Cosmic strings coupled with gravitational and electromagnetic waves

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Two families of (time-dependent) cylindrically symmetric solutions of the Einstein-Maxwell electrovacuum equations are obtained. They admit a regular axis, they are free from curvature singularities, and they are asymptotically flat away from the axis. For general values of the parameters the solutions exhibit an angle deficit in going around the axis and they are interpreted as describing stable, open, cosmic strings, coupled with gravity and electromagnetism in the manner of general relativity. For particular values of the parameters the angle deficit is eliminated and the solution describes the propagation of nonradiating gravitational and electromagnetic waves. The two families are algebraically special, of Petrov type D.

I. INTRODUCTION

The cosmic strings, produced in the symmetry breaking of grand unified theories, have been proposed as providing mechanisms for galaxy formation and for gravitational lensing.[1] The open strings are usually modeled[2-7] as cylindrically symmetric perfect-fluid solutions of the Einstein equations (with equation of state "energy density plus pressure in the axial direction equal to zero") matched, at some very small radius from the axis, to an asymptotically flat exterior solution. However, the presence of the string can be detected, and its mass per unit length measured, even when we stay entirely in the exterior region, in much the same way as a current is measured from the induced magnetic flux by Ampères law: The string results in the appearance of a topological defect, or an angle deficit, in going around the axis. Equivalently, we may say for such situations that a wedge has been cut out of the spacetime or that, although the spacetime is smooth and the curvature is finite near the axis, a conical singularity is present.

Leaving aside the matching of an exterior to an interior solution, to construct a cosmic string would require one to obtain a cylindrically symmetric solution of the Einstein equations which (i) admits a regular axis, (ii) is asymptotically flat away from the axis, and (iii) is free everywhere from curvature singularities. It is usually difficult for a solution to combine all three properties and the merit of the two families of solutions obtained in the present paper is that they succeed in all three. For any such solution it is then straightforward to introduce an arbitrary angle deficit: we multiply the $(dt)^2$-$(d\omega)^2$ part of the metric by a positive constant; and this is allowed by the field equations.

Recently, we have obtained a solution for a rotating cosmic string[8,9] (Ref. 9 will be referred to hereafter as paper I). The solution solves the vacuum Einstein equations, it satisfies the three requirements mentioned in the previous paragraph, and it is nonradiating, which implies the stability of the string. In the present paper we generalize this solution, by coupling the rotating string with both gravity and electromagnetism. We obtain two families of solutions. The first family, studied in Sec. III, is a one-parameter generalization of the rotating string of paper I. In addition to the parameters p and q which control the degrees of freedom of the gravitational field, it also involves the parameters α and k which determine the mass per unit length of the string and the relative strength of the electromagnetic to the gravitational field, respectively; for $k=1$ there is no electromagnetic field and the solution reduces to that of paper I.

The second family, studied in Sec. IV, is a nonstraightforward $k \to 0$ limit of the first family. There are two parameters controlling the degrees of freedom of the gravitational and electromagnetic field and one for the mass density of the string. A physical characterization of the solution is that it cannot support a gravitational field when there is no electromagnetic field present. Both families are algebraically special, of Petrov type D, with coinciding the repeated principal null directions of the Weyl and the Maxwell tensors.

For special values of the parameter α the angle deficit (and the string) disappears and the two families describe the propagation of cylindrically symmetric gravitational and electromagnetic waves which are regular everywhere.

Twenty years ago Stachel[10] raised the question whether there exist cylindrically symmetric gravitational waves of Petrov type D. Paper I answered the question affirmatively for gravitational waves and the present paper answers it, again affirmatively, for gravitational and electromagnetic waves.

II. THE EINSTEIN-MAXWELL EQUATIONS

We consider the Einstein-Maxwell equations (without currents) in spacetimes with two spacelike commuting Killing fields. In this section we shall keep the presentation a little more general than needed, giving the equations for any of the gauge choices

$$\Delta=\eta^2\pm 1, \quad \delta=\mu^2\pm 1 ; \tag{2.1}$$

these gauges were found useful in Ref. 11 (to be referred to hereafter as paper II) for the description of inhomogeneous cosmological solutions. Eventually, we shall adopt the choice of gauge $\Delta=\eta^2+1$, $\delta=\mu^2-1$, which in paper I was demonstrated to be suitable for the description of cylindrically symmetric spacetimes and of rotating cosmic strings. Of course, one has to perform the additional coordinate transformation (3.19) to write the spacetime metric in a manifestly cylindrically symmetric form.

We adopt the notation of Chandrasekhar and Xanthopoulos[12] (to be referred to hereafter as paper III). Equations (18)—(30) of paper III are the Einstein-Maxwell equations, in the presence of two spacelike commuting Killing fields, before the specification of any gauge. Then imposing, compatibly with Eq. (27) of paper III, the gauge condition

$$e^{\psi+\mu_2}=\sqrt{\Delta\delta},\quad \Delta=\eta^2\pm1,\quad \delta=\mu^2\pm1\ , \tag{2.2}$$

the Einstein-Maxwell equations eventually reduce to the two complex Ernst equations

$$(\mathrm{Re}Z-|H|^2)[(\Delta Z_{,\eta})_{,\eta}-(\delta Z_{,\mu})_{,\mu}] =\Delta(Z_{,\eta})^2-\delta(Z_{,\mu}{}^2)-2H^*(\Delta Z_{,\eta}H_{,\eta}-\delta Z_{,\mu}H_{,\mu})\ , \tag{2.3a}$$

$$(\mathrm{Re}Z-|H|^2)[(\Delta H_{,\eta})_{,\eta}-(\delta H_{,\mu})_{,\mu}] =\Delta H_{,\eta}Z_{,\eta}-\delta H_{,\mu}Z_{,\mu}-2H^*[\Delta(H_{,\eta})^2-\delta(H_{,\mu})^2]\ . \tag{2.3b}$$

Note that

$$Z=\Psi+HH^*+i\Phi,\quad \chi=(\Delta\delta)^{1/2}/\Psi\ , \tag{2.4}$$

the line element is

$$ds^2=e^{\nu+\mu_3}\sqrt{\Delta}\left[\frac{(d\eta)^2}{\Delta}-\frac{(d\mu)^2}{\delta}\right] -\frac{\Delta\delta}{\Psi}(d\phi)^2-\Psi(dz-q_2d\phi)^2\ , \tag{2.5}$$

and the complex potential H measures the electromagnetic field. For any solution of Eqs. (2.3) the metric coefficient q_2 is obtained from

$$q_{2,\eta}=\frac{\delta}{\Psi^2}[\Phi_{,\mu}+2\,\mathrm{Im}(HH^*_{,\mu})]\ , \tag{2.6a}$$

$$q_{2,\mu}=\frac{\Delta}{\Psi^2}[\Phi_{,\eta}+2\,\mathrm{Im}(HH^*_{,\eta})]\ , \tag{2.6b}$$

while $\nu+\mu_3$ is determined by

$$\frac{\mu}{\delta}(\nu+\mu_3)_{,\mu}+\frac{\eta}{\Delta}(\nu+\mu_3)_{,\mu}=\frac{1}{\chi^2}(\chi_{,\eta}\chi_{,\mu}+q_{2,\eta}q_{2,\mu}) +\frac{2\chi}{\sqrt{\Delta\delta}}(H_{,\eta}H^*_{,\mu}+H^*_{,\eta}H_{,\mu})\ , \tag{2.7a}$$

$$2\eta(\nu+\mu_3)_{,\eta}+2\mu(\nu+\mu_3)_{,\mu} =(4-\frac{3\eta^2}{\Delta}-\frac{\mu^2}{\delta}) +\frac{1}{\chi^2}[\Delta(\chi_{,\eta}{}^2+q_{2,\eta}{}^2)+\delta(\chi_{,\mu}{}^2+q_{2,\mu}{}^2)] +\frac{4\chi}{\sqrt{\Delta\delta}}(\Delta|H_{,\eta}|^2+\delta|H_{,\mu})^2\ . \tag{2.7b}$$

Quite often we work with Ψ and Φ, which are simpler than χ and q_2. Then $\nu+\mu_3$ is most easily obtained, equivalently, by using the equations

$$\frac{\mu}{\delta}M_{,\eta}+\frac{\eta}{\Delta}M_{,\mu}=\frac{1}{\Psi^2}[\Psi_{,\eta}\Psi_{,\mu}+(\Phi_{,\eta}+I_{(\eta)})(\Phi_{,\mu}+I_{(\mu)})]+\frac{2}{\Psi}(H_{,\eta}H^*_{,\mu}+H^*_{,\eta}H_{,\mu})\ , \tag{2.8a}$$

$$2\eta M_{,\eta}+2\mu M_{,\mu}=\left[4-\frac{3\eta^2}{\Delta}-\frac{\mu^2}{\delta}\right]+\frac{4}{\Psi}(\Delta H_{,\eta}H^*_{,\eta}+\delta H_{,\mu}H^*_{,\mu}) +\frac{1}{\Psi^2}\{\Delta[\Psi_{,\eta}{}^2+(\Phi_{,\eta}+I_{(\eta)})^2]+\delta[\Psi_{,\mu}{}^2+(\Phi_{,\mu}+I_{(\mu)})^2]\}\ , \tag{2.8b}$$

where

$$M=\nu+\mu_3+\ln\left[\frac{\Psi}{\sqrt[4]{\Delta\delta}}\right]\ , \tag{2.9}$$

$$I_{(\alpha)}=2\,\mathrm{Im}(HH^*_{,\alpha}) =-i(HH^*_{,\alpha}-H^*H_{,\alpha}),\quad \alpha=\eta,\mu\ . \tag{2.10}$$

[Note that, in accordance with paper I, we have changed Φ into $-\Phi$ in the definition (2.4) of Z. Changes in signs are also introduced in Eqs. (2.6) due to passing from the gauge $\Delta=1-\eta^2,\delta=1-\mu^2$, to $\Delta=\eta^2\pm1,\delta=\mu^2\pm1$.]

III. THE FIRST FAMILY

The first family of electromagnetic strings is obtained by looking for solutions of the complex Ernst equations (2.3) for which the complex potentials H and Z are linearly dependent. Precisely, we look for solutions of Eqs. (2.3) satisfying

$$H=Q(Z+1)\ , \tag{3.1}$$

where Q is generally a complex constant. Note that the ansatz (3.1) is used, in the framework of stationary spacetimes, for the construction of the Kerr-Newman from the Kerr family of black-hole solutions. Similarly, the same ansatz has been used in paper III for the construction of a spacetime describing the collision of gravitational and electromagnetic waves. The aim now is to use the same procedure to "electrify" the rotating cosmic string obtained in paper I.

Having imposed the ansatz (3.1) we readily conclude, as in paper III, that whenever E is the Ernst potential corresponding to a vacuum solution

$$\Psi+i\Phi=\frac{1+E}{1-E} \tag{3.2}$$

and satisfying the vacuum Ernst equation

$$(1-EE^*)[(\Delta E_{,\eta})_{,\eta}-(\delta E_{,\mu})_{,\mu}] = -2E^*(\Delta E_{,\eta}{}^2-\delta E_{,\mu}{}^2) \tag{3.3}$$

then

$$\Psi_{(e)}=\frac{k^2(1-EE^*)}{|1-kE|^2},\quad \Phi_{(e)}=\frac{ik(E^*-E)}{|1-kE|^2},$$
$$H=\frac{2Q}{1-kE},\quad Z_{(e)}=\frac{1+kE}{1-kE},\quad k=(1-4QQ^*)^{1/2}, \tag{3.4}$$
$$e^{(\nu+\mu_3)_{(e)}}=\frac{1}{4k^2}[(1-k)^2(\Psi^2+\Phi^2)+2(1-k^2)\Psi+(1+k)^2]e^{(\nu+\mu_3)_{(v)}}$$

represent a solution of the Einstein-Maxwell equations (2.3) and (2.8).

From now on we shall adopt the choice of gauge (as in paper I)

$$\Delta=\eta^2+1,\quad \delta=\mu^2-1\ ; \tag{3.5}$$

and we shall choose as a vacuum solution of Eq. (3.3) the solution

$$E=\frac{p\eta+iq\mu}{p^2\eta^2+q^2\mu^2},\quad p,q \text{ const},\quad q^2-p^2=1\ , \tag{3.6}$$

which leads to the (vacuum) rotating string of paper I.

For the vacuum solution (3.6) we have (paper I, Sec. IV) that

$$\Psi=\frac{Y}{X},\quad \Phi=\frac{2q\mu}{X}, \qquad q_{2(v)}=\frac{2q\delta(1-p\eta)}{pY},\quad e^{(\nu+\mu_3)_{(v)}}=\frac{\alpha^2X}{\sqrt{\Delta}}, \tag{3.7}$$

where

$$X=(1-p\eta)^2+q^2\mu^2\ , \qquad Y=p^2\eta^2+q^2\mu^2-1=p^2\Delta+q^2\delta\ ; \tag{3.8}$$

the constants p and q specify the degrees of freedom of the gravitational field while the constant α is a measure of the "strength (or mass) per unit length" of the string.

Next we determine the metric coefficients of the electromagnetic string. Using the identity

$$Y^2+4q^2\mu^2=X(X+4p\eta) \tag{3.9}$$

we readily find that

$$(1-k)^2(\Psi^2+\Phi^2)+2(1-k^2)\Psi+(1+k)^2=\frac{4\Pi}{X}, \tag{3.10}$$

$$e^{(\nu+\mu_3)_{(e)}}=\frac{\alpha^2\Pi}{k^2\sqrt{\Delta}},\quad \Psi_{(e)}=\frac{k^2Y}{\Pi},\qquad \Phi_{(e)}=\frac{2kq\mu}{\Pi}, \tag{3.11}$$

where

$$\Pi=p^2\eta^2+q^2\mu^2+k^2-2kp\eta\ . \tag{3.12}$$

To complete the metric we need to evaluate $q_{2(e)}$ as well, from Eqs. (2.6) where, of course, as Ψ and Φ we should use the corresponding electromagnetic scalars $\Psi_{(e)}$ and $\Phi_{(e)}$. Proceeding as in paper III, Sec. 5(a) we find that

$$q_{2(e)}=\frac{(1+k)^2}{4k^2}q_{2(v)}+\frac{(1-k)^2}{4k^2}q_2^{(e)}, \tag{3.13}$$

where

$$q_{2,\eta}^{(e)}=\frac{\delta}{\Psi^2}[(\Phi^2-\Psi^2)\Phi_{,\mu}+2\Phi\Psi\Psi_{,\mu}] = -\frac{2q\delta}{Y^2}[(1+p\eta)^2-q^2\mu^2]\ , \tag{3.14a}$$

$$q_{2,\mu}^{(e)}=\frac{\Delta}{\Psi^2}[(\Phi^2-\Psi^2)\Phi_{,\eta}+2\Phi\Psi\Psi_{,\eta}] = \frac{4pq\mu\Delta(1+p\eta)}{Y^2}\ . \tag{3.14b}$$

Integration gives

$$q_2^{(e)}=\frac{2q\delta(1+p\eta)}{pY}, \tag{3.15}$$

where the constant of integration is suitably chosen so that $q_2^{(e)}$ vanishes for $\mu=1$; as we shall see in Sec. III A, $\mu=1$ will describe the azimuthal axis. From Eqs. (3.7), (3.13), and (3.15) we obtain that

$$q_{2(e)}=\frac{q\delta}{k^2pY}(1+k^2-2kp\eta)\ . \tag{3.16}$$

In conclusion, the line element reads

$$ds^2=\frac{\alpha^2\Pi}{k^2}\left[\frac{(d\eta)^2}{\Delta}-\frac{(d\mu)^2}{\delta}\right] -\frac{\Delta\delta\Pi}{k^2Y}(d\phi)^2-\frac{k^2Y}{\Pi}(dz-q_{2(e)}d\phi)^2\ , \tag{3.17}$$

where Π and $q_{2(e)}$ are given by Eqs. (3.12) and (3.16).

The remarkable similarity between the line element (3.17), which solves the Einstein-Maxwell equations, with that of Eq. (52) of paper I which solves the Einstein vacuum equations, should be noted: The former is obtained from the latter by the changes

$$X\to\frac{\Pi}{k^2}\quad\text{and}\quad q_2=q_{2(v)}\to q_{2(e)}\ . \tag{3.18}$$

It should be emphasized, however, that this similarity is not a general characteristic of the ansatz (3.1); instead, the similarity depends crucially on the particular E chosen for the vacuum solution and it is not expected to prevail in other electromagnetic solutions obtained from different vacuum solutions.

A. Physical interpretation

To interpret the metric (3.17) we express it in cylindrical coordinates given by

$$\omega=(\Delta\delta)^{1/2}, \quad t=\eta\mu, \quad \omega\in[0,\infty), \quad t\in\mathbb{R} . \tag{3.19}$$

Note that $\mu=1,\eta\in\mathbb{R}$ corresponds to the axis $\omega=0,t\in\mathbb{R}$. By using that

$$\frac{(d\eta)^2}{\Delta}-\frac{(d\mu)^2}{\delta}=\frac{1}{\eta^2+\mu^2}[(dt)^2-(d\omega)^2] \tag{3.20}$$

the metric (3.17) can be written in its cylindrically symmetric form

$$ds^2=\frac{\alpha^2\Pi}{k^2(\eta^2+\mu^2)}[(dt)^2-(d\omega)^2] -\frac{\omega^2\Pi}{k^2Y}(d\phi)^2-\frac{k^2Y}{\Pi}(dz-q_{2(e)}d\phi)^2 . \tag{3.21}$$

Since $Y=p^2(\eta^2+1)+q^2(\mu^2-1)$ and $\mu\geq 1,\eta\in\mathbb{R},Y$ is always positive and the metric (3.21) is nowhere singular.

The expressions giving (η,μ) interms of (t,ω) are quite involved [paper I, Eqs. (28)] but they simplify considerably near the axis, $\omega\to 0+$, and asymptotically, $\omega\to+\infty$. We find that:

Near the axis, $\omega \ll |t|$, $t=$ *finite*:

$$\begin{aligned}\eta&\simeq t-\frac{\omega^2 t}{2(1+t^2)}+O(\omega^4) ,\\ \mu&\simeq 1+\frac{\omega^2}{2(1+t^2)}+O(\omega^4) .\end{aligned} \tag{3.22}$$

Asymptotically, $\omega \gg |t|$:

$$\begin{aligned}\eta&\simeq\frac{t}{\omega}+\frac{t(t^2-1)}{2\omega^3}+O(\omega^{-4}) ,\\ \mu&\simeq\omega+\frac{(1-t^2)}{2\omega}+O(\omega^{-2}) .\end{aligned} \tag{3.23}$$

Using the expansions (3.22) and (3.23) we find the following behavior of the metric (3.21).

(i) Near the axis

$$\begin{aligned}&\Pi\simeq(pt-k)^2+q^2+O(\omega^2) ,\\ &Y=p^2(1+t^2)+O(\omega^2) ,\\ &\eta^2+\mu^2=1+t^2+O(\omega^2) ,\end{aligned} \tag{3.24}$$

which implies that

$$q_{2(e)}=\frac{q(1+k^2-2pkt)}{k^2p^3(1+t^2)^2}\omega^2+O(\omega^3) . \tag{3.25}$$

Hence

$$ds^2=\frac{\alpha^2A(t)}{k^2}\left[(dt)^2-(d\omega)^2-\frac{\omega^2}{\alpha^2p^2}(d\phi)^2\right]-\frac{k^2p^2}{A}(dz)^2 +\frac{2q(1+k^2-2kpt)}{pA(1+t^2)^2}\omega^2(d\phi)(dz) , \tag{3.26}$$

where

$$A=A(t)=\frac{(pt-k)^2+q^2}{1+t^2} . \tag{3.27}$$

Obviously, the curvature is smooth near the axis and the two Killing fields behave like

$$\left|\frac{\partial}{\partial z}\right|^2=O(1), \quad \left|\frac{\partial}{\partial\phi}\right|^2=O(\omega^2), \quad \frac{\partial}{\partial z}\cdot\frac{\partial}{\partial\phi}=O(\omega^2) . \tag{3.28}$$

The metric (3.21) exhibits a *conical* singularity on the axis, characterized by the angle deficit

$$(\delta\phi)_{\rm ax}=2\pi\left[1-\frac{1}{|\alpha p|}\right] , \tag{3.29}$$

which signals the presence of a cosmic string on the axis with mass density[2–4,13]

$$\mu_0=\tfrac{1}{4}\left[1-\frac{1}{|\alpha p|}\right] . \tag{3.30}$$

For the choice

$$|\alpha|=|p|^{-1} \tag{3.31}$$

the angle deficit—and the string—disappear, the axis is *locally flat,* and the metric (3.21) represents the propagation of gravitational and electromagnetic cylindrical waves.

(ii) Asymptotically

$$\Pi\simeq q^2\omega^2, \quad Y\simeq q^2\omega^2, \quad \eta^2+\mu^2\simeq\omega^2 , \tag{3.32}$$

which implies that

$$q_{2(e)}\simeq\frac{1+k^2}{k^2pq}+O(\omega^{-1}) . \tag{3.33}$$

Hence

$$ds^2\simeq\frac{\alpha^2q^2}{k^2}\left[(dt)^2-(d\omega)^2-\frac{\omega^2}{\alpha^2q^2}(d\phi)^2\right] -k^2\left[dz-\frac{1+k^2}{k^2pq}d\phi\right]^2 . \tag{3.34}$$

The metric (3.21), therefore, is *asymptotically flat in all directions away from the axis.* The original choice of Killing fields $(\partial/\partial\phi)$ and $(\partial/\partial z)$, which are orthogonal on the axis, are no longer orthogonal in the asymptotic region. Instead $(\partial/\partial\phi)$ and $(\partial/\partial\tilde{z})$ given by

$$z\to\tilde{z}=z-\frac{1+k^2}{k^2pq}\phi \tag{3.35}$$

are orthogonal, and $(\partial/\partial\tilde{z})$ becomes a spatial translation:

$$\left|\frac{\partial}{\partial \tilde{z}}\right|^2 \sim 1, \quad \left|\frac{\partial}{\partial \phi}\right|^2 = O(\omega^2), \quad \frac{\partial}{\partial \tilde{z}} \cdot \frac{\partial}{\partial \phi} = O(\omega^2) . \tag{3.36}$$

The "dragging" of the $(\partial/\partial z)$ Killing field is attributed to the rotation of the spacetime.

Asymptotically the spacetime exhibits an angle deficit

$$(\delta\phi)_{\rm as} = 2\pi\left[1-\frac{1}{|\alpha q|}\right] \tag{3.37}$$

which, since $|q| > |p|$, is always greater than the deficit $(\delta\phi)_{\rm ax}$ determined near the axis. The increase in the angle deficit is attributed to the positive contribution of the gravitational and electromagnetic waves to the deficit. The choice $|\alpha| = |q|^{-1}$ would eliminate the asymptotic angle deficit but it would impose a negative deficit—an angle surplus—near the axis.

It should be noticed that both angle deficits, near the axis and asymptotically, are independent of the parameter k which determines the strength of the electromagnetic field (for $k=1$ there is no electromagnetic field). The independence of $(\delta\phi)_{\rm ax}$ from k is physically reasonable: it expresses that the deficit near the axis is solely due to the string. The independence of $(\delta\phi)_{\rm as}$ from k however, is surprising. It suggests the conclusion that, although the intervening gravitational waves do contribute to the angle deficit, the intervening electromagnetic waves do not. However, the electromagnetic waves do contribute to the rotation of the spacetime by affecting the "dragging" of the Killing field $(\partial/\partial z)$.

Is the string really rotating? Or is all the rotation actually stored in the surrounding gravitational and electromagnetic field? To understand the details of the string one should consider a cylindrically symmetric perfect fluid solution with a regular axis—smooth curvature and no angle deficit—and match it, at some distance $\omega=\omega_0$, to the exterior solution (3.21). (By this procedure, for instance, the relation $\delta\phi = 8\pi\mu_0$, relating the angle deficit to the mass per unit length for static cylindrically symmetric strings has been established.) It is very unlikely, however, that a C^2 matching would exist with hypersurface orthogonal Killing fields in the interior and nonorthogonal Killing fields in the exterior spacetime. This is the main reason for attributing rotation to the string as well as to the surrounding gravitational and electromagnetic fields.

If we could choose $q=0$, $q_{2(e)}=0$ and the metric (3.21) would have been hypersurface orthogonal. But since $q^2-p^2=1$, $q=0$ is not in the permitted range of the parameters. The conclusion is that the metric (3.21) does not belong to a one-parameter family of solutions continuously connected to a solution with hypersurface orthogonal Killing fields, i.e., to one exhibiting no rotation. Surprisingly enough, we cannot switch off the rotation of the string.

B. The Weyl and Maxwell scalars

We change to the (ψ,θ) coordinates given by

$$\eta = \sinh\psi, \quad \mu = \cosh\theta . \tag{3.38}$$

The metric (3.17) takes the alternative form

$$ds^2 = U^2[(d\psi)^2-(d\theta)^2] - \frac{V^2}{1-\mathscr{E}\mathscr{E}^*}|(1-\mathscr{E})(dz)+i(1+\mathscr{E})(d\phi)|^2 , \tag{3.39}$$

where

$$U^2 = \frac{\alpha^2}{k^2}(p^2\sinh^2\psi + q^2\cosh^2\theta + k^2 - 2pk\sinh\psi) , \tag{3.40}$$

$$V = (\Delta\delta)^{1/4} = (\cosh\psi\sinh\theta)^{1/2} , \tag{3.41}$$

and

$$\chi_{(e)} + iq_{2(e)} = \frac{1+\mathscr{E}}{1-\mathscr{E}} . \tag{3.42}$$

Proceeding as in Chandrasekhar and Xanthopoulos[14] (to be referred hereafter as paper IV), Appendix A, we find that a suitable null tetrad is given by Eq. (A4) of that reference, and that for any metric of the form (3.39) which satisfies the Einstein-Maxwell equations and is obtained from any vacuum solution described by $\Psi + i\Phi = (1+E)/(1-E)$ via the ansatz (3.4), the only nonvanishing Weyl and Maxwell scalars are given by

$$4U^2\Psi_2 = \frac{E-k}{(1-EE^*)(1-kE^*)}\left[S(E^*_{,\psi}-E^*_{,\theta}) + D(E^*_{,\psi}+E^*_{,\theta}) + \frac{2(E-k)(E^*_{,\psi}-E^*_{,\theta})(E^*_{,\psi}+E^*_{,\theta})}{(1-EE^*)(1-kE^*)}\right] , \tag{3.43a}$$

$$\begin{aligned} 4U^2\frac{1-\mathscr{E}^*}{1-\mathscr{E}}\Psi_4 = {} & D^2 + D_{,\psi} - D_{,\theta} - D\bar{D} + \frac{2(E-k)(E^*_{,\psi\psi}+E^*_{,\theta\theta}-2E^*_{,\psi\theta})}{(1-EE^*)(1-kE^*)} \\ & + \frac{1}{1-EE^*}\left[\frac{D(E^*-k)}{1-kE}(E_{,\psi}-E_{,\theta}) + (D-2\bar{D})\frac{E-k}{1-kE^*}(E^*_{,\psi}-E^*_{,\theta})\right] \\ & + \frac{2[(1+k^2)(1+EE^*)-2k(E+E^*)]\,|E_{,\psi}-E_{,\theta}|^2}{(1-EE^*)^2|1-kE|^2} + \frac{4k(E-k)(E^*_{,\psi}-E^*_{,\theta})^2}{(1-EE^*)(1-kE^*)^2} , \end{aligned} \tag{3.43b}$$

$$4U^2\frac{1-\mathscr{E}^*}{1-\mathscr{E}}\Psi_0^*=S^2+S_{,\psi}+S_{,\theta}-S\bar{S}+\frac{2(E^*-k)(E_{,\psi\psi}+E_{,\theta\theta}+2E_{,\psi\theta})}{(1-EE^*)(1-kE)}$$
$$+\frac{1}{1-EE^*}\left[\frac{S(E-k)}{1-kE^*}(E^*_{,\psi}+E^*_{,\theta})+(S-2\bar{S})\frac{E^*-k}{1-kE}(E_{,\psi}+E_{,\theta})\right]$$
$$+\frac{2[(1+k^2)(1+EE^*)-2k(E+E^*)]\,|E_{,\psi}+E_{,\theta}|^2}{(1-EE^*)^2|1-kE|^2}+\frac{4k(E^*-k)(E_{,\psi}+E_{,\theta})^2}{(1-EE^*)(1-kE)^2}\,, \tag{3.43c}$$

$$-4U^2\Phi_{22}=D^2+D_{,\psi}-D_{,\theta}-D\bar{D}+\frac{2|E-k|^2|E_{,\psi}-E_{,\theta}|^2}{(1-EE^*)^2|1-kE|^2}$$
$$+\frac{D}{1-EE^*}\left[\frac{(E-k)(E^*_{,\psi}-E^*_{,\theta})}{1-kE^*}+\frac{(E^*-k)(E_{,\psi}-E_{,\theta})}{1-kE}\right], \tag{3.43d}$$

$$-4U^2\Phi_{00}=S^2+S_{,\psi}+S_{,\theta}-S\bar{S}+\frac{2|E-k|^2|E_{,\psi}+E_{,\theta}|^2}{(1-EE^*)^2|1-kE|^2}$$
$$+\frac{S}{1-EE^*}\left[\frac{(E^*-k)(E_{,\psi}+E_{,\theta})}{1-kE}+\frac{(E-k)(E^*_{,\psi}+E^*_{,\theta})}{1-kE^*}\right], \tag{3.43e}$$

$$-4U^2\frac{(1-\mathscr{E}^*)}{(1-\mathscr{E})}\Phi_{20}=SD+D_{,\psi}+D_{,\theta}+\frac{2(E-k)(E^*_{,\psi\psi}-E^*_{,\theta\theta})}{(1-EE^*)(1-kE^*)}+\frac{E-k}{(1-EE^*)(1-kE^*)}[S(E^*_{,\psi}-E^*_{,\theta})+D(E^*_{,\psi}+E^*_{,\theta})]$$
$$+\frac{2(1-k^2)(E_{,\psi}+E_{,\theta})(E^*_{,\psi}-E^*_{,\theta})}{(1-EE^*)|1-kE|^2}+\frac{4E(E-k)(E^*_{,\psi}+E^*_{,\theta})(E^*_{,\psi}-E^*_{,\theta})}{(1-EE^*)^2(1-kE^*)}\,, \tag{3.43f}$$

where

$$\begin{aligned}S&=(\ln V^2)_{,\psi}+(\ln V^2)_{,\theta}\,,\\ D&=(\ln V^2)_{,\psi}-(\ln V^2)_{,\theta}\,,\\ \bar{S}&=(\ln U^2)_{,\psi}+(\ln U^2)_{,\theta}\,,\\ \bar{D}&=(\ln U^2)_{,\psi}-(\ln U^2)_{,\theta}\,.\end{aligned} \tag{3.44}$$

For the solution considered in this section we have

$$\begin{aligned}S&=\tanh\psi+\coth\theta\,,\\ D&=\tanh\psi-\coth\theta\,,\\ U^2&=\frac{\alpha^2}{k^2EE^*}(1-kE)(1-kE^*)\,,\\ E_{,\psi\psi}+E_{,\theta\theta}\pm 2E_{,\psi\theta}&=-E+\frac{2}{E}(E_{,\psi}\pm E_{,\theta})^2\,,\\ |E_{,\psi}\pm E_{,\theta}|^2&=EE^*(1-EE^*)\,,\\ E_{,\psi\psi}-E_{,\theta\theta}&=-\frac{E^2}{E^*}+\frac{2}{E}({E_{,\psi}}^2-{E_{,\theta}}^2)\,,\\ {E_{,\psi}}^2-{E_{,\theta}}^2&=(1-EE^*)E^3(E^*)^{-1}\,;\end{aligned} \tag{3.45}$$

hence we find, after some major simplifications, that

$$\Psi_2=-\frac{k^2}{2\alpha^2}\frac{(E-k)(E^*)^3}{(1-kE)(1-kE^*)^3}\,, \tag{3.46a}$$

$$\frac{1-\mathscr{E}}{1-\mathscr{E}^*}\Psi_0=\frac{3k^2}{2\alpha^2}\frac{E(E-k)(E^*_{,\psi}+E^*_{,\theta})^2}{(1-EE^*)(1-kE)(1-kE^*)^3}\,, \tag{3.46b}$$

$$\frac{1-\mathscr{E}^*}{1-\mathscr{E}}\Psi_4=\frac{3k^2}{2\alpha^2}\frac{E(E-k)(E^*_{,\psi}-E^*_{,\theta})^2}{(1-EE^*)(1-kE)(1-kE^*)^3}\,, \tag{3.46c}$$

$$\Phi_{00}=\Phi_{22}=\frac{k^2(1-k^2)}{2\alpha^2}\frac{(EE^*)^2}{(1-kE)^2(1-kE^*)^2}\,, \tag{3.47a}$$

$$\frac{1-\mathscr{E}^*}{1-\mathscr{E}}\Phi_{20}=-\frac{k^2(1-k^2)}{2\alpha^2}\times\frac{EE^*(E_{,\psi}+E_{,\theta})(E^*_{,\psi}-E^*_{,\theta})}{(1-EE^*)(1-kE)^2(1-kE^*)^2}\,. \tag{3.47b}$$

Since $\Phi_{00}=\phi_0\phi_0^*\geq 0$, the parameter k should take values in the interval $[0,1]$.

It is easy to see that the consistency condition among the Maxwell scalars

$$\Phi_{20}\cdot\Phi_{20}^*=\Phi_{00}\cdot\Phi_{22} \tag{3.48}$$

is satisfied, by means of the identities (3.45). Similarly we find that the Weyl and Maxwell scalars satisfy the two identities

$$\Psi_0\cdot\Psi_4=9{\Psi_2}^2 \quad\text{and}\quad 3\Phi_{00}\cdot\Psi_2=\Phi_{20}\cdot\Psi_0\,, \tag{3.49}$$

which are the necessary and sufficient conditions[15] (Ref. 15 will be referred to hereafter as paper V) for the metric (3.17) to be of Petrov type D and the twice repeated principal null directions of the Weyl and the Maxwell tensors to coincide.

Using the asymptotic relationships (3.23) between the original (η,μ) and the cylindrical (t,ω) coordinates we readily find that the Weyl and the Maxwell scalars asymptotically behave like

$$\Psi_2 \simeq \frac{ik^3}{2\alpha^2 q^3 \omega^3} + O(\omega^{-4}) ,$$
$$\frac{1-\mathscr{E}_a}{1-\mathscr{E}_a^*}\Psi_0 \simeq \frac{3ik^3}{2\alpha^2 q^3\omega^3} + O(\omega^{-4}) , \tag{3.50}$$
$$\frac{1-\mathscr{E}_a^*}{1-\mathscr{E}_a}\Psi_4 \simeq \frac{3ik^3}{2\alpha^2 q^3\omega^3} + O(\omega^{-4}) ,$$

and

$$\Phi_{00} = \Phi_{22} \simeq \frac{k^2(1-k^2)}{2\alpha^2 q^4 \omega^4} + O(\omega^{-5}) ,$$
$$\frac{1-\mathscr{E}_a^*}{1-\mathscr{E}_a}\Phi_{20} \simeq -\frac{k^2(1-k^2)}{2\alpha^2 q^4\omega^4} + O(\omega^{-5}) , \tag{3.51}$$

where

$$\frac{1-\mathscr{E}_a}{1-\mathscr{E}_a^*} \simeq \frac{ik^2pq+k^2+1}{ik^2pq-k^2-1} + O(\omega^{-1}) . \tag{3.52}$$

A fall-off like ω^{-3} for the Weyl and ω^{-4} for the Maxwell scalars suggests that the spacetime, although time dependent, it does *not* radiate gravitational or electormagnetic energy. (It should be noted, however, that this fall-off is along spacelike, and not along null directions.) And this suggests the stability of the cosmic string predicted by the present solution. Alternatively we could argue that the stability of the string is implied from the fact [see Eq. (3.30)] that its mass per unit length is constant (i.e., time independent) and view the conclusion that the spacetime is *nonradiating* as a consistency test of the solution.

Near the axis $\omega=0$ all the Weyl and Maxwell scalars tend to finite, nonzero, time-dependent values.

C. Relation to the Kerr-Newman metric

The conclusions that (i) the metric (3.21) is of Petrov type D with the repeated principal null directions of the Weyl and the Maxwell tensors coinciding and (ii) it is obtained form the metric for the vacuum string by exactly the same ansatz which leads from the Kerr to the Kerr-Newman metric raise the question how the Kerr-Newman and the metric (3.21) are actually related. We shall now establish that the metric (3.21) is locally isometric to an analytic continuation of the Kerr-Newman metric (but not isometric to the real Kerr-Newman metric) with purely imaginary charge.

The Kerr-Newman metric with $a>m$ and signature $(-+++)$ is

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2} \times \left[dt + \frac{a(r^2+a^2-\Delta)\sin^2\theta}{\Delta - a^2\sin^2\theta}d\phi\right]^2 + \frac{\Delta\rho^2\sin^2\theta}{\Delta - a^2\sin^2\theta}(d\phi)^2 + \frac{\rho^2}{\Delta}[(dr)^2+\Delta(d\theta)^2] , \tag{3.53}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2mr + a^2 + Q_*{}^2 , \tag{3.54}$$

and m, a, and Q_* are the mass, the angular momentum per unit mass, and the charge. We perform the complex transformation

$$\theta \to i\theta ; \tag{3.55}$$

since $\cos(i\theta)=\cosh\theta$ and $\sin(i\theta)=i\sinh\theta$, we obtain a new real metric with signature $(+---)$. Next we set

$$-Q_*^2 = \frac{(1-k^2)m^2}{k^2}, \quad q = -\frac{ka}{m} ,$$
$$p = -\frac{(k^2a^2-m^2)^{1/2}}{m} , \tag{3.56}$$

where k, p, and q are real parameters subject to

$$0<k<1 \quad \text{and} \quad q^2-p^2=1 ; \tag{3.57}$$

since $Q_*^2<0$, the first of the identifications (3.56) requires that Q_* is imaginary. Moreover set

$$r-m = -\frac{pm}{k}\eta, \quad \cosh\theta = \mu , \tag{3.58}$$

which implies that

$$r^2+a^2\cosh^2\theta = \frac{m^2}{k^2}\Pi, \quad \Delta + a^2\sinh^2\theta = \frac{m^2}{k^2}Y ,$$
$$\Delta = \frac{1}{k^2}(k^2a^2-m^2)(\eta^2+1) , \tag{3.59}$$
$$\frac{a(r^2+a^2-\Delta)\sinh^2\theta}{\Delta + a^2\sinh^2\theta} = k(k^2a^2-m^2)^{1/2}q_{2(e)} .$$

And finally change, in the Kerr-Newman metric

$$t \to kmz/\alpha, \quad \phi \to m\phi\alpha^{-1}(k^2a^2-m^2)^{-1/2} . \tag{3.60}$$

The metric (3.53) transforms to a constant multiple of the metric (3.21),

$$\frac{\alpha^2}{m^2}(ds)^2_{(K-N)} = (ds)^2_{\text{string}} , \tag{3.61}$$

and the claim is established.

IV. THE SECOND FAMILY

The second family of solutions, also describing an electromagnetic string, is characterized in the notation of Sec. II by the ansatz

$$Z=1 . \tag{4.1}$$

Then, clearly, the first of the two complex Ernst equations (2.3) is identically satisfied while the second reduces to

$$(1-HH^*)[(\Delta H_{,\eta})_{,\eta} - (\delta H_{,\mu})_{,\mu}] = -2H^*(\Delta H_{,\eta}{}^2 - \delta H_{,\mu}{}^2) , \tag{4.2}$$

which is precisely the vacuum Ernst equation (3.3). The ansatz (4.1), therefore, corresponds to an alternative way for constructing, from any vacuum solution, an electromagnetic solution with two commuting Killing fields. The physical characterization of these solutions is that there is no gravitational field when there is no electromagnetic field. Such solutions have been recently investigated in paper V in the study of colliding gravitational and electromagnetic waves that develop timelike singularities. We shall here show that within the same family (4.1) there are solutions describing a rotating cosmic string coupled with gravity and electromagnetism.

As in Sec. III we shall adopt the choice of gauge

$$\Delta=\eta^2+1,\quad \delta=\mu^2-1\ ; \tag{4.3}$$

and we shall consider the solution

$$H=\frac{p\eta+iq\mu}{p^2\eta^2+q^2\mu^2}=\frac{1}{p\eta-iq\mu},\quad q^2-p^2=1\ . \tag{4.4}$$

From the first of Eqs. (2.4) we find that

$$\Phi=0,\quad \Psi=1-HH^*=\frac{Y}{p^2\eta^2+q^2\mu^2}\ , \tag{4.5}$$

where Y is again given by Eq. (3.8). (Since there is no interplay between vacuum and electromagnetic solutions in this section, we shall no longer use the subscript e to distinguish the metric coefficients of the electromagnetic solution.)

The determination of q_2 is straightforward from Eqs. (2.6), (4.4), and (4.5). We find that

$$q_{2,\eta}=-\frac{2pq\eta\delta}{Y^2},\quad q_{2,\mu}=\frac{2pq\mu\Delta}{Y^2}\ , \tag{4.6}$$

and therefore that

$$q_2=\frac{q\delta}{pY}\ , \tag{4.7}$$

where, as in Sec. III, q_2 has been made to vanish on the axis $\mu=1$ by a suitable choice of the integration constant.

Next we turn to the determination of $\nu+\mu_3$. For solutions of the family characterized by the condition (4.1) the right-hand sides of Eqs. (2.8) simplify considerably. Expressing Ψ interms of H one finds, as in paper V, Sec. 2 but with some differences in signs because of the different choice of Δ and δ that

$$\frac{\mu}{\delta}M_{,\eta}+\frac{\eta}{\Delta}M_{,\mu}=\frac{2(H_{,\eta}H^*_{,\mu}+H^*_{,\eta}H_{,\mu})}{(1-HH^*)^2}\ , \tag{4.8a}$$

$$2\eta M_{,\eta}+2\mu M_{,\mu}=\left[4-\frac{3\eta^2}{\Delta}-\frac{\mu^2}{\delta}\right]+\frac{4(\Delta H_{,\eta}H^*_{,\eta}+\delta H_{,\mu}H^*_{,\mu})}{(1-HH^*)^2}\ , \tag{4.8b}$$

where M is given by Eq. (2.9). For the solution (4.4) we find that (the easiest way: set $H=h^{-1}$, $h=p\eta-iq\mu$, and note that the right-hand sides of Eqs. (4.8) remain invariant under inversion and complex conjugation)

$$\frac{\mu}{\delta}M_{,\eta}+\frac{\eta}{\Delta}M_{,\mu}=0\ , \tag{4.9a}$$

$$2\eta M_{,\eta}+2\mu M_{,\mu}=4-\frac{3\eta^2}{\Delta}-\frac{\mu^2}{\delta}+\frac{4}{Y}\ . \tag{4.9b}$$

Integration gives

$$M=\ln Y-\tfrac{3}{4}\ln\Delta-\tfrac{1}{4}\ln\delta+\text{const}\ . \tag{4.10}$$

Hence

$$e^{\nu+\mu_3}\sqrt{\Delta}=\alpha^2(p^2\eta^2+q^2\mu^2)\ , \tag{4.11}$$

where α is a constant of integration.

To summarize, the metric corresponding to the solution (4.4) is

$$ds^2=\alpha^2\Pi\left[\frac{(d\eta)^2}{\Delta}-\frac{(d\mu)^2}{\delta}\right]-\frac{\Delta\delta\Pi}{Y}(d\phi)^2-\frac{Y}{\Pi}(dz-q_2d\phi)^2\ , \tag{4.12}$$

where

$$\Pi=p^2\eta^2+q^2\mu^2,\quad q_2=\frac{q\delta}{pY}\ ,\qquad Y=p^2\eta^2+q^2\mu^2-1\ . \tag{4.13}$$

Comparison of Eqs. (3.12), (3.16), (3.17), (4.12), and (4.13) show that the metric of this section could be obtained from a suitable

$$k\to 0,\quad \alpha\to 0,\quad k/\alpha\to\text{finite} \tag{4.14}$$

limit of the metric of Sec. III. Since the limiting procedure is not always well defined in relativity[16] it is safer to work the new solution *ab initio*.

A. Physical interpretation

Expressed in the cylindrical coordinates (3.19) the metric (4.12) becomes

$$ds^2=\frac{\alpha^2\Pi}{\eta^2+\mu^2}[(dt)^2-(d\omega)^2]-\frac{\omega^2\Pi}{Y}(d\phi)^2-\frac{Y}{\Pi}(dz-q_2d\phi)^2\ . \tag{4.15}$$

(i) Near the axis.

From Eqs. (3.22) we find that

$$\Pi\simeq p^2t^2+q^2+O(\omega^2)\ ,\qquad q_2\simeq\frac{q}{p^3(1+t^2)^2}\omega^2+O(\omega^4)\ . \tag{4.16}$$

Hence

$$ds^2\simeq\alpha^2A\left[(dt)^2-(d\omega)^2-\frac{\omega^2}{\alpha^2p^2}(d\phi)^2\right]-\frac{p^2}{A}(dz)^2+\frac{2q}{pA(1+t^2)^2}\omega^2(dz)(d\phi)\ , \tag{4.17}$$

where

$$A=A(t)=\frac{p^2t^2+q^2}{1+t^2}\ . \tag{4.18}$$

As in Sec. III A we find that the two Killing fields exhibit the behaviors (3.28), the curvature is smooth, there is a conical singularity (a cosmic string) on the axis, and the angle deficit is given by the expression (3.29).

And finally, for the choice $|\alpha|=|p|^{-1}$ the string disappears and the metric (4.15) describes cylindrically symmetric gravitational and electromagnetic waves.

(ii) Asymptotically.
From Eqs. (3.23) we find that

$$\Pi \simeq q^2\omega^2, \quad q_2 \simeq \frac{1}{pq} + O(\omega^{-1}) . \tag{4.19}$$

Hence

$$ds^2 \simeq \alpha^2 q^2 \left[(dt)^2 - (d\omega)^2 - \frac{\omega^2}{\alpha^2 q^2}(d\phi)^2 \right] - \left[dz - \frac{1}{pq} d\phi \right]^2 . \tag{4.20}$$

We conclude that the metric (4.15) is *asymptotically flat in all directions away from the axis.* The Killing fields $(\partial/\partial\phi)$ and $(\partial/\partial\tilde{z})$, where

$$z \to \tilde{z} = z - \frac{1}{pq}\phi , \tag{4.21}$$

become asymptotically orthogonal and they exhibit the behaviors (3.36). The asymptotic angle deficit is given by Eq. (3.37) and it is greater than the deficit determined near the axis.

B. The Weyl and Maxwell scalars

For any solution of the Einstein-Maxwell equations of the form (3.39) obtained by the ansatz $Z=1$ ($\Phi=0$, $\Psi=1-HH^*$) the Weyl and the Maxwell scalars in a suitable null tetrad have been evaluated in paper V, Appendix A, Eqs. (A6) and (A7). Here

$$\begin{aligned} E = H &= \frac{p\eta + iq\mu}{p^2\eta^2 + q^2\mu^2} = \frac{p\sinh\psi + iq\cosh\theta}{p^2\sinh^2\psi + q^2\cosh^2\theta} , \\ V &= (\cosh\psi \sinh\theta)^{1/2} , \\ U^2 &= \alpha^2(p^2 \sinh^2\psi + q^2\cosh^2\theta) = \frac{\alpha^2}{EE^*} . \end{aligned} \tag{4.22}$$

We find after some long reductions and very remarkable simplifications that

$$\begin{aligned} &\Psi_2 = -\frac{1}{2\alpha^2 h(h^*)^3}, \quad \Phi_{00} = \Phi_{22} = \frac{1}{2\alpha^2(hh^*)^2} , \\ &\frac{1-\mathcal{E}}{1-\mathcal{E}^*}\Psi_0 = \frac{3}{2\alpha^2}\,\frac{(p\cosh\psi + iq\sinh\theta)^2}{h(h^*)^3(hh^*-1)} , \\ &\frac{1-\mathcal{E}^*}{1-\mathcal{E}}\Psi_4 = \frac{3}{2\alpha^2}\,\frac{(p\cosh\psi - iq\sinh\theta)^2}{h(h^*)^3(hh^*-1)} , \\ &\frac{1-\mathcal{E}^*}{1-\mathcal{E}}\Phi_{20} = -\frac{(p\cosh\psi - iq\sinh\theta)^2}{2\alpha^2(hh^*)^2(hh^*-1)} , \end{aligned} \tag{4.23}$$

where

$$h = \frac{1}{H} = p\sinh\psi - iq\cosh\theta . \tag{4.24}$$

The scalars (4.23) satisfy the conditions (3.49) which imply that the metric (4.15) is of Petrov type D with the twice repeated principal null directions of the Weyl and the Maxwell scalars coinciding.

Asymptotically we find that

$$\begin{aligned} &\Psi_2 \simeq \frac{1}{2\alpha^2 q^4\omega^4} + O(\omega^{-5}) , \\ &\Psi_0 \simeq \frac{3}{2\alpha^2 q^4\omega^4} + O(\omega^{-5}) , \\ &\Psi_4 \simeq \frac{3}{2\alpha^2 q^4\omega^4} + O(\omega^{-5}) , \\ &\Phi_{00} = \Phi_{22} \simeq \frac{1}{2\alpha^2 q^4\omega^4} + O(\omega^{-5}) , \\ &\Phi_{20} = \frac{1}{2\alpha^2 q^4\omega^4} + O(\omega^{-5}) . \end{aligned} \tag{4.25}$$

The spacetime is not radiating and the string is stable. Note that the fall off of the Weyl curvature is faster in the present solution that the solution of Sec. III. Near the axis, on the other hand, all the Weyl and Maxwell scalars tend to finite, nonzero, time-dependent values.

V. DISCUSSION

Why is that the solution for rotating cosmic strings, vacuum and electromagnetic, would turn out to be of Petrov type D? And why should they be so closely related to the Kerr and Kerr-Newman black-hole solutions, being locally isometric to their analytic continuations? Is it merely the fact that we were looking for the simplest solutions of the Ernst equation and it was a matter of luck that we found some; or, are there uniqueness theorems for the string solutions, based mainly on the three properties mentioned in Sec. I, exactly as for the black-hole solutions?[17,18] And why, contrary to the situation with the black-hole solutions, are the ones we are currently considering not connected, by one-parameter family, to solutions with hypersurface orthogonal Killing fields? Why, in other words, cannot we switch-off the rotation of the strings?

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