}\cdot\mathbf{r})^2\right] + \frac{4m(\mathbf{r}\cdot\mathbf{v})\mathbf{v}}{r^3}-\frac{2\mu(\mathbf{r}\cdot\mathbf{v})\mathbf{v}}{r^3}+\frac{2K\mathbf{r}}{r^6} , \tag{4.23}$$

where

$$\mathbf{r}_{12}\equiv\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2,\quad \mathbf{v}=\mathbf{v}_1-\mathbf{v}_2 ,$$
$$\mathbf{v}_1=\frac{m_2}{m}\mathbf{v},\quad \mathbf{v}_2=-\frac{m_1}{m}\mathbf{v} , \tag{4.24}$$
$$\mathbf{a}=\mathbf{a}_1-\mathbf{a}_2,\quad m=m_1+m_2 .$$

Moreover, we have used the notation $k_i=l_i^2$ for $i=1,2$, $\mu=m_1m_2/m$, and

$$K=(k_1-k_2)^2+\frac{m_2}{m_1}(k_1^2-k_1k_2)+\frac{m_1}{m_2}(k_2^2-k_1k_2) . \tag{4.25}$$

In the test-particle limit, $m_1\to 0$, $k_1\to 0$, and $k_1/m_1\neq 0$, we get for the NGT part of (4.23) the same result as in (4.20). The force law for two point particles obtained from Eq. (4.23) satisfies Newton's third law $\mathbf{F}_{12}=-\mathbf{F}_{21}$ when the center-of-mass coordinates are redefined in the post-Newtonian approximation. We obtain $K=0$ for equal k's, $k_1=k_2=k$. When $k_1/m_1>k_2/m_2$ and $k_1>k_2$ or $k_2/m_2>k_1/m_1$ and $k_2>k_1$, then $K>0$ and we will obtain a retrograde periastron shift contribution in the NGT for a double star, as we shall find in the next section. The velocity-dependent NGT contributions are expected to occur first in the post-post-Newtonian order of approximation.

V. THE PERIHELION PRECESSION OF A TEST PARTICLE AND THE DEFLECTION OF LIGHT

We can obtain astronomical predictions in the NGT in a direct way by using perturbation theory to solve the equations of motion instead of using (4.19). Let us set $u=1/r$ and by using (4.11), we obtain $dr/d\tau = -J\,du/d\phi$. Substituting this relation into (4.17) gives, after some manipulations of terms, the equation

$$\frac{d^2u}{d\phi^2}+u=N+3m_s u^2-2Cu^3 , \tag{5.1}$$

where $N=m_sE/J^2$ and $C=(l_s^4-A)/J^2$. We solve (5.1) by successive approximations with

$$u=u_0+v , \tag{5.2}$$

where u_0 is a solution to the first-order equation

$$\frac{d^2u_0}{d\phi^2}+u_0=N \tag{5.3}$$

given by

$$u_0=N+B\cos\phi , \tag{5.4}$$

where B is a constant. Substituting the first approximation into the right-hand side of (5.1) and solving for v in terms of the nonlinear contributions, we get

$$u=N+B\cos\phi+D\phi\sin\phi+(\text{periodic terms}) , \tag{5.5}$$

where D is the constant

$$D=3m_sBN-3CBN^2-\tfrac{3}{4}CB^3 . \tag{5.6}$$

We can use the trigonometric identity

$$\cos(\phi-\epsilon\phi)\simeq\cos\phi+\epsilon\phi\sin\phi$$

in which $\epsilon=D/B$ can be treated as a small quantity. Thus for the solution of u, we obtain

$$u=N+B\cos(\phi-\epsilon\phi)+(\text{periodic terms}) . \tag{5.7}$$

Perihelion occurs when $u=1/r$ is a maximum which gives, for one revolution,

$$\phi=2\pi(1+\epsilon\phi) . \tag{5.8}$$

Denoting by $\delta\omega$ the change in the angle of perihelion ω, we get

$$\delta\omega=2\pi\epsilon \tag{5.9}$$

and using $J=(pm_s)^{1/2}$, $E=1$ gives $N=m_s/J^2=1/p$ and $B=e/p$, where p is the semilatus rectum of the Keplerian orbit $p=a(1-e^2)$ with a denoting the semimajor axis and e the eccentricity. Then (5.9) yields the result for the perihelion precession of a planet orbiting the Sun:

$$\Delta\omega = \frac{6\pi G M_\odot}{c^2 a(1-e^2)}\lambda , \tag{5.10}$$

where λ is given by

$$\lambda = 1 - \frac{L_\odot{}^4 c^4(1+e^2/4)}{G^2 M_\odot{}^2 a^2(1-e^2)^2} . \tag{5.11}$$

Here the constant $L_\odot$ is

$$L_\odot = (l_\odot{}^2 M_\odot d)^{1/4} , \tag{5.12}$$

where

$$d = l_\odot{}^2/M_\odot - l_p{}^2/m_p . \tag{5.13}$$

We regain the standard result for the perihelion precession of a test particle in GR when $L_\odot = 0$ in (5.11), giving $\lambda = 1$. The predictions of the NGT for the perihelion precession of Mercury are consistent with observations for $L_\odot \leq 3000$ km (Ref. 8).

For a double star system, the periastron shift is given by[17]

$$\dot{\omega} = \frac{3G^{2/3} m^{2/3}}{(P/2\pi)^{5/3} c^2(1-e^2)}\lambda_b , \tag{5.14}$$

where

$$\lambda_b = 1 - \frac{K c^4(1+e^2/4)}{(Gm)^{8/3}(P/2\pi)^{4/3}(1-e^2)^2} . \tag{5.15}$$

P is the period of the double star, $m = m_1 + m_2$ e is the eccentricity, and K is given by the formula (4.25). For equal-mass and equal-l stars in a binary system, the constant $K = 0$. When $K > 0$ the NGT periastron shift contribution is negative.

Let us now calculate the deflection of light grazing the limb of the Sun. For a massless photon $E = N = 0$ and (5.1) becomes

$$\frac{d^2u}{d\phi^2} + u = 3m_s u^2 - 2Cu^3 , \tag{5.16}$$

where now $C = l_s{}^4/J^2$, since l_p is zero for a photon. We can solve (5.16) by successive approximations as before using (5.2). The first-order solution is now of the form

$$u_0 = \frac{1}{r_0}\sin\phi \tag{5.17}$$

which is the straight-line solution for a light ray passing at a distance r_0 from the Sun's center. Substituting the first-order solution u_0 into the right-hand side of (5.16) and solving for v gives

$$u = \frac{1}{r_0}\sin\phi + \frac{a}{2r_0{}^2}(1 + \tfrac{1}{3}\cos 2\phi) + \frac{b}{r_0{}^3}(\tfrac{1}{32}\sin 3\phi - \tfrac{3}{8}\phi\cos\phi) , \tag{5.18}$$

where $a = 3m_s$ and $b = -2l_s{}^4/J^2$.

The asymptotes of the trajectory will correspond to those values of the angle ϕ for which $u = 1/r$ becomes zero. These asymptotes occur when ϕ is close to zero or π. Let us denote by δ_1 the small angle between the asymptote near $\phi = 0$ and the x axis. We approximate $\sin\phi$ by δ_1 and $\cos 2\phi$ by 1 and set $u = 0$ in (5.18) to give

$$\delta_1 = -\frac{2a}{3r_0}\left[1 - \frac{9}{32}\frac{b}{r_0{}^2}\right]^{-1} . \tag{5.19}$$

For the second asymptote, we take $\phi = \pi - \delta$ and following the same procedure the result is obtained:

$$\delta_2 = -\left[\frac{2a}{3r_0} + \frac{3\pi b}{8r_0{}^2}\right]\left[1 - \frac{9b}{32r_0{}^2}\right]^{-1} . \tag{5.20}$$

The small angle between the asymptotes, yielding the total light deflection, is given by[8]

$$\Delta = \frac{4GM_\odot}{c^2 R_\odot}\left[1 - \frac{9l_\odot{}^4}{16R_\odot{}^4} - \frac{3\pi l_\odot{}^4 c^2}{16R_\odot{}^3 G M_\odot}\right] \tag{5.21}$$

a result obtained previously.

By using light deflection data obtained from quasars, we obtain the bound $l_\odot \leq 6\times 10^3$ km. The second-order bending of light effect in the NGT is, for $l_\odot = 3\times 10^3$ km, given by

$$\Delta_{\rm NGT} = -1.9\times 10^2 \text{ arc}\,\mu\text{sec} , \tag{5.22}$$

while for GR with $l_\odot = 0$ and with a quadrupole moment coefficient for the Sun: $J_2 = 0$, we obtain

$$\Delta_{\rm GR} = 4 \text{ arc}\,\mu\text{sec} . \tag{5.23}$$

Choosing $J_2 = 6\times 10^{-6}$, we also find that

$$\Delta_{J_2} = 10 \text{ arc}\,\mu\text{sec} . \tag{5.24}$$

An accurate second-order observation of the bending of light by the Sun could produce useful bounds on $l_\odot$ and J_2.

VI. THE RED-SHIFT AND RADAR SIGNAL TIME DELAY EFFECTS

From Eq. (3.32) we find that

$$d\tau^2 = g_{(\mu\nu)}dx^\mu dx^\nu \tag{6.1}$$

is a constant of the motion. For a clock at rest the measure of proper time is $d\tau = (g_{00})^{1/2}dt$ and the red-shift of a spectral line emitted from the surface of a star is

$$\frac{\Delta\lambda}{\lambda} = -\frac{\Delta\nu}{\nu} = \frac{GM}{c^2 R} - \frac{l^4}{2R^4} , \tag{6.2}$$

where R is the radius of the star. For the Sun this yields the bound

$$l_\odot \leq 2\times 10^4 \text{ km} . \tag{6.3}$$

The NGT force cannot in general be screened away like the electromagnetic force and may affect the composition of a clock. However, for ordinary cesium or maser clocks, the new forces will not affect the difference between one energy level and another, and any Stark-type effects can be shown to be very small and undetectable. For a particle moving in a circular orbit of radius R, we have

$dr/dt=0$ and (4.17) gives

$$\frac{J^2}{R^2}-\frac{1}{\gamma(R)}\left[1+\frac{A}{2R^4}\right]^2+E=0 . \tag{6.4}$$

For a system in equilibrium, the differential of (6.4) vanishes and yields

$$J^2=\frac{R^3\gamma'(R)}{2\gamma(R)^2}\left[1+\frac{A}{2R^4}\right]^2+\frac{A}{\gamma(R)R^2}\left[2+\frac{A}{R^4}\right] \tag{6.5}$$

and

$$E=-\frac{J^2}{R^2}+\frac{1}{\gamma(R)}\left[1+\frac{A}{2R^4}\right]^2 . \tag{6.6}$$

Substituting (6.5) into Eq. (4.11) and using (4.14) gives the rate of revolution as

$$\frac{d\phi}{dt}=\left[1+\frac{A}{2R^4}\right]^{-1}\left[\frac{\gamma'(R)}{2R}\left[1+\frac{A}{2R^4}\right]^2+\frac{\gamma(R)A}{R^6}\left[2+\frac{A}{R^4}\right]\right]^{1/2} . \tag{6.7}$$

By using the result that $ds^2=E\,d\tau^2$, (6.5) and (6.6), we get, for the proper time,

$$d\tau=\gamma(R)^{1/2}\left[1+\frac{A}{2R^4}\right]^{-1}\left\{\left[1+\frac{A}{2R^4}\right]^2-\left[\frac{R\gamma'(R)}{2\gamma(R)}\left[1+\frac{A}{2R^4}\right]^2+\frac{A}{R^4}\left[2+\frac{A}{R^4}\right]\right]\right\}^{1/2}dt . \tag{6.8}$$

For $A=0$ this expression reduces correctly to the GR result for the proper time.[18] We see that when we transform from the rest frame in the NGT to another frame, such as the one associated with a particle moving in a circular orbit, the NGT force affects the proper time of the clock.

For completeness, we will give the result in NGT for the time delay of travel time of radar signals. Let us adopt the line element in isotropic coordinates, obtained by introducing the new radius variable

$$r=\rho(1+m_s/2\rho)^2 . \tag{6.9}$$

Using r in place of the notation ρ, we have

$$ds^2=\left[\frac{(1+m_s/2r)^8+l_s^4/r^4}{(1+m_s/2r)^8}\right]\frac{(1-m_s/2r)^2}{(1+m_s/2r)^2}dt^2-(1+m_s/2r)^4d\sigma^2 , \tag{6.10}$$

where

$$d\sigma^2=dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\phi^2 . \tag{6.11}$$

We now expand the metric coefficients in powers of m_s/r and l_s/r. In Cartesian coordinates x, y, and z, we get

$$ds^2=\left[1-\frac{2m_s}{r}+\frac{l_s^4}{r^4}+2\left[\frac{m_s}{r}\right]^2\right]dt^2-\left[1+\frac{2m_s}{r}\right](dx^2+dy^2+dz^2) , \tag{6.12}$$

where

$$r=(x^2+y^2+z^2)^{1/2} ,\quad \theta=\arctan[z/(x^2+y^2)^{1/2}] ,\quad \phi=\arctan(y/x) . \tag{6.13}$$

The Sun is taken to be at the origin of coordinates and the transmitter (Earth) and the reflector (planet) lie in the $z=0$ plane and the transmitter-reflector line lies along the x direction. For a null ray

$$ds^2=g_{00}dt^2+g_{11}dx^2=0 . \tag{6.14}$$

The delay of the coordinate time t between transmission and reflection is

$$\Delta t=\int_{-r_\oplus}^{r_p}(-g_{11}/g_{00})^{1/2}dx=\int_{-r_\oplus}^{r_p}\left[1+\frac{2m_s}{(x^2+r_0{}^2)^{1/2}}-\frac{l_s{}^4}{2(x^2+r_0{}^2)^2}\right]dx , \tag{6.15}$$

where $r_\oplus$ and r_p are the distances of the Sun from Earth and the planet, respectively, and $r_0=y=\text{const}$. Integration yields

$$\Delta t=r_\oplus+r_p+2m_s\ln\left[\frac{[r_p+(r_p{}^2+r_0{}^2)^{1/2}][r_\oplus+(r_\oplus{}^2+r_0{}^2)^{1/2}]}{r_0{}^2}\right]-\frac{l_s{}^4}{4r_0{}^2}\left[\frac{r_p}{r_p{}^2+r_0{}^2}+\frac{r_\oplus}{r_\oplus{}^2+r_0{}^2}+\frac{1}{r_0}\arctan\left[\frac{r_p}{r_0}\right]+\frac{1}{r_0}\arctan\left[\frac{r_\oplus}{r_0}\right]\right] . \tag{6.16}$$

The excess time delay is a maximum when the planet is at superior conjunction and the radar signal just grazes the limb of the Sun. In this case $r_0 \simeq R_\odot$ and the maximum round-trip delay is

$$(\Delta\tau)_{\max}=\frac{4GM_\odot}{c^3}\ln\left[\frac{4r_p r_\oplus}{R_\odot^2}\right]-\frac{l_\odot^4}{2cR_\odot^2}\left[\frac{1}{r_p}+\frac{1}{r_\oplus}+\frac{1}{R_\odot}\arctan\left[\frac{r_p}{R_\odot}\right]+\frac{1}{R_\odot}\arctan\left[\frac{r_\oplus}{R_\odot}\right]\right]. \quad (6.17)$$

By using the Viking data for Mars,[19] we obtain

$$l_\odot \le 10^4 \text{ km}. \quad (6.18)$$

From these results we see that NGT is consistent with all the solar system relativity tests that have been performed to date provided that $L_\odot \le 3000$ km.

VII. MAXIMALLY EXTENDED COORDINATES AND BLACK-HOLE RADIATION

One way to avoid coordinate singularities in GR at the horizon is to use Kruskal coordinates.[20,21] In the NGT we can also find a set of maximally extended coordinates by using methods analogous to those used in GR (Refs. 22 and 23). We seek a coordinate system in which light rays everywhere have the slope $dr'/dt'=\pm 1$, and the line element has the form

$$ds^2=f^2(r',t')(dt'^2-dr'^2)-r^2(r',t')d\Omega^2, \quad (7.1)$$

where

$$d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2. \quad (7.2)$$

By means of the transformation law

$$g_{\mu\nu}=\frac{\partial x^{*\alpha}\partial x^{*\beta}}{\partial x^\mu \partial x^\nu}g^*_{\alpha\beta} \quad (7.3)$$

we find that

$$\frac{\partial r'}{\partial t}=\pm(1-2m_s/r)(1+l_s^4/r^4)^{1/2}\frac{\partial t'}{\partial r}, \quad (7.4)$$

$$\frac{\partial t'}{\partial t}=\pm(1-2m_s/r)(1+l_s^4/r^4)^{1/2}\frac{\partial r'}{\partial r}. \quad (7.5)$$

The solutions for r' and t' are given by

$$r'=\exp(br^*)\cosh bt, \quad t'=\exp(br^*)\sinh bt, \quad (7.6)$$

where

$$\frac{dr^*}{dr}=\frac{1}{(1+l_s^4/r^4)^{1/2}(1-2m_s/r)} \quad (7.7)$$

and

$$f^2(r',t')=\frac{r-2m_s}{b^2 r}(1+l_s^4/r^4)\exp(-2br^*). \quad (7.8)$$

Here b is a constant given by

$$b=\frac{1}{4m_s}\left[1+\left[\frac{l_s}{2m_s}\right]^4\right]^{1/2} \quad (7.9)$$

and $f^2(r',t')$ remains finite at $=2m_s$.

The integration of (7.7) for r^* gives[23]

$$r^*(r)=\frac{r^3}{(r^4+l_s^4)^{1/2}}+m_s\ln\left[\frac{r^2+(r^4+l_s^4)^{1/2}}{l_s^2}\right]+H(r,2m_s,l_s)+\frac{(2m_s)^3}{2[(2m_s)^4+l_s^4]^{1/2}}\ln\left[\left[\frac{F_-(2m_s)F_+(r)-F_+(2m_s)F_-(r)}{F_-(2m_s)F_+(r)+F_+(2m_s)F_-(r)}\right]\times\left[\frac{(2m_s-l_s)(r+l_s)F_+(2m_s)F_-(r)-(2m_s+l_s)(r-l_s)F_-(2m_s)F_+(r)}{(2m_s-l_s)(r+l_s)F_+(2m_s)F_-(r)+(2m_s+l_s)(r-l_s)F_-(2m_s)F_+(r)}\right]\right], \quad (7.10)$$

where

$$F_\pm(r)=[(r-l_s)^2+(3\pm 22^{1/2})(r+l_s)^2]^{1/2}, \quad (7.11)$$

and $H(r,2m_s,l_s)$ is a linear combination of elliptical integrals of the first, second, and third kinds, which is well behaved everywhere. The result (7.10) is obtained in the following way. We have

$$r^*(r)=\int\frac{r^3dr}{(r-2m_s)(r^4+l_s^4)^{1/2}}=\int\frac{r^2dr}{(r^4+l_s^4)^{1/2}}+2m_s\int\frac{r\,dr}{(r^4+l_s^4)^{1/2}}+(2m_s)^2\int\frac{dr}{(r^4+l_s^4)^{1/2}}+(2m_s)^3\int\frac{dr}{(r-2m_s)(r^4+l_s^4)^{1/2}}, \quad (7.12)$$

where

$$\int\frac{r^2dr}{(r^4+l_s^4)^{1/2}}=\frac{r^3}{(r^4+l_s^4)^{1/2}}+\frac{l_s}{2}\left[I\left[\kappa,\frac{1}{2^{1/2}}\right]-2E\left[\kappa,\frac{1}{2^{1/2}}\right]\right]. \quad (7.13)$$

Here I and E are elliptic integrals of the first and second kinds, respectively, and $\sin\kappa=(r^2-l_s^2)/(r^4+l_s^4)$. Moreover,

$$\int \frac{r\,dr}{(r^4+l_s^4)^{1/2}}=\frac{1}{2}\ln\left[\frac{r^2+(r^4+l_s^4)^{1/2}}{l_s^2}\right], \tag{7.14}$$

$$\int \frac{dr}{(r^4+l_s^4)}=\frac{1}{2l_s}I\left[\kappa,\frac{1}{2^{1/2}}\right]=-\frac{2^{1/2}}{al_s}I(\epsilon,\lambda)\ , \tag{7.15}$$

where $a=2^{1/2}+1$, $\lambda^2=42^{1/2}/a^2$, and $\epsilon=a(r+l_s)/[(r-l_s)^2+a^2(r+l_s)^2]^{1/2}$. The last integral in (7.12) is

$$\int \frac{dr}{(r-2m_s)(r^4+l_s^4)^{1/2}}=-\frac{2^{1/2}[(2m_s-l_s)-a^2(2m_s+l_s)]}{aF_+^2(2m_s)}I(\epsilon,\lambda)+\frac{22^{1/2}l_s(l_s-2m_s)}{aF_+^2(2m_s)(l_s+2m_s)}\Pi(\epsilon,n,\lambda)$$
$$+\frac{1}{2[(2m_s)^4+l_s^4]^{1/2}}\ln\left[\frac{F_-(2m_s)F_+(r)-F_+(2m_s)F_-(r)}{F_-(2m_s)F_+(r)+F_+(2m_s)F_-(r)}\right], \tag{7.16}$$

where $n=1+(l_s-2m_s)/a^2(l_s+2m_s)^2$ and Π is an elliptic integral of the third kind. Π is only well defined for $n<1$, so it is necessary to use

$$\Pi(\epsilon,n,\lambda)=-\Pi(\epsilon,N,\lambda)+I(\epsilon,\lambda)$$
$$+\frac{a(l_s+2m_s)F_+(2m_s)}{2(l_s-2m_s)F_-(2m_s)}\ln\left[\frac{(l_s+2m_s)(r-l_s)F_+(2m_s)F_-(r)-(l_s-2m_s)(r+l_s)F_-(2m_s)F_+(r)}{(l_s+2m_s)(r-l_s)F_+(2m_s)F_-(r)-(l_s-2m_s)(r+l_s)F_-(2m_s)F_+(r)}\right], \tag{7.17}$$

where $N=42^{1/2}(l_s+2m_s)^2/F_+^2(2m_s)$. Collecting together these contributions, we obtain the result (7.10). Expanding (7.10) in small values of $r-2m_s$ and substituting into (7.8), we get

$$f^2(r',t')\simeq(r-2m_s)^{1-\{2b(2m_s)^3/[(2m_s)^4+l_s^4]^{1/2}\}} \tag{7.18}$$

which leads to the result (7.9). In the GR limit $b\to 1/4m_s$ as $l_s\to 0$.

As in GR, $r^*(r)$ goes to $-\infty$ at $r=2m_s$, becomes $+\infty$ as $r\to\infty$, and is well behaved everywhere in between. When $l_s\to 0$ these results for the maximally extended coordinates reduce to those of GR.

Except for the constant b and the functional form of $r^*(r)$ the results of the NGT are the same as GR. In the NGT a black hole with no charge or angular momentum will emit massless, scalar particles thermally with an effective temperature

$$T=\frac{b}{2\pi k}=\frac{1}{8\pi k m_s}\left[1+\left[\frac{l_s}{2m_s}\right]^4\right]^{1/2}. \tag{7.19}$$

From the Lagrangian for a massless scalar field in the NGT,

$$\mathscr{L}_\phi=(-g)^{1/2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\ , \tag{7.20}$$

we obtain the Klein-Gordon equation

$$[(-g)^{1/2}g^{\mu\nu}\phi_{,\mu}]_{,\nu}=0\ . \tag{7.21}$$

The expansion

$$\phi=\sum_{n,m}\int_{-\infty}^{\infty}dk\,e^{-ikt}Y_{nm}(\theta,\phi)\frac{1}{r}\psi_{nmk}(r^*)\ , \tag{7.22}$$

where $Y_{nm}(\theta,\phi)$ is a spherical harmonic, leads to the equation

$$\frac{\partial^2\psi}{\partial r^{*2}}+\frac{1}{r}\frac{\partial\psi}{\partial r^*}\left[\frac{2l_s^4}{r^4(1+l_s^4/r^4)^{1/2}}(1-2m_s/r)\right]+[k^2-U(r^*)]\psi=0\ , \tag{7.23}$$

where

$$U(r^*)=(1-2m_s/r)\left[\frac{n(n+1)}{r^2}+\frac{2m_s}{r^3}(1+l_s^4/r^4)-\frac{4l_s^4}{r^6}(1-2m_s/r)\right]. \tag{7.24}$$

Let us make the transformation

$$\psi=(1+l_s^4/r^4)^{-1/4}R\ . \tag{7.25}$$

Then (7.23) becomes

$$\frac{\partial^2 R}{\partial r^{*2}}+[k^2-V(r^*)]R=0\ , \tag{7.26}$$

where

$$V(R^*)=(1-2m_s)\left[\frac{n(n+1)}{r^2}+\frac{2m_s}{r^3}+(1-2m_s/r)(1+l_s^4/r^4)\frac{l_s^4}{r^6}\right]. \tag{7.27}$$

The power output of gravitational synchrotron radiation in mode n, m is given by

$$P_{\rm out}=\left[\frac{1}{8\pi}\right]\frac{|H|^2}{(1+l^4/r_0^4)^{1/2}}|R(r_0^*)|^2\ , \tag{7.28}$$

where $u^0=dt/d\tau$ and

$$H\equiv H_{nm}=-\left[\frac{4\pi\mu}{r_0u^0}\right]Y_{nm}(\pi/2,0)\ . \tag{7.29}$$

By using Eqs. (A2) and (1.2), we get

$$g^{\alpha+\beta-}{}_{;\beta}\equiv g^{\alpha\beta}{}_{,\beta}+g^{\alpha\lambda}\Gamma^{\beta}_{\lambda\beta}+g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta}=0\ . \tag{7.30}$$

With the help of this equation, the Klein-Gordon equation (7.21) can be written in covariant form as

$$(g^{\mu+\nu-}\phi_{,\mu})_{;\nu}=0\ , \tag{7.31}$$

where a semicolon denotes covariant differentiation with respect to the $\Gamma^{\lambda}_{\mu\nu}$ connection and the + and − notation stands for

$$A_{\mu+;\nu}=A_{\mu,\nu}-A_{\rho}\Gamma^{\rho}_{\mu\nu} \tag{7.32}$$

and

$$A_{\mu-;\nu}=A_{\mu,\nu}-A_{\rho}\Gamma^{\rho}_{\nu\mu}\ . \tag{7.33}$$

A solution of (7.31) for ϕ can be written in the form

$$\phi=\mathrm{Re}[e^{i\theta/\epsilon}(a+b\epsilon+\cdots)]\ , \tag{7.34}$$

where ϵ is a small expansion parameter, and we have

$$\phi_{,\mu}=\mathrm{Re}\left[\frac{i}{\epsilon}\theta_{,\mu}(a+b\epsilon+\cdots)e^{i\theta/\epsilon}+(a_{,\mu}+b_{,\mu}\epsilon+\cdots)e^{i\theta/\epsilon}\right]. \tag{7.35}$$

It can easily be shown that (7.34) and (7.35) yield the same geometrical optics equation

$$g^{(\mu\nu)}k_{\mu}k_{\nu}=0 \tag{7.36}$$

to order $O(1/\epsilon^2)$, as the Klein-Gordon equation

$$(g^{(\mu\nu)}\theta_{,\mu})_{|\nu}=0\ , \tag{7.37}$$

where $k_{\mu}=\theta_{,\mu}$ and the subscript vertical bar denotes the covariant derivative with respect to the Christoffel symbol

$$\begin{Bmatrix}\mu\\ \alpha\beta\end{Bmatrix}.$$

Let us now consider radially inward falling test particles corresponding to the gravitational collapse of the envelope of a star. We choose $J=0$ in (4.17) and obtain the equation

$$\frac{dr}{d\tau}=\pm\left[\frac{1}{\alpha\gamma}\left[1+\frac{A}{2r^4}\right]^2-\frac{E}{\alpha}\right]^{1/2}. \tag{7.38}$$

Integrating this equation gives

$$\tau=\int^{r}dr\left[\frac{r^4+l_s{}^4}{G(r)}\right]^{1/2}+\mathrm{const}\ , \tag{7.39}$$

where

$$G(r)=\frac{A^2}{4r^4}+2m_s r^3+l_s{}^4\left[\frac{2m_s}{r}-1\right]+A\ . \tag{7.40}$$

We see that $G(r)>0$ for $r<2m_s$ so that a test particle that falls through the event horizon at $r=2m_s$ must continue to fall toward $r=0$ as in GR, thus avoiding any violation of causality. Also $G(r)\leq 0$ and $G''(r)>0$ for $r\geq 2m_s$ which means that for some value $r=r_c$ for which $G(r_c)=0$, test particles will be prevented from forming black-hole event horizons, i.e., test particles will be "bounced" away from $r=r_c$. We have $A=0$ when $l_p=0$ and $G'(r)=0$ and $G''(r)>0$ for

$$r=(\tfrac{1}{3})^{1/4}l_s \tag{7.41}$$

corresponding to a minimum in the potential. Also $G(r)=0$ for

$$2m_s r^4+2l_s{}^4 m_s=l_s{}^4 r\ . \tag{7.42}$$

Substituting (7.41) into (7.42) shows that $G(r)\leq 0$ for $l_s\geq 8/3^{3/4}m_s$. For $m_s=GM_{\odot}/c^2$ test particles are bounced away from the black-hole event horizon when $l_s\geq 5.16$ km. For photon $l_p=E=0$ and they will fall to $r=0$ in finite proper time.

VIII. CONCLUSIONS

From the conservation laws of the NGT, we have derived the equations of motion of a test particle using the field equations in the presence of matter, described by a perfect fluid with the two sources $T^{\mu\nu}$ and S^{μ} corresponding to the conserved energy-momentum tensor and the current of the fluid, respectively. The field equations of non-Riemannian space-time are nonlinear and, as in GR, they determine the motion of the particles. Given the form of the perfect-fluid energy-momentum tensor, as determined by a variational principle, the vector-current source S^{μ} and the definition of a test particle, the equations of motion follow uniquely from the conservation laws of the NGT. The problem of the motion of two massive bodies such as are found in a binary system can be solved in a similar way using the post-Newtonian method of expansion for the NGT.[17]

Only particles with $l_p=0$, such as photons, obey geodesic equations of motion in the NGT, and only for these particles is the equivalence principle satisfied; particles with $l_p\neq 0$ do not follow geodesic paths in space-time because of the existence of a force that behaves like $1/r^5$. This consequence of the NGT is in accord with the results obtained by Coleman and Korte,[24] who used a conformal causal structure of space-time. Matter with $S^{\mu}\neq 0$ is responsible for producing a non-Riemannian geometry that is characterized by the nonsymmetric affine displacement field $\Gamma^{\lambda}_{\mu\nu}$, which replaces, in the most general manner consistent with our notions of the space-time continuum, the inertial frame of special relativity. The use of a nonsymmetrical $g_{\mu\nu}$ leads to a complete dynamical theory of non-Riemannian space-time, a possibility that is lost if one insists on adhering to a Riemannian symmetric metric and a nonsymmetric affine connection. This non-Riemannian geometry also points to a fundamental difference between matter and antimatter; according to the predictions of the NGT, a particle falls at a faster rate in a gravitational field than an antiparticle.[25]

In analogy with the early developments of thermodynamics, we must attempt to provide a microscopic interpretation of the NGT. As a classical theory, it describes a structure with a high degree of mathematical and logical consistency, but a microscopic particle interpretation of the current density S^{μ} is needed to complete the theory. Any such interpretation depends on having a fundamental unified theory of particles and their interactions and since such a theory is at present not available, we must resort to models of the charge density S^0. One possibility is that S^0 corresponds to conserved fermion number.[6,7] The fermion number could be a linear combination of baryon number and lepton number as considered in Ref. 25. Another possibility is that for neutral macroscopic matter S^0 is proportional to proton number Z, which would distinguish between a neutron star and the Sun, since a neutron star consists predominantly of neutrons.

In the linear approximation of the NGT,[26,27] we can write

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} , \tag{8.1}$$

where the $h_{\mu\nu}$ are small quantities of the first order. An analysis of the linear approximation reveals that the current S^μ occurs in the Lagrangian with the coupling $h_{[\mu\nu]}S^{[\mu,\nu]}$, and because of (1.20) the current S^μ does not couple to the massless spin-0^+ particle associated with $h_{[\mu\nu]}$ in first order. A calculation of the propagator also shows that the massless spin-1^- auxiliary vector field W_μ does not propagate in lowest order; W_μ is a contact field. There are no ghost poles in the real (or hyperbolic complex) version of the theory.[27] The $h_{[\mu\nu]}$ can couple to intrinsic spin through the matter tensor $T^{[\mu\nu]}$. In this case, we have

$$T^{[\mu\nu]}=\kappa\epsilon^{\mu\nu\rho\sigma}J_{\rho,\sigma} , \tag{8.2}$$

where J_μ is the intrinsic spin pseudovector and κ is a constant. However, this coupling is expected to be very small for macroscopic systems.

In the linear approximation, we obtain Newton's force law for a static system of masses except for possible intrinsic spin effects.[26] The new NGT force arises in the higher nonlinear orders with a range of $1/r^5$, due to multiple particle exchanges. A similar situation occurs in GR, since there we obtain the Newtonian force law for static systems in the linear approximation, but the relativistic effects are produced by multiple graviton exchanges in the higher nonlinear orders. This absence of an interaction between the S^μ currents of two bodies in the linear order of approximation, explains why there is no $1/r^2$ force between S^0 charges in the NGT.

It is hoped that future experiments can test the idea that space-time is a dynamical non-Riemannian structure and confirm that matter carrying S^0 charge plays a special role in determining the geometry of space-time.

ACKNOWLEDGMENTS

I thank L. Campbell, P. Savaria, and E. Woolgar for helpful discussions. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

APPENDIX

For completeness, we shall give some details on the derivation of the static, spherically symmetric solution of the vacuum field equations and the derivation of Eqs. (4.4)–(4.6). We shall follow the method of solution given by Papapetrou in Ref. 16. The components of the real nonsymmetric tensor $g_{\mu\nu}$ are displayed in (4.1) and (4.2). By using the relation (1.2), we find that

$$\begin{gathered} g^{11}=-\frac{\gamma}{\alpha\gamma-w^2},\quad g^{22}=-\frac{1}{r^2} ,\\ g^{33}=-\frac{1}{r^2\sin^2\theta},\quad g^{00}=\frac{\alpha}{\alpha\gamma-w^2} . \end{gathered} \tag{A1}$$

The field equations in vacuum are obtained from (1.14)–(1.16) and (1.18) and (1.19) by setting $S^\mu=T^{\mu\nu}=0$. We obtain

$$g_{\mu\nu,\sigma}-g_{\rho\nu}\Gamma^\rho_{\mu\sigma}-g_{\mu\rho}\Gamma^\rho_{\sigma\nu}=0 , \tag{A2}$$

$$\mathfrak{g}^{[\mu\nu]}{}_{,\nu}=0 , \tag{A3}$$

$$R_{(\mu\nu)}(\Gamma)=0 , \tag{A4}$$

$$R_{[\mu\nu],\sigma}(\Gamma)+R_{[\nu\sigma],\mu}(\Gamma)+R_{[\sigma\mu],\nu}(\Gamma)=0 . \tag{A5}$$

The 64 linear, homogeneous equations (A2) have been solved by Papapetrou,[16] Wyman,[28] and Bonnor[29] for the static case. For the components of $g_{\mu\nu}$ in (4.1) and (4.2), we have

$$\begin{aligned} &\Gamma^1_{11}=\frac{\alpha'}{2\alpha},\quad \Gamma^1_{22}=\csc^2\theta\Gamma^1_{33}=-\frac{r}{\alpha} ,\\ &\Gamma^1_{10}=-\Gamma^1_{01}=\frac{2w}{r\alpha} ,\\ &\Gamma^2_{12}=\Gamma^2_{21}=1/r,\quad \Gamma^2_{33}=-\sin\theta\cos\theta ,\\ &\Gamma^2_{02}=-\Gamma^2_{20}=\frac{w}{r\alpha} ,\\ &\Gamma^3_{13}=\Gamma^3_{31}=1/r,\quad \Gamma^3_{23}=\Gamma^3_{32}=\cot\theta ,\\ &\Gamma^3_{03}=-\Gamma^3_{30}=\frac{w}{r\alpha},\quad \Gamma^0_{10}=\Gamma^0_{01}=\frac{\gamma'}{2\gamma}+\frac{2w^2}{r\alpha\gamma} . \end{aligned} \tag{A6}$$

The prime denotes differentiation with respect to r. Substituting (A6) into (1.10), we find the expressions

$$\begin{aligned} R_{11}(\Gamma)=&-\frac{1}{2}\left[\frac{\gamma'}{\gamma}\right]'-\frac{\gamma'}{4\gamma}\left[\frac{\gamma'}{\gamma}-\frac{\alpha'}{\alpha}\right]+\frac{\alpha'}{r\alpha}\\ &-2\left[\frac{w^2}{r\alpha\gamma}\right]'-\frac{2w^2}{r\alpha\gamma}\left[\frac{\gamma'}{\gamma}-\frac{\alpha'}{2\alpha}+\frac{2w^2}{r\alpha\gamma}\right] , \end{aligned} \tag{A7}$$

$$\begin{aligned} R_{22}(\Gamma)&=\csc^2\theta R_{33}(\Gamma)\\ &=-\frac{r}{2\alpha}\left[\frac{\gamma'}{\gamma}-\frac{\alpha'}{\alpha}\right]-\frac{1-\alpha}{\alpha}-\frac{2w^2}{\alpha^2\gamma} , \end{aligned} \tag{A8}$$

$$\begin{aligned} R_{00}(\Gamma)=&\frac{1}{2}\left[\frac{\gamma'}{\alpha}\right]'-\frac{\gamma'}{4\alpha}\left[\frac{\gamma'}{\gamma}-\frac{\alpha'}{\alpha}-\frac{4}{r}\right]+4\left[\frac{w^2}{r\alpha^2}\right]'\\ &-\frac{w^2}{r\alpha^2}\left[\frac{3\gamma'}{\gamma}-\frac{2\alpha'}{\alpha}-\frac{14}{r}+\frac{8w^2}{r\alpha\gamma}\right] , \end{aligned} \tag{A9}$$

$$R_{10}(\Gamma)=-R_{01}(\Gamma)=2\left[\frac{w}{r\alpha}\right]'+\frac{6w}{r^2\alpha} . \tag{A10}$$

By contracting (A2), it can be shown that (A3) is equivalent to

$$\Gamma^\sigma_{[\mu\sigma]}=0 . \tag{A11}$$

A calculation gives

$$(-g)^{1/2}=r^2\sin\theta(\alpha\gamma-w^2)^{1/2} \tag{A12}$$

and the only nonvanishing component of $\mathfrak{g}^{[\mu\nu]}$ is

$$\mathfrak{g}^{[10]}=-\mathfrak{g}^{[01]}=-\frac{wr^2\sin\theta}{(\alpha\gamma-w^2)^{1/2}} . \tag{A13}$$

Equation (A3) then yields

$$\frac{d}{dr}\left[\frac{wr^2\sin\theta}{(\alpha\gamma-w^2)^{1/2}}\right]=0 . \tag{A14}$$

Integrating this equation leads to the solution

$$\frac{w^2r^4}{\alpha\gamma-w^2}=l^4 , \tag{A15}$$

where l is a constant of integration. The solution for w is

$$w^2=\alpha\gamma C , \tag{A16}$$

where

$$C=\frac{l^4}{l^4+r^4} . \tag{A17}$$

Let us consider the scalar quantity

$$S=\frac{1}{\alpha}R_{11}+\frac{2}{r^2}R_{22}-\frac{1}{\gamma}R_{00} \tag{A18}$$

and the linear combinations

$$\frac{1}{\alpha}R_{11}+\tfrac{1}{2}S=0 , \tag{A19}$$

$$\frac{1}{r^2}R_{22}+\tfrac{1}{2}S=0 , \tag{A20}$$

$$\frac{1}{\gamma}R_{00}-\tfrac{1}{2}S=0 . \tag{A21}$$

We shall set $\alpha=e^{\mu}$ and $\gamma=e^{\nu}$ and from (A19)–(A21), we get the set of equations

$$e^{-\mu}\left[\frac{\nu'}{r}+\frac{1}{r^2}+\frac{4}{r^2}C-\frac{2}{r^2}C^2-\frac{\mu'+\nu'}{2r}C\right]-\frac{1}{r^2}=0 , \tag{A22}$$

$$e^{-\mu}\left[\frac{\nu''}{2}-\frac{\mu'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\mu'}{2r}+\frac{3(\nu'-\mu')}{2r}C+\frac{2}{r^2}C^2-\frac{8r^2}{l^4}C^2\right]=0 , \tag{A23}$$

$$e^{-\mu}\left[-\frac{\mu'}{r}+\frac{1}{r^2}+\frac{2}{r^2}C^2+\frac{\mu'+\nu'}{2r}C\right]-\frac{1}{r^2}=0 . \tag{A24}$$

We shall now subtract (A22) from (A23) to give

$$\frac{\mu'+\nu'}{r}+\frac{4}{r^2}C-\frac{4}{r^2}C^2-\frac{\mu'+\nu'}{r}C=0 \tag{A25}$$

or, equivalently,

$$\frac{\mu'+\nu'}{r}+\frac{4}{r^2}C=0 , \tag{A26}$$

since $1-C>0$. By using this equation in (A24),

$$e^{-\mu}\left[\frac{1}{r^2}-\frac{\mu'}{r}\right]-\frac{1}{r^2}=0 . \tag{A27}$$

This is the same equation as in GR, and leads to the same solution for α:

$$\alpha=(1-2m/r)^{-1} . \tag{A28}$$

We can write (A26) in the form

$$\frac{d}{dr}\left[\mu+\nu+\ln\left[\frac{r^4}{r^4+l^4}\right]\right]=0 . \tag{A29}$$

Integration of this equation gives

$$\alpha\gamma\left[\frac{r^4}{r^4+l^4}\right]=A^2 , \tag{A30}$$

where A is a constant of integration. Using (A22) and (A24), we can immediately verify that (A23) is satisfied. Substituting (A30) into (A16) and using (A17), gives the solution for w:

$$w=\pm A\frac{l^2}{r^2} . \tag{A31}$$

We adopt the boundary conditions that as $r\to\infty$, $w\to 0$, and $g_{(\mu\nu)}\to\eta_{\mu\nu}$. These boundary conditions require that $A=1$ and γ and w reduce to the solutions in (4.1) and (4.2). Equation (A12) now becomes

$$(-g)^{1/2}=r^2\sin\theta , \tag{A32}$$

which is the same expression obtained in flat Minkowski space and GR. By substituting the solutions for α and w into (A10), we get the result quoted in Eq. (4.4).

[1]A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, London, 1923), p. 125.

[2]H. Weyl, *Space-Time-Matter* (Dover, New York, 1952), p. 300.

[3]A. Einstein and J. Grommer, Sitz. Preuss. Akad. Wiss. 1, (1927); K. Lanczos, Z. Phys. **44**, 773 (1927).

[4]L. Infeld and A. Schild, Rev. Mod. Phys. **21**, 408 (1949).

[5]J. W. Moffat, Phys. Rev. D **19**, 3554 (1979); **19**, 3562 (1979).

[6]J. W. Moffat, J. Math. Phys. **21**, 1978 (1980).

[7]J. W. Moffat, in *The Origin and Evolution of Galaxies,* edited by V. De Sabbata (World Scientific, Singapore, 1982), p. 127.

[8]J. W. Moffat, Phys. Rev. Lett. **50**, 709 (1983).

[9]G. Kunstatter, J. W. Moffat, and J. Malzan, J. Math. Phys. **24**, 886 (1983); J. W. Moffat, Found. Phys. **14**, 1217 (1984).

[10]J. W. Moffat, Astrophys. J. Lett. **287**, L77 (1984).

[11]J. W. Moffat, Can. J. Phys. **64**, 178 (1986).

[12]A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1956), Appendix II.

[13]D. Vincent, Class. Quant. Gravit. **2**, 409 (1985).

[14]R. B. Mann and J. W. Moffat, Can. J. Phys. **59**, 1723 (1981); **61**, 656(E) (1983).

[15]A. Papapetrou, Proc. R. Soc. London **A209**, 248 (1951); *Lectures in General Relativity* (Reidel, Dordrecht-Holland/Boston-U. S. A., 1974), p. 149.

[16]A. Papapetrou, Proc. R. Ir. Acad. A **52**, 69 (1948).

[17]J. W. Moffat and E. Woolgar (in preparation).

[18]S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972), p. 187.

[19]R. D. Reasenberg *et al.*, Astrophys. J. **234**, L219 (1979).

[20]M. D. Kruskal, Phys. Rev. **119**, 1743 (1960); G. Szekeres, Publ. Mat. Debrecen **7**, 285 (1960).

[21]J. C. Graves and D. R. Brill, Phys. Rev. **120**, 1507 (1960).

[22]G. Kunstatter and J. W. Moffat, Phys. Rev. D **17**, 396 (1978).

[23]L. Campbell and J. W. Moffat, University of Toronto report 1982 (unpublished).

[24]R. A. Coleman and H. Korte, J. Math. Phys. (to be published).

[25]J. W. Moffat, P. Savaria, and E. Woolgar, University of Toronto report 1986 (unpublished).

[26]R. B. Mann and J. W. Moffat, J. Phys. A **14**, 2367 (1981).

[27]R. B. Mann and J. W. Moffat, Phys. Rev. D **26**, 1858 (1982).

[28]M. Wyman, Can. J. Math. **2**, 427 (1950).

[29]W. B. Bonnor, Proc. R. Soc. London **A209**, 353 (1951); **A210**, 427 (1952).