Scalar soliton stars and black holes

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Explicit solutions of scalar soliton stars and scalar black holes are given. The former has no horizon and the latter does. The soliton stars are cold, stable, and coherent states of very large mass $M \sim (l_P m)^{-4} m$, with l_P the Planck length, m the mass of the relevant scalar field, and $\hbar = c = 1$.

I. INTRODUCTION

In this paper we give the analysis of scalar stars and scalar black holes. We first discuss the scalar soliton stars. In order to have such a solution, the system should consist of (minimally) a complex scalar field ϕ and the gravitational field $g_{\mu\nu}$. The necessary and sufficient conditions are as follows.

(i) There must be invariance under a space-independent phase transformation

$$\phi \rightarrow e^{i\theta}\phi \; ; \tag{1.1}$$

consequently, we have the conservation of its generator N, called the particle number.

(ii) In the absence of the gravitational field, the theory has nontopological soliton solutions.[1,2] In contrast, for the mini-soliton stars,[3,4] one requires the theory to satisfy only (i) but not (ii), and that makes the characteristics of a mini-soliton star quite different from those of a soliton star.

If the scalar field is a fundamental (i.e., not phenomenological) field, then in order to have a renormalizable theory and the existence of nontopological solitons [condition (ii)], one assumes, besides ϕ, an additional Hermitian scalar field χ. The simplest example is when the self-interaction of χ is of the degenerate vacuum form (in units $\hbar = c = 1$):

$$U(\chi) = \tfrac{1}{2} m^2 \chi^2 \left[1 - \frac{\chi}{\chi_0} \right]^2 , \tag{1.2}$$

with $m = \chi$ mass. We may assign $\chi = 0$ to the normal vacuum state, and $\chi = \chi_0$ to the false (or degenerate) vacuum state. (Theories of this type have been extensively studied in the literature, e.g., in connection with the bag model,[5,6] and with spontaneous T violation.[7,8])

For orientation purposes, we repeat here the qualitative feature of a scalar soliton star already discussed in I. Consider first the example of a nontopological soliton without gravity. The soliton contains an interior in which $\chi \simeq \chi_0$, a shell of width $\sim m^{-1}$, over which χ changes from χ_0 to 0, and an exterior that is essentially the vacuum. The N-carrying field ϕ is confined to the interior; this produces a kinetic energy E_k:

$$E_k \simeq \pi N / R \tag{1.3}$$

(assuming for simplicity that the mass of ϕ is zero when $\chi = \chi_0$ the false-vacuum state, but nonzero when $\chi = 0$ the normal-vacuum state). The shell contains a surface energy

$$E_s = 4\pi s R^2 ,$$

where s is the surface tension, related to χ_0 and m by

$$s \simeq \tfrac{1}{6} m \chi_0^2 . \tag{1.4}$$

The radius R can be calculated by minimizing the total energy $E = E_k + E_s$. Setting $\partial E / \partial R = 0$, we have the equipartition

$$E_k = 2E_s . \tag{1.5}$$

Hence, the soliton mass M (which is the minimum of E) can be written as

$$M = 3E_s = 12\pi s R^2 , \tag{1.6}$$

the total conserved quantum number is

$$N = 8 s R^3 , \tag{1.7}$$

and therefore, for large N,

$$M \propto N^{2/3} . \tag{1.8}$$

Because the exponent of N is < 1, when N is large the soliton mass is always less than that of the free particle solution, and that ensures its stability.

Next, we include the gravitational field. For configurations with R much greater than the Schwarzschild radius $2GM$, the effects of gravity can be treated as a perturbation. Gravity becomes important when R becomes of the same order as $2GM$. Hence, the critical mass M_c for the formation of a black hole may be estimated by simply equating R with the Schwarzschild radius

$$R \sim 2GM_c ,$$

which leads to, because of (1.6),

$$M_c \sim (48\pi G^2 s)^{-1} . \tag{1.9}$$

Since Newton's constant G is the square of the Planck length $l_P \simeq 10^{-33}$ cm, whereas a typical Higgs-type field χ may have $\chi_0 \sim m$ about, or higher than, 30 GeV (in any case, much less than the Planck mass), we have

$$M_c \sim (l_P m)^{-4} m \tag{1.10}$$

which is $\sim 10^{15} M_\odot$, with a corresponding radius $R \sim 10^2$

light years, for $m \sim 30$ GeV. In addition, the remarkable feature is for such a system, with a giant mass and a large radius, to be described by discrete coherent quantum states.

At present, very little is known concerning the nature of the Higgs-type bosons, except that they should be massive, spin 0, and have expectation values which modify the masses of other fields. Thus, M_c for the soliton star could also be much less than the above estimate, depending on the theory.

It is quite likely that all these scalar fields are in fact phenomenological fields, and their interactions should be described by an effective Lagrangian. In that case, the effective Lagrangian does not have to satisfy the renormalization condition; hence, a single complex field ϕ (without the Hermitian field χ) is sufficient to generate a nontopological soliton solution. The simplest self-interaction of ϕ with a degenerate vacuum is

$$U = m^2\phi^\dagger\phi[1-(2\phi^\dagger\phi/\sigma_0^2)]^2 , \tag{1.11}$$

where $\phi^\dagger$ is the Hermitian conjugate of ϕ and σ_0, m are constants. In the normal vacuum, $|\phi|=0$ and m is the mass of the plane-wave solution. We note that invariance under (1.1) requires U to be a function of $\phi^\dagger\phi$. In order to have a nontopological soliton solution (in the absence of gravity), the self-interaction of ϕ must contain an attractive component; this is why the coefficient of $(\phi^\dagger\phi)^2$ in (1.11) is negative. The stability condition further constrains U to be positive when $|\phi| \to \infty$, and that leads to, minimally, a sixth-order function of ϕ for the interaction U.

In Sec. II we give the general formalism of the problem for a spherically symmetric system consisting of ϕ, χ, and $g_{\mu\nu}$. At any given N, the energy levels are discrete. These are computed in Sec. III for the simple case (1.11) with $\chi=0$. In Sec. IV we extend our analysis to black holes. The solutions derived show explicitly the relationship between matter distributions and the horizon; it is hoped that such prototype calculations may lead to a better understanding of these unusual and interesting physical objects.

II. GENERAL FORMULAS

A. Lagrangian

Consider a theory consisting of a complex field ϕ, a Hermitian field χ, and a gravitational field $g_{\mu\nu}$, which is also the metric of space-time, in accordance with general relativity. The total Lagrangian is

$$L = L(g) + L(m) , \tag{2.1}$$

where $L(g)$ refers to the gravitational field Lagrangian and $L(m)$ to the matter-field Lagrangian, under the influence of $g_{\mu\nu}$. We write

$$\int L(m)dt = -\int (\phi^{\dagger\mu}\phi_\mu + \tfrac{1}{2}\chi^\mu\chi_\mu + U)|g|^{1/2}d^4x , \tag{2.2}$$

where

$$|g| = \text{absolute value of the determinant of the matrix } (g_{\mu\nu}) , \tag{2.3}$$

$d^4x = dx^0dx^1dx^2dx^3$ with $x^0 =$ time t, a dagger denotes Hermitian conjugation,

$$\phi_\mu = \partial\phi/\partial x^\mu, \quad \phi^\dagger_\mu = \partial\phi^\dagger/\partial x^\mu , \quad \chi_\mu = \partial\chi/\partial x^\mu , \tag{2.4}$$

$\phi^\mu = g^{\mu\nu}\phi_\nu$, $\phi^{\dagger\mu} = g^{\mu\nu}\phi^\dagger_\mu$, and $\chi^\mu = g^{\mu\nu}\chi_\nu$. For the general formulas derived in this section we do not fix the specific form of U, except that it is a function of $\phi^\dagger\phi$ and χ, so that the invariance requirement given by (1.1) holds. The current j^μ, defined by

$$j^\mu \equiv -i(\phi^\dagger\phi^\mu - \phi^{\dagger\mu}\phi) , \tag{2.5}$$

satisfies

$$j^\mu{}_{;\mu} = 0 ;$$

therefore

$$\frac{\partial}{\partial x^\mu}(|g|^{1/2}j^\mu) = 0 , \tag{2.6}$$

and the particle number

$$N \equiv \int j^0|g|^{1/2}dx^1dx^2dx^3 \tag{2.7}$$

is conserved. For $N \neq 0$, it is clear that ϕ must be time dependent.

For the gravitational field, we shall restrict ourselves to a time-independent and spherically symmetric metric. As in II, the square of the length differential can be written in terms of the spherical coordinates (t,ρ,α,β) as

$$ds^2 = -e^{2u}dt^2 + e^{2\bar{v}}d\rho^2 + \rho^2(d\alpha^2 + \sin^2\alpha\, d\beta^2) \tag{2.8}$$

or in terms of the isotropic coordinates (t,r,α,β) as

$$ds^2 = -e^{2u}dt^2 + e^{2v}(dr^2 + r^2d\alpha^2 + r^2\sin^2\alpha\, d\beta^2) , \tag{2.9}$$

where α,β are the standard polar and azimuthal angles, and ρ is $(2\pi)^{-1}$ times the circumference (i.e., the length of the great circle) of a two-sphere, related to r by

$$\rho = re^v . \tag{2.10}$$

The functions u, v, and $\bar{v}$ depend only on r, or, equivalently, only on ρ. Hence,

$$e^{-\bar{v}} = d\ln\rho/d\ln r = 1 + r\frac{dv}{dr} . \tag{2.11}$$

For the time-independent metric, the gravitational energy $E(g)$ is simply the negative of $L(g)$. In the spherical coordinates,

$$L(g) = -E(g) = (2G)^{-1}\int e^u\left[e^{\bar{v}} - 2\left[1+\rho\frac{du}{d\rho}\right] + e^{-\bar{v}}\left[1+2\rho\frac{du}{d\rho}\right]\right]d\rho . \tag{2.12}$$

In the isotropic coordinates,

$$L(g)=-E(g)=(2G)^{-1}\int e^{u+v}(2u'v'+v'^2)r^2dr\ , \tag{2.13}$$

where

$$u'=du/dr \text{ and } v'=dv/dr\ . \tag{2.14}$$

As shown in the Appendix of II, the minimum energy solution requires ϕ to have a harmonic dependence on t: $\phi \propto e^{-i\omega t}$. For the plane-wave solution, $\omega \geq \phi$ mass, whereas for the soliton solution

$$\omega < \phi \text{ mass}\ , \tag{2.15}$$

so that ϕ is confined. In the spherically symmetric case, we may write

$$\phi = 2^{-1/2}\sigma e^{-i\omega t}\ , \tag{2.16}$$

where σ is real and depends only on ρ (i.e., only on r). Similar reasoning gives χ as a function of ρ (or equivalently r) only, independent of time. Hence $L(m)$ can be written as

$$L(m)=\int(-U-V+W)|g|^{1/2}dx^1dx^2dx^3\ , \tag{2.17}$$

where $U=U(\sigma,\chi)$ is the same function as in (2.2),

$$V=\tfrac{1}{2}e^{-2\bar{v}}\left[\left[\frac{d\sigma}{d\rho}\right]^2+\left[\frac{d\chi}{d\rho}\right]^2\right] = \tfrac{1}{2}e^{-2v}\left[\left[\frac{d\sigma}{dr}\right]^2+\left[\frac{d\chi}{dr}\right]^2\right] \tag{2.18}$$

and

$$W=\tfrac{1}{2}\omega^2e^{-2u}\sigma^2\ . \tag{2.19}$$

The particle number, defined by (2.7), is

$$N=(2/\omega)\int W|g|^{1/2}dx^1dx^2dx^3\ . \tag{2.20}$$

The matter energy $E(m)$ is given by the corresponding Hamiltonian:

$$E(m)=N\omega-L(m) = \int(U+V+W)|g|^{1/2}dx^1dx^2dx^3\ . \tag{2.21}$$

The total energy of the system is

$$E=E(g)+E(m)\ . \tag{2.22}$$

The field equations can be derived either by taking the extremity of E with N fixed, or the extremity of L with ω fixed.

B. Basic equations

Let $\mathcal{R}_{\mu\nu}$ be the Ricci tensor and $\mathcal{R}=g^{\mu\nu}\mathcal{R}_{\mu\nu}$ the scalar curvature. The Einstein equation relates

$$G_{\mu\nu}=\mathcal{R}_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}\mathcal{R}$$

to the matter tensor: in the spherical coordinates (t,ρ,α,β) we have

$$\rho^2G_t{}^t=e^{-2\bar{v}}-1-2e^{-2\bar{v}}\rho\frac{d\bar{v}}{d\rho} = -8\pi G\rho^2(W+V+U)\ , \tag{2.23}$$

$$\rho^2G_\rho{}^\rho=e^{-2\bar{v}}-1+2e^{-2\bar{v}}\rho\frac{du}{d\rho} = 8\pi G\rho^2(W+V-U)\ , \tag{2.24}$$

and

$$\rho^2G_\alpha{}^\alpha=e^{-2\bar{v}}\left[\rho^2\frac{d^2u}{d\rho^2}+\left[1+\rho\frac{du}{d\rho}\right]\rho\frac{d}{d\rho}(u-\bar{v})\right] = 8\pi G\rho^2(W-V-U)\ ; \tag{2.25}$$

the last one is identical to that for $\rho^2G_\beta{}^\beta$. The field equations of the spin-0 fields are

$$e^{-2\bar{v}}\left[\frac{d^2\sigma}{d\rho^2}+\left[\frac{2}{\rho}+\frac{du}{d\rho}-\frac{d\bar{v}}{d\rho}\right]\frac{d\sigma}{d\rho}\right]+\omega^2e^{-2u}\sigma-\frac{\partial U}{\partial\sigma}=0 \tag{2.26}$$

and

$$e^{-2\bar{v}}\left[\frac{d^2\chi}{d\rho^2}+\left[\frac{2}{\rho}+\frac{du}{d\rho}-\frac{d\bar{v}}{d\rho}\right]\frac{d\chi}{d\rho}\right]-\frac{\partial U}{\partial\chi}=0\ . \tag{2.27}$$

It is straightforward to verify that (2.23)–(2.27) are consistent with $G^\mu_{\nu;\mu}=0$; i.e.,

$$\frac{du}{d\rho}G_t{}^t-\left[\frac{2}{\rho}+\frac{du}{d\rho}+\frac{d}{d\rho}\right]G_\rho{}^\rho+\frac{2}{\rho}G_\alpha{}^\alpha=0\ . \tag{2.28}$$

Substituting (2.23)–(2.25) into (2.28), we obtain

$$\frac{d}{d\rho}(W+V-U)=-\frac{4}{\rho}V-2(W+V)\frac{du}{d\rho}\ , \tag{2.29}$$

which is simply the sum of (2.26) times $d\sigma/d\rho$ and (2.27) times $d\chi/d\rho$. These equations can be readily derived by using either

$$\left[\frac{\delta L}{\delta u}\right]_\omega=\left[\frac{\delta L}{\delta\bar{v}}\right]_\omega=\left[\frac{\delta L}{\delta\sigma}\right]_\omega=\left[\frac{\delta L}{\delta\chi}\right]_\omega=0$$

at a constant ω, or

$$\left[\frac{\delta E}{\delta u}\right]_N=\left[\frac{\delta E}{\delta\bar{v}}\right]_N=\left[\frac{\delta E}{\delta\sigma}\right]_N=\left[\frac{\delta E}{\delta\chi}\right]_N=0$$

at a constant N, where L and E are given by (2.1) and (2.22).

In the isotropic coordinates (t,r,α,β), it is useful to introduce

$$x\equiv\dot{u}\equiv ru'\ ,\quad y\equiv1+\dot{v}\equiv1+rv'=e^{-\bar{v}}\ ,\quad \dot{x}\equiv rx',\ \dot{y}=ry'\ , \tag{2.30}$$

where, as well as throughout the paper,

$$u'=du/dr,\quad v'=dv/dr\ ,$$
$$u''=d^2u/dr^2,\quad v''=d^2v/dr^2\ , \tag{2.31}$$
$$x'=dx/dr,\quad y'=dy/dr\ .$$

Hence (2.23)–(2.25) become

$$2v''+v'^2+\frac{4}{r}v'=-8\pi G e^{2v}(W+V+U)\ , \tag{2.32}$$

$$2u'v'+v'^2+\frac{2}{r}(u'+v')=8\pi G e^{2v}(W+V-U)\ , \tag{2.33}$$

$$u''+v''+u'^2+\frac{1}{r}(u'+v')=8\pi G e^{2v}(W-V-U)\ , \tag{2.34}$$

or, alternatively,

$$2\dot{y}+y^2-1=-8\pi G r^2 e^{2v}(W+V+U)\ , \tag{2.35}$$

$$2xy+y^2-1=8\pi G r^2 e^{2v}(W+V-U)\ , \tag{2.36}$$

$$\dot{x}+\dot{y}+x^2=8\pi G r^2 e^{2v}(W-V-U)\ , \tag{2.37}$$

and (2.26) and (2.27) can be written as

$$e^{-2v}\left[\sigma''+\left[u'+v'+\frac{2}{r}\right]\sigma'\right]+\omega^2e^{-2u}\sigma-\frac{\partial U}{\partial\sigma}=0 \tag{2.38}$$

and

$$e^{-2v}\left[\chi''+\left[u'+v'+\frac{2}{r}\right]\chi'\right]-\frac{\partial U}{\partial\chi}=0 \tag{2.39}$$

with $\sigma'=d\sigma/dr$, $\chi'=d\chi/dr$, $\sigma''=d^2\sigma/dr^2$, and $\chi''=d^2\chi/dr^2$.

Substitute these solutions into the integrals (2.12), (2.13), and (2.17) for $L(g)$ and $L(m)$. Regarding $L=L(g)+L(m)$ as a function of ω, we have

$$\frac{dL}{d\omega}=\left[\frac{\partial L}{\partial\omega}\right]_{u,v,\sigma,\chi}=N\ . \tag{2.40}$$

Likewise, we may use these solutions to evaluate the corresponding $E(g)$ and $E(m)$. By regarding the soliton mass

$$M\equiv E=E(g)+E(m) \tag{2.41}$$

as a function of the particle number N, we have, on account of $E=N\omega-L$,

$$\frac{dM}{dN}=\left[\frac{\partial E}{\partial N}\right]_{u,v,\sigma,\chi}=\omega\ . \tag{2.42}$$

From the estimate (1.8), it follows that $dM/dN\propto N^{-1/3}$ and therefore

$$\frac{\omega}{m}\propto N^{-1/3}\ . \tag{2.43}$$

Consequently, for a large soliton star

$$\frac{\omega}{m}\ll 1\ . \tag{2.44}$$

By using (2.38) we see that at ∞

$$\sigma=O(e^{-(m^2-\omega^2)^{1/2}r})=O(e^{-mr})\to 0\ ; \tag{2.45}$$

correspondingly,

$$\chi\to 0 \tag{2.46}$$

its vacuum value, also exponentially. Likewise, u, v, and $\overline{v}$ approach exponentially the Schwarzschild solution. In the spherical coordinates, as $\rho\to\infty$ we have

$$e^u\sim\left[1-\frac{4a}{\rho}\right]^{1/2}\ ,\qquad e^{\overline{v}}\sim\left[1-\frac{4a}{\rho}\right]^{-1/2}\ , \tag{2.47}$$

and in the isotropic coordinates, as $r\to\infty$

$$e^u\sim\frac{r-a}{r+a}\ ,\qquad e^v\sim\left[\frac{r+a}{r}\right]^2\ , \tag{2.48}$$

where

$$a=\tfrac{1}{2}GM\ . \tag{2.49}$$

C. Soliton mass

The soliton mass M is defined by (2.41). As is well known,[9] the same M can also be derived by using the asymptotic behavior of the metric $g_{\rho\rho}=e^{2\overline{v}}$ or $g_{tt}=-e^{2u}$ at $\rho=\infty$:

$$M=\lim_{\rho\to\infty}\rho\overline{v}/G\quad\text{or}\quad M=-\lim_{\rho\to\infty}\rho u/G\ . \tag{2.50}$$

These formulas can be established by using (2.23); we find

$$\frac{d}{d\rho}[\rho e^u(1-e^{-\overline{v}})]=G\frac{d}{d\rho}[E(g)+E(m)]\ , \tag{2.51}$$

where, because of (2.12) and (2.21),

$$\frac{d}{d\rho}E(g)\equiv(2G)^{-1}e^u\left[-e^{\overline{v}}+2\left[1+\rho\frac{du}{d\rho}\right]-e^{-\overline{v}}\left[1+2\rho\frac{du}{d\rho}\right]\right]$$

and

$$\frac{d}{d\rho}E(m)\equiv 4\pi\rho^2e^{u+\overline{v}}(W+V+U)\ .$$

The integration of (2.51) gives the top equation in (2.50) directly, on account of $M=E(g)+E(m)$ and $u=\overline{v}=0$ at ∞. The Schwarzschild form (2.47) gives then the second equation in (2.50). (Although many of the arguments given here are essentially the same as in Sec. II B of II, they are repeated to make this paper self-contained.)

There are many alternative formulas for M: (2.23) and

(2.32) can also be written as

$$\frac{d}{d\rho}[\rho(1-e^{-2\bar{v}})]=8\pi G\rho^2(W+V+U) \tag{2.52}$$

and

$$\frac{d}{dr}\left[e^{v/2}r^2\frac{dv}{dr}\right]=-4\pi Gr^2e^{5v/2}(W+V+U)\ . \tag{2.53}$$

Upon integration and using the Schwarzschild solution at ∞, we obtain

$$M=4\pi\int_0^\infty (W+V+U)\rho^2 d\rho \tag{2.54}$$

and

$$M=4\pi\int_0^\infty (W+V+U)e^{5v/2}r^2dr\ . \tag{2.55}$$

Either expression establishes the positivity of M. In addition, from (2.52) and $\bar{v}=0$ at ∞, it follows that (for soliton stars)

$$0\le y=e^{-\bar{v}}\le 1\ . \tag{2.56}$$

By taking the combination $G_\alpha{}^\alpha+\frac{1}{2}(G_\rho{}^\rho-G_t{}^t)$, we have

$$\frac{d}{d\rho}\left[\rho^2e^{u-\bar{v}}\frac{du}{d\rho}\right]=8\pi G\rho^2e^{u+\bar{v}}(2W-U)\ . \tag{2.57}$$

This leads to still another formula for M:

$$\begin{aligned}M&=8\pi\int_0^\infty (2W-U)e^{u+\bar{v}}\rho^2d\rho\\ &=8\pi\int_0^\infty (2W-U)e^{u+3v}r^2dr\ ,\end{aligned} \tag{2.58}$$

which is the virial theorem

$$E(g)+4\pi\int(-3W+V+3U)e^{u+\bar{v}}\rho^2d\rho=0\ .$$

The same result can also be derived by making a scale transformation changing $\rho,u(\rho),\bar{v}(\rho),\sigma(\rho)$ to $\lambda\rho,u(\lambda\rho)$, $\bar{v}(\lambda\rho),\sigma(\lambda\rho)$ in (2.12), (2.20), and (2.21), and then setting $(\partial E/\partial\lambda)_N=0$.

Another relation can be obtained by considering the difference $G_\rho{}^\rho-G_t{}^t$; this gives

$$\frac{d}{d\rho}(u+\bar{v})=8\pi G\rho e^{2\bar{v}}(W+V) \tag{2.59}$$

and is always positive. Because $u+\bar{v}=0$ at ∞, we have (for soliton stars)

$$u+\bar{v}<0 \tag{2.60}$$

at all finite ρ.

D. Behavior near the origin

From (2.23)–(2.27), we see that, as $\rho\to 0$ (therefore, also $r\to 0$),

$$\begin{aligned}&u=u(0)+O(\rho^2)\ ,\\ &\bar{v}=O(\rho^2)\ ,\\ &\sigma=\sigma(0)+O(\rho^2)\ ,\\ &\chi=\chi(0)+O(\rho^2)\ .\end{aligned} \tag{2.61}$$

These imply that in the isotropic coordinates, as $r\to 0$ the variables x and y, defined by (2.30), are of the form

$$x=\tfrac{1}{2}ar^2+O(r^4)$$

and (2.62)

$$y=1+\tfrac{1}{2}br^2+O(r^4)\ ,$$

where a and b are constants. By using (2.35)–(2.37), we find

$$\left[\frac{dx}{dy}\right]_0\equiv\frac{a}{b}=\frac{2U(0)-4W(0)}{U(0)+W(0)}\ge -4\ , \tag{2.63}$$

where $U(0)$ and $W(0)$ are the values of U and W at $r=0$. [Note that $V(0)=0$ because of (2.61).]

III. A SIMPLE EXAMPLE

As noted in Sec. I, very likely all these scalar fields are in reality only phenomenological fields. Hence the interaction U does not have to satisfy Dyson's renormalization condition; in that case, the simplest example for a scalar soliton star is for U to be given by (1.11) and

$$\chi=0\ . \tag{3.1}$$

As in (2.16), we set

$$\phi=2^{-1/2}\sigma e^{-i\omega t}$$

for the minimum energy solution; (1.11) then becomes

$$U=\tfrac{1}{2}m^2\sigma^2\left[1-\left[\frac{\sigma}{\sigma_0}\right]^2\right]^2\ . \tag{3.2}$$

The normal vacuum is described by $|\phi|=0$, and therefore, $\sigma=0$; the "false" vacuum refers to $\sigma=\sigma_0$, which is related to $\sigma=-\sigma_0$ through a phase change $\phi\to e^{i\pi}\phi$.

By substituting $\chi=0$ into (2.2), one sees that all formulas, (2.1)–(2.63), in Sec. II remain applicable. On the other hand, some of the approximate expressions given in Sec. I have to be modified, as we shall see.

A. Order of magnitude estimates

As in Sec. I, we first give a qualitative description of the soliton solution in the absence of gravity. One may take a trial function

$$\sigma=\begin{cases}\sigma_0, & \rho<R+O(m^{-1})\ ,\\ 0, & \rho>R+O(m^{-1})\ ,\end{cases} \tag{3.3}$$

and, when $\rho=R+O(m^{-1})$,

$$\sigma\simeq\frac{\sigma_0}{(1+e^{2m(\rho-R)})^{1/2}}\ . \tag{3.4}$$

The latter expression (3.4) is chosen because it satisfies

$$\frac{d^2\sigma}{d\rho^2}-\frac{dU}{d\sigma}=0 \tag{3.5}$$

and the boundary condition (3.3). Using this trial function, one finds that the energy of the soliton is given approximately by

$$E=E_k+E_s\ , \tag{3.6}$$

where E_k is the kinetic energy due to the interior $\rho<R$,

$$E_k \simeq \frac{1}{2}\left[\frac{4\pi}{3}\right]R^3\omega^2\sigma_0^2\ , \tag{3.7}$$

and E_s is the surface energy due to the transition region $\rho=R+O(m^{-1})$,

$$E_s \simeq 4\pi s R^2\ , \tag{3.8}$$

where

$$s=\tfrac{1}{4}m\sigma_0^2\ . \tag{3.9}$$

The frequency ω is related to the particle number N by

$$N \simeq \left[\frac{4\pi}{3}\right]R^3\omega\sigma_0^2\ . \tag{3.10}$$

Keeping N fixed, and setting $\partial E/\partial R=0$ and $M=E_k+E_s$ we see that

$$\begin{aligned} &E_k=\tfrac{2}{3}E_s\ ,\\ &M=\tfrac{5}{3}E_s=5\left[\frac{4\pi}{3}\right]sR^2\ . \end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned} &M\propto N^{4/5}\ ,\\ &R\propto N^{2/5}\propto M^{1/2}\ ,\\ &\omega\propto N^{-1/5}\propto M^{-1/4}\ . \end{aligned} \tag{3.12}$$

Because $M\propto N^{4/5}$, for sufficiently large N, M has to be $<Nm$; that ensures the stability of the soliton solution.

Next, we turn on the gravitational field. It is clear that gravity becomes important only if R is of the order of the Schwarzschild radius $2GM$. By using the second equation in (3.11), we obtain an estimate of the critical mass

$$M_c \sim (G^2 s)^{-1} \sim (l_P{}^4 m\sigma_0{}^2)^{-1} \tag{3.13}$$

the same as in (1.9) and (1.10). The only modification to the two-field case, ϕ and χ, discussed in Sec. I is the different powers of N between (1.8) and (3.12).

Using (3.12) and (3.13) and treating $\sigma_0 \sim m \ll l_P{}^{-1}$, $M\sim M_c$, we can estimate the relative importance of the three matter energies U, V, and W introduced in (2.17)—(2.19). Because V and W are proportional to $(d\sigma/d\rho)^2$ and $\omega^2\sigma^2$, in the interior $\rho<R+O(m^{-1})$ we expect

$$\frac{V}{W}\sim\frac{1}{\omega^2R^2}\sim\left[\frac{m}{M}\right]^{1/2}\ll 1\ . \tag{3.14}$$

Neglecting $d\sigma/d\rho$ in the field equation of σ, (2.26), we find

$$1-\left[\frac{\sigma}{\sigma_0}\right]^2 \simeq -\frac{1}{2}\left[\frac{\omega^2}{m^2}\right]e^{-2u}\ , \tag{3.15}$$

and therefore

$$\frac{U}{W}\sim\frac{\omega^2}{m^2}\sim\left[\frac{m}{M}\right]^{1/2}\ll 1\ .$$

For $m\sim 30$ GeV and M of the same order as the critical value M_c, we have

$$\frac{V}{W}\sim\frac{U}{W}\sim l_P{}^2m^2\sim 10^{-35}\ . \tag{3.16}$$

As we shall discuss, in the transition region $\rho\simeq R$, because σ makes a rapid transition in the form (3.4) from near σ_0 to near 0 over a distance $O(m^{-1})$, $V\simeq U\gg W$. In the exterior, when ρ is $>R+O(m^{-1})$, all matter energies U, V, and W go to zero exponentially.

B. Field equations

In the spherical coordinates, the equation for σ is given by (2.26):

$$e^{-2\bar v}\left[\frac{d^2\sigma}{d\rho^2}+\left[\frac{2}{\rho}+\frac{du}{d\rho}-\frac{d\bar v}{d\rho}\right]\frac{d\sigma}{d\rho}\right]+\omega^2e^{-2u}\sigma-\frac{dU}{d\sigma}=0, \tag{3.17}$$

where, on account of (3.2),

$$\frac{dU}{d\sigma}=m^2\sigma\left[1-\left[\frac{\sigma}{\sigma_0}\right]^2\right]\left[1-3\left[\frac{\sigma}{\sigma_0}\right]^2\right]\ . \tag{3.18}$$

In the isotropic coordinates, the equation for σ is given by (2.38):

$$e^{-2v}\left[\sigma''+\left[u'+v'+\frac{2}{r}\right]\sigma'\right]+\omega^2e^{-2u}\sigma-\frac{dU}{d\sigma}=0\ , \tag{3.19}$$

where, as before, $\sigma'=d\sigma/dr$ and $\sigma''=d^2\sigma/dr^2$. Likewise, the equations for the gravitational field are given by (2.23)—(2.25), or equivalently (2.32)—(2.34), where U is given by (3.2),

$$\begin{aligned} &V=\tfrac{1}{2}e^{-2\bar v}\left[\frac{d\sigma}{d\rho}\right]^2=\tfrac{1}{2}e^{-2v}\sigma'^2\ ,\\ &W=\tfrac{1}{2}\omega^2e^{-2u}\sigma^2\ , \end{aligned} \tag{3.20}$$

in accordance with (2.18) and (2.19) and $\chi=0$. In the present case, the identity (2.28), which comes from $G^{\mu}_{\nu;\mu}=0$, is the same as (3.17) multiplied by $d\sigma/d\rho$; the matter equation is not independent of Einstein's equations.

In the following, we regard σ_0 as being of the same order as m, both in the range of 30 GeV or thereabouts. The parameter

$$\lambda\equiv(8\pi G)^{1/2}\sigma_0 \tag{3.21}$$

is, therefore, extremely small $\sim 10^{-17}$. (σ_0 could be considerably larger than 30 GeV, and λ would still be quite small.)

C. Interior: $\rho < R + O(m^{-1})$

Define

$$\bar{\rho} \equiv \lambda^2 m \rho \quad \text{and} \quad e^{-\bar{u}} \equiv \frac{\omega}{\lambda m} e^{-u} . \tag{3.22}$$

For M of the same order as $M_c \sim (G^2 m \sigma_0^2)^{-1}$, we have $R \sim GM \sim (Gm\sigma_0^2)^{-1} \sim (\lambda^2 m)^{-1}$; therefore, $\bar{\rho}$ is ~ 1. In the interior, because σ is near σ_0, we expect $d\sigma/d\rho = (d\sigma/d\bar{\rho})\lambda^2 m$ to be $O(\lambda^2)$. Equation (3.17) becomes

$$\omega^2 e^{-2u}\sigma - \frac{dU}{d\sigma} = O(\lambda^4) \tag{3.23}$$

which, together with (3.18) and (3.22), leads to

$$\frac{\sigma}{\sigma_0} = 1 + \frac{\lambda^2}{4} e^{-2\bar{u}} + O(\lambda^4) . \tag{3.24}$$

Since, according to (3.12) and (3.13),

$$\frac{\omega}{m} \sim \left[\frac{m}{M} \right]^{1/4} \sim (Gm^2)^{-1/2} \sim \lambda , \tag{3.25}$$

we expect $\bar{u}$ and u both to be ~ 1; as we shall see, this is also consistent with the field equation of u.

Because of (3.16), we have

$$W \pm V \pm U = W[1 + O(\lambda^2)] ; \tag{3.26}$$

therefore,

$$8\pi G \rho^2 (W \pm V \pm U) = \frac{1}{2} \left[\frac{\sigma}{\sigma_0} \right]^2 e^{-2\bar{u}} \bar{\rho}^2 + O(\lambda^2) . \tag{3.27}$$

In the following, we adopt the approximation that the small parameter $\lambda^2 \sim 10^{-34}$ (if $\sigma_0 \sim m \sim 30$ GeV) is

$$\lambda^2 = 0+ . \tag{3.28}$$

Hence, for $\rho < R$

$$V = U = 0, \quad \sigma = \sigma_0 ,$$

and

$$W = \tfrac{1}{2}\omega^2 \sigma_0^2 e^{-2u} , \tag{3.29}$$

because $d\bar{u}/d\bar{\rho} = du/d\bar{\rho}$, (2.23) and (2.24) become

$$2\bar{\rho}\frac{d\bar{v}}{d\bar{\rho}} = (\tfrac{1}{2} e^{-2\bar{u}}\bar{\rho}^2 - 1)e^{2\bar{v}} + 1$$

and

$$2\bar{\rho}\frac{d\bar{u}}{d\bar{\rho}} = (\tfrac{1}{2} e^{-2\bar{u}}\bar{\rho}^2 + 1)e^{2\bar{v}} - 1 . \tag{3.30}$$

When $\bar{\rho} \to 0$, these equations determine

$$\bar{u} = \bar{u}(0) + \tfrac{1}{6} e^{-2\bar{u}(0)}\bar{\rho}^2 + O(\bar{\rho}^4)$$

and

$$\bar{v} = \tfrac{1}{12} e^{-2\bar{u}(0)}\bar{\rho}^2 + O(\bar{\rho}^4) . \tag{3.31}$$

The interior solution can be obtained by first assigning at $\bar{\rho} = 0$ an initial value

$$e^{-\bar{u}(0)} = \frac{\omega}{\lambda m} e^{-u(0)} , \tag{3.32}$$

and then integrating (3.30) numerically from $\rho = 0$ to R; i.e., from $\bar{\rho} = 0$ to

$$\bar{\rho} = \bar{\rho}_{\text{in}} \equiv \lambda^2 m R , \tag{3.33}$$

which is regarded as ~ 1.

D. (x,y) trajectory

It is convenient to express the solution in terms of the variables x and y, introduced in (2.30). For the interior solution, we may substitute $V = U = 0$, given by the approximation (3.28) and (3.29), into (2.35)–(2.37) and eliminate W between them. The result is, for $\rho < R$,

$$\dot{y} = 1 - xy - y^2 ,$$
$$\dot{x} = -2 - x^2 + 3xy + 2y^2 , \tag{3.34}$$

and

$$\frac{dy}{dx} = \frac{1 - xy - y^2}{-2 - x^2 + 3xy + 2y^2} . \tag{3.35}$$

It is convenient to think of

$$\tau \equiv \ln r \tag{3.36}$$

as a fictitious time, $x(\tau)$ and $y(\tau)$ as the trajectory of a "particle," and $\dot{x} = dx/d\tau$ and $\dot{y} = dy/d\tau$ as its velocity components. Each solution describes a trajectory in the (x,y) plane. As $\bar{\rho} \to 0$ (therefore $r \to 0$ and $\tau \to -\infty$), it follows from (3.31) that

$$x = \tfrac{1}{3} e^{-2\bar{u}(0)}\bar{\rho}^2 + O(\bar{\rho}^4)$$

and

$$y = 1 - \tfrac{1}{12} e^{-2\bar{u}(0)}\bar{\rho}^2 + O(\bar{\rho}^4) . \tag{3.37}$$

There are two critical points of (3.34), defined by $\dot{x} = \dot{y} = 0$:

(i) $x = 0$ and $y = 1$

and

(ii) $x = y = 2^{-1/2}$. (3.38)

From (3.37), we see that at $\rho = 0$, the trajectory begins at (i) with an initial slope

$$\left[\frac{dy}{dx} \right]_0 = -\frac{1}{4} . \tag{3.39}$$

[This agrees with (2.63), since $U(0)/W(0) = O(\lambda^2)$ can be neglected.] When ρ increases from 0 to R, the interior solution moves along a universal trajectory, called I (the interior), in the (x,y) plane; I is completely determined by the first-order differential equation (3.35) with the initial condition (3.39), and is shown by the solid curve in Fig. 1.

At $\rho = R-$, the solution is at a point, called "in" (denoting the inner face of the surface) on I, with

$$x = x_{\text{in}} \quad \text{and} \quad y = y_{\text{in}} . \tag{3.40}$$

As shown in Fig. 2 (and as we shall see), in the surface re-

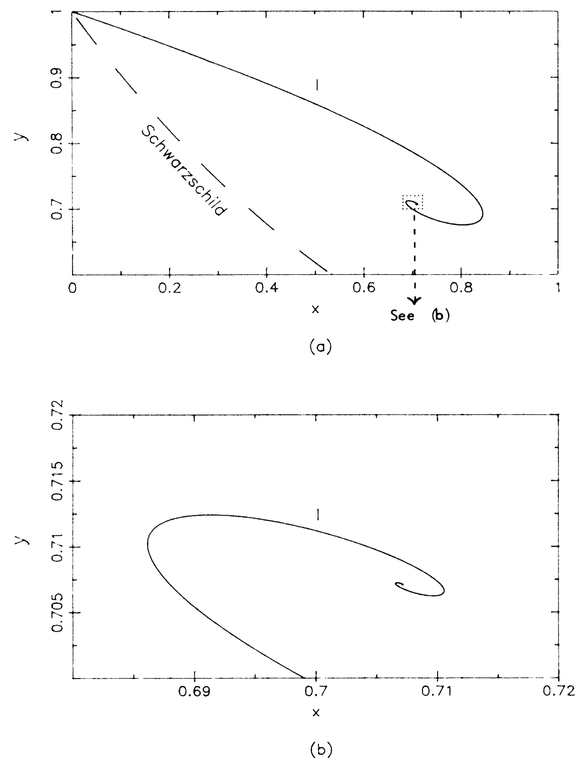

FIG. 1. The universal trajectory I, determined by (3.35) with the boundary condition (3.39) at $x=0$ and $y=1$. (The dashed curve is the Schwarzschild hyperbola $2xy+y^2-1=0$.)

gion when ρ increases from $R-$ to $R+$, the solution leaves I abruptly, moves along the straight line

$$x-x_{\text{in}}=y-y_{\text{in}} , \tag{3.41}$$

and ends at a point, called

$$A:(x_A,y_A) \tag{3.42}$$

on the Schwarzschild hyperbola

$$2xy+y^2-1=0 , \tag{3.43}$$

with x_A and y_A both >0. Afterwards, we are in the exterior region $\rho>R$, which is described by the Schwarzschild solution (2.47)–(2.49), i.e.,

$$x=\frac{2ar}{r^2-a^2}$$

and (3.44)

$$y=e^u=\frac{r-a}{r+a} .$$

The trajectory then moves along the hyperbola (3.43) from A when $\rho=R+$, back to point (i) when $\rho=\infty$. Different solutions are characterized only by different points $(x_{\text{in}},y_{\text{in}})$ when the transition occurs; neither I nor the Schwarzschild hyperbola depend on the particularities of the individual solution.

In Fig. 1 the dashed curve is the Schwarzschild hyperbola, and the solid curve is I. We see that as $\rho\to\infty$

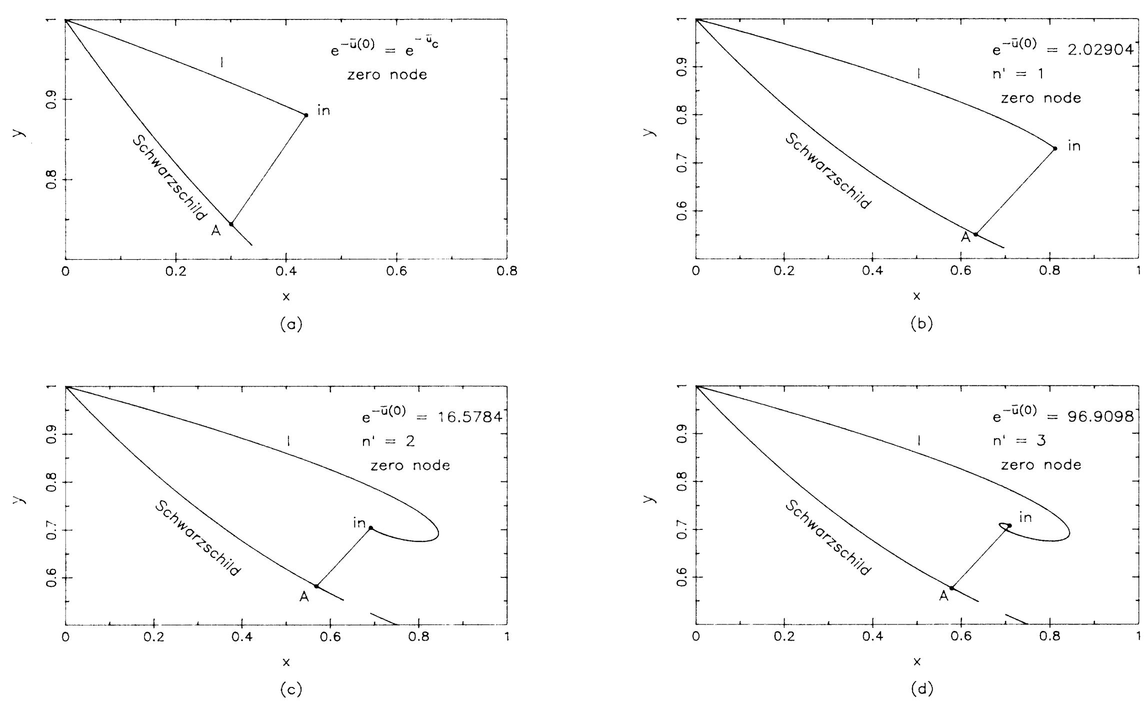

FIG. 2. Four examples of the (x,y) trajectory of the soliton star solution. Each trajectory consists of three sections: (i) interior, from $x=0,y=1$ to "in" along I, (ii) surface, from "in" to A, (iii) exterior, from A back to $x=0,y=1$ along the Schwarzschild hyperbola $2xy+y^2-1=0$. (a) refers to the critical solution c of (3.90) and (3.91). (b)–(d) are the first three (zero-node, $n=0$) cusp solutions, with $n'=1$, 2, and 3.

(therefore r and $\tau=\ln r$ also $\to\infty$), I spirals indefinitely towards the point (ii), $x=y=2^{-1/2}$. This can be understood by expanding the solution near (ii):

$$x=2^{-1/2}+\xi \text{ and } y=2^{-1/2}+\eta . \tag{3.45}$$

Treating ξ and η as infinitesimals, we can write (3.34) as

$$\frac{d}{d\tau}\begin{bmatrix}\xi\\ \eta\end{bmatrix}=\mathcal{M}\begin{bmatrix}\xi\\ \eta\end{bmatrix}, \tag{3.46}$$

where

$$\mathcal{M}=2^{-1/2}\begin{bmatrix}1 & 7\\ -1 & -3\end{bmatrix}.$$

The eigenvalues of $\mathcal{M}$ are

$$(-1\pm\sqrt{3}i)/\sqrt{2} \tag{3.47}$$

which shows that as $\tau=\ln r\to\infty$, the trajectory oscillates indefinitely, with an exponentially decreasing amplitude for ξ and η.

The surface (i.e., transition region) begins at the point "in" when the solution leaves I. As we shall prove in the next section, this point is determined by

$$\tfrac{1}{8}\bar{\rho}_{\text{in}}=\tfrac{1}{3}\{x_{\text{in}}+2y_{\text{in}}-[(x_{\text{in}}-y_{\text{in}})^2+3]^{1/2}\} , \tag{3.48}$$

where $\bar{\rho}_{\text{in}}$, x_{in}, and y_{in} are defined by (3.33) and (3.40).

E. Surface region $\rho=R+O(m^{-1})$

For M of the same order as M_c, (3.13), the ratio of the surface width to the radius R is only $(mR)^{-1}\sim\lambda^2$ ($\sim10^{-34}$ if $\sigma_0\sim m\sim30$ GeV). As in the previous section, the extreme smallness of λ^2 greatly simplifies the solution within the surface. In this region it is more convenient to use the isotropic coordinates, given by (2.9). When $\rho=R$ (and therefore $\bar{\rho}=\bar{\rho}_{\text{in}}$), denote

$$u=u_s,\quad v=v_s ,$$

and

$$r=r_s=Re^{-v_s} . \tag{3.49}$$

Within the surface $d\sigma/dr$ is $O(m\sigma_0)$, but because $x=r\,du/dr$ and $y=1+r\,dv/dr$ are ~1, du/dr and dv/dr are both $O(r_s^{-1})$, i.e., $\lambda^2O(m)$. Hence, neglecting $O(\lambda^2)$, we can regard $u=u_s$ and $v=v_s$ as constants across the surface; in addition, since ω^2/m^2 is also $O(\lambda^2)$, Eq. (3.19) becomes then

$$e^{-2v_s}\sigma''-\frac{dU}{d\sigma}=0 . \tag{3.50}$$

This gives the solution, valid for $r=r_s+O(m^{-1})$,

$$\sigma\simeq\frac{\sigma_0}{(1+e^{\mu(r-r_s)})^{1/2}} , \tag{3.51}$$

where

$$\mu=2me^{v_s} . \tag{3.52}$$

[Without the gravitational field, (3.51) reduces to (3.4).] To the same accuracy, we have within the surface

$$U=V=O(m^2\sigma_0^2)$$

but

$$W=O(\omega^2\sigma_0^2)=\lambda^2O(U) . \tag{3.53}$$

By using (3.51), we find the integrals of U and V across the shell (i.e., the surface):

$$\int_{\text{shell}}U\,dr=\int_{\text{shell}}V\,dr=\tfrac{1}{8}m\sigma_0^2e^{-v_s} . \tag{3.54}$$

Hence, in the approximation (3.28), $\lambda^2=0+$, we may write (valid in the surface region)

$$U=V=\tfrac{1}{8}m\sigma_0^2e^{-v_s}\delta(r-r_s)$$

and

$$W=0 \tag{3.55}$$

in (2.35) and (2.37); these lead to, for $r_s+>r>r_s-$,

$$\frac{dx}{dr}=\frac{dy}{dr}=-\pi Gr_se^{v_s}m\sigma_0^2\delta(r-r_s) . \tag{3.56}$$

In the exterior region $r>r_s$, the same approximation $\lambda^2=0+$ leads to zero matter density, and therefore

$$U=V=W=0 . \tag{3.57}$$

The solution has to lie on the Schwarzschild hyperbola (3.43).

Integrating (3.56) across the surface from $r=r_s-$ to r_s+, we see that the discontinuities in x and y from "in" to A are

$$\Delta x=x_{\text{in}}-x_A=\tfrac{1}{8}\bar{\rho}_{\text{in}}=\pi Gr_se^{v_s}m\sigma_0^2$$

and

$$\Delta y=y_{\text{in}}-y_A=\tfrac{1}{8}\bar{\rho}_{\text{in}}=\pi Gr_se^{v_s}m\sigma_0^2 , \tag{3.58}$$

where $\bar{\rho}_{\text{in}}=\lambda^2mR$ is given by (3.33), with $\lambda^2=8\pi G\sigma_0^2$ and $R=r_se^{v_s}$. Because x_A and y_A are on the Schwarzschild hyperbola, it follows that

$$2x_Ay_A+y_A^2-1=0 . \tag{3.59}$$

Expressing x_A and y_A in terms of x_{in}, y_{in}, and $\bar{\rho}_{\text{in}}$, we derive the condition (3.48).

F. Soliton mass

Since A is on the Schwarzschild hyperbola, we have, in accordance with (3.44) and (2.48),

$$x_A=\frac{2ar_s}{r_s^2-a^2},\quad y_A=\frac{r_s-a}{r_s+a} ,$$
$$e^{u_s}=\frac{r_s-a}{r_s+a},\quad e^{v_s}=\left[\frac{r_s+a}{r_s}\right]^2 . \tag{3.60}$$

Therefore from x_A and y_A, one can deduce r_s/a, e^{u_s}, and e^{v_s}. From (3.58), r_s is also determined; therefore, so is M, since $a=\frac{1}{2}GM$. The same M can also be computed by using the mass formulas (2.54) and (2.55).

We discuss first the evaluation of (2.55) in the isotropic coordinates. In the approximation (3.28), $\lambda^2=0+$, (2.55) can be written as

$$M = M_I + M_S ,$$

where

$$M_I = 4\pi \int_0^{r_s -} (W+V+U)e^{5v/2}r^2 dr \tag{3.61}$$

and

$$M_S = 4\pi \int_{r_s -}^{r_s +} (W+V+U)e^{5v/2}r^2 dr . \tag{3.62}$$

In the interior $r<r_s$, we have (3.29). Therefore, (2.32) becomes

$$2v''+v'^2+\frac{4}{r}v' = -8\pi G e^{2v} W ;$$

i.e.,

$$(r^2 v' e^{v/2})' = -4\pi G r^2 e^{5v/2} W . \tag{3.63}$$

From the definitions (2.30) and (3.40) of y and $y_{\rm in}$, it follows that

$$y_{\rm in} = 1 + rv' \quad \text{at } r = r_s - .$$

Integrating (3.63) from $r=0$ to $r_s -$, we find

$$M_I = G^{-1}(1-y_{\rm in}) r_s e^{v_s/2} . \tag{3.64}$$

The surface integral, on account of (3.55) and (3.58), is

$$M_S = \pi m \sigma_0^2 r_s^2 e^{3v_s/2}$$
$$= G^{-1} r_s e^{v_s/2}(y_{\rm in} - y_A) . \tag{3.65}$$

The sum of M_I and M_S gives the mass of the soliton star:

$$M = G^{-1}(1-y_A) r_s e^{v_s/2} . \tag{3.66}$$

Using (3.60), we find the right-hand side of (3.66) to be $G^{-1}2a = M$, in agreement with the expected answer.

Of course, the soliton mass can also be computed by using (2.54) in the spherical coordinates:

$$M = \overline{M}_I + \overline{M}_S , \tag{3.67}$$

where

$$\overline{M}_I = 4\pi \int_0^{R-} (W+V+U)\rho^2 d\rho$$

and (3.68)

$$\overline{M}_S = 4\pi \int_{R-}^{R+} (W+V+U)\rho^2 d\rho .$$

In the spherical coordinates, $\rho < R$ refers to the interior and $\rho > R$ to the exterior. Because of (2.10) and (2.30) we have

$$\rho = re^{v}, \quad y = e^{-\overline{v}} ,$$
$$\frac{d\rho}{dr} = ye^{v} . \tag{3.69}$$

When $\rho < R$, we have the interior solution $U=V=0$ (in the approximation $\lambda^2 = 0+$). Hence, (2.23) becomes

$$\frac{d}{d\rho}[(e^{-2\overline{v}}-1)\rho] = -8\pi G \rho^2 W ,$$

which gives, upon integration from $\rho=0$ to $R-$,

$$\overline{M}_I = (2G)^{-1}(1-y_{\rm in}^2)R . \tag{3.70}$$

It is important to note that, because $y=e^{-\overline{v}}$, across the surface $R+ > \rho > R-$, the metric $g_{\rho\rho} = e^{2\overline{v}}$, like y, is a step function. In contrast, the metrics $g_{tt} = -e^{2u}$ and $g_{rr} = e^{2v}$ in the isotropic coordinates are all continuous. Only their derivatives (with respect to r) are step functions at the surface; from (3.55), (3.56), and (3.58), it follows that

$$\frac{dx}{dr} = -\delta(r-r_s)\Delta x = -8\pi G r_s e^{2v_s} V$$

and (3.71)

$$\frac{dy}{dr} = -\delta(r-r_s)\Delta y = -8\pi G r_s e^{2v_s} V .$$

Using (3.69), we write

$$V\rho\, d\rho = Ve^{2v} ry\, dr = Ve^{2v} ry \frac{dr}{dy} dy$$
$$= -(8\pi G)^{-1} y\, dy .$$

When ρ varies from $R-$ to $R+$, y changes from $y_{\rm in}$ to y_A. Consequently,

$$\overline{M}_S = 4\pi \int_{R-}^{R+} (V+U)\rho^2 d\rho = -G^{-1}R \int_{y_{\rm in}}^{y_A} y\, dy$$
$$= (2G)^{-1}(y_{\rm in}^2 - y_A^2)R .$$

The sum $\overline{M}_I + \overline{M}_S$ is

$$M = (2G)^{-1}(1-y_A^2)R . \tag{3.72}$$

At $\rho = R+$, the Schwarzschild solution (2.47) holds. Since $y = e^{-\overline{v}}$, we find

$$y_A^2 = 1 - \frac{4a}{R} ,$$

and (3.72) gives the correct relation $a = \frac{1}{2}GM$, as expected.

G. A self-consistency check

On the inside face of the surface, $r = r_s -$ (or equivalently $\rho = R-$), we have $(x,y) = (x_{\rm in}, y_{\rm in})$. Let the corresponding value of W be $W_{\rm in}$. As a further check on the self-consistency of our solution (in the approximation $\lambda^2 = 0+$) we may compute $W_{\rm in}$ by using either (a) the interior solution, or (b) the exterior and the surface solution. In method (a), (2.36) and (3.29) give, at $r = r_s -$,

$$8\pi G r_s^2 e^{2v_s} W_{\rm in} = 2x_{\rm in} y_{\rm in} + y_{\rm in}^2 - 1 . \tag{3.73}$$

In method (b), we note that in the isotropic coordinates, (2.29) can be written as

$$r\frac{d}{dr}(U-V-W) = 2[(ru'+2rv'+2)V + ru'W]$$
$$= 2[(x+2y)V + xW] . \tag{3.74}$$

In the outside region when r is $> r_s$, we have $U=V=W=0$; on the surface, for $r_s+ > r > r_s-$, the solution is $W=0$ and $U=V$ given by (3.55); at $r = r_s-$, we have $U=V=0$ and $W=W_{\rm in}$. Thus, integrating (3.74) from $r=r_s-$ to r_s+, we obtain

$$r_s W_{\rm in}=2\int_{r_s-}^{r_s+}(x+2y)V\,dr\ . \tag{3.75}$$

Note that V is a δ function and $x+2y$ a step function. Because of (3.71), (3.75) becomes

$$r_s W_{\rm in}=-(4\pi G r_s)^{-1}e^{-2v_s}\int_{r_s-}^{r_s+}\left[x\frac{dx}{dr}+2y\frac{dy}{dr}\right]dr\ ,$$

which gives

$$8\pi G r_s{}^2 e^{2v_s}W_{\rm in}=x_{\rm in}{}^2-x_A{}^2+2(y_{\rm in}{}^2-y_A{}^2)\ . \tag{3.76}$$

Using (3.58) and (3.59), we find

$$(x_{\rm in}-y_{\rm in})^2=(x_A-y_A)^2=x_A{}^2+2y_A{}^2-1\ ,$$

and therefore (3.73)=(3.76), confirming the consistency between the two methods (a) and (b).

H. Numerical results

Only the *zero*-node solutions (node number $n=0$) will be given here. As in the mini-soliton stars, σ can have nodes. We have also calculated the $n\neq 0$ solutions; those results will be given elsewhere.

Following the method outlined in Sec. III C, we first assign at $\bar\rho=\lambda^2 m\rho=0$ an initial value $e^{-\bar u(0)}$, in accordance with (3.32). The two coupled first-order differential equations (3.30) are then integrated from $\bar\rho=0$ to $\bar\rho>0$. The actual task of integration can be facilitated by observing the invariance of (3.30) under the transformation

$$e^{-\bar u}\to\kappa e^{-\bar u},\quad \bar\rho\to\bar\rho/\kappa,\quad \bar v\to\bar v\ , \tag{3.77}$$

where κ is a constant. Of course, the boundary condition (3.32) must vary accordingly, with

$$\bar u(0)\to\bar u(0)-\ln\kappa\ . \tag{3.78}$$

Consequently, solutions with different initial values $e^{-\bar u(0)}$ are related to each other.

Define $\hat u(\hat\rho)$ and $\hat v(\hat\rho)$ to be the solution of

$$2\hat\rho\, d\hat v/d\hat\rho=(\tfrac12 e^{-2\hat u}\hat\rho^2-1)e^{2\hat v}+1$$

and (3.79)

$$2\hat\rho\, d\hat u/d\hat\rho=(\tfrac12 e^{-2\hat u}\hat\rho^2+1)e^{2\hat v}-1\ ,$$

with the boundary condition

$$\hat u=\hat v=0\quad \text{at } \hat\rho=0\ . \tag{3.80}$$

Any solution of (3.30) with the boundary condition $\bar u(0)\neq 0$ can then be derived from $\hat u(\hat\rho)$ and $\hat v(\hat\rho)$ through

$$\exp[-\bar u(\bar\rho)]=\exp[-\hat u(\hat\rho)-\bar u(0)]$$

and (3.81)

$$\exp[-\bar v(\bar\rho)]=\exp[-\hat v(\hat\rho)]\ ,$$

where

$$\bar\rho=\hat\rho e^{\bar u(0)}\ . \tag{3.82}$$

Because $y=e^{-\bar v}$ and $x=y\bar\rho\, d\bar u/d\bar\rho$, we have

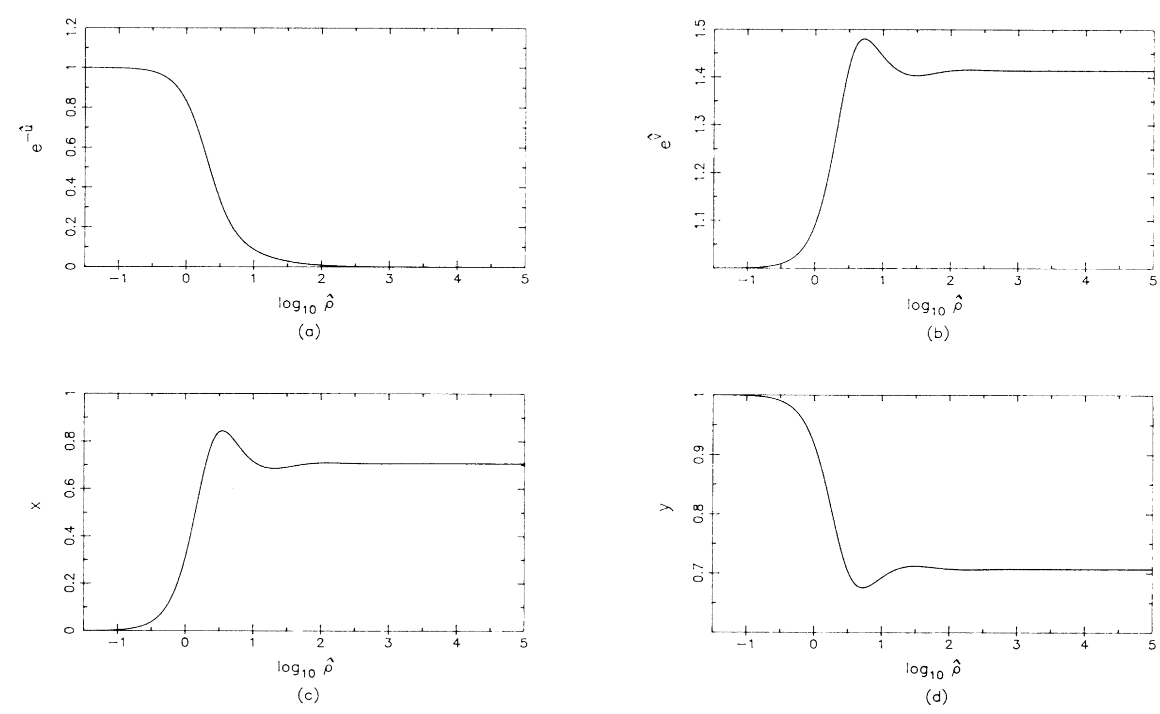

FIG. 3. From the solutions of $\hat u(\hat\rho)$, $\hat v(\hat\rho)$, $x(\hat\rho)$, and $y(\hat\rho)$ of (3.79) and (3.80), one can derive $\bar u(\bar\rho)$, $\bar v(\bar\rho)$, $x(\bar\rho)$, and $y(\bar\rho)$ through (3.81) and (3.82), for any initial value $\bar u(0)$.

$$y = e^{-\hat{v}}$$
and $\qquad$ (3.83)
$$x = y\hat{\rho}\, d\hat{u}/d\hat{\rho}\ .$$

Thus, from $\hat{u}(\hat{\rho})$ and $\hat{v}(\hat{\rho})$, we also derive $x(\hat{\rho})$ and $y(\hat{\rho})$. These four functions are plotted in Fig. 3.

In order to have a solution of the soliton star, we must satisfy (3.48):

$$\tfrac{1}{8}\bar{\rho} = \Delta_-(x,y)\ , \tag{3.84}$$

where

$$\Delta_-(x,y) \equiv \tfrac{1}{3}\{x + 2y - [(x-y)^2 + 3]^{1/2}\}\ . \tag{3.85}$$

(The significance of the subscript $-$ will become clear when we discuss the black-hole solution.) Substituting the solutions $x(\hat{\rho})$ and $y(\hat{\rho})$ into (3.85) we define

$$\Delta_-(\hat{\rho}) \equiv \Delta_-(x(\hat{\rho}), y(\hat{\rho}))\ ; \tag{3.86}$$

hence, (3.84) becomes

$$\tfrac{1}{8} e^{\bar{u}(0)}\hat{\rho} = \Delta_-(\hat{\rho})\ , \tag{3.87}$$

whose solution $\hat{\rho} = \hat{\rho}_{\text{in}}$ determines a $\bar{\rho}_{\text{in}}$ through (3.82); i.e.,

$$\bar{\rho}_{\text{in}} = \hat{\rho}_{\text{in}} e^{\bar{u}(0)}\ . \tag{3.88}$$

From (3.40) and (3.58), it follows that

$$x_{\text{in}} = x(\hat{\rho}_{\text{in}}),\quad y_{\text{in}} = y(\hat{\rho}_{\text{in}})\ ,$$
$$x_A = x_{\text{in}} - \Delta_-(\hat{\rho}_{\text{in}}),\quad y_A = y_{\text{in}} - \Delta_-(\hat{\rho}_{\text{in}})\ . \tag{3.89}$$

These and (3.60) then determine M, R, r_s, and other physical characteristics of the soliton star.

In Fig. 4 the solid curve is $\Delta_-(\hat{\rho})$ which is independent of $e^{-\bar{u}(0)}$, and the dashed line is $e^{\bar{u}(0)}\hat{\rho}/8$. For $e^{-\bar{u}(0)} = 2.5$, there are two solutions of (3.87). It is clear that if we decrease $e^{-\bar{u}(0)}$, the dashed line will swing counterclockwise, until it reaches a critical point, called c, when

$$e^{-\bar{u}(0)} = e^{-\bar{u}_c} = 1.266\,289\,8\ . \tag{3.90}$$

At c, (3.87) has only one unique solution; its physical characteristics are

$$\omega = 0.7116 m\lambda = 2.013(\pi G)^{1/2} m\sigma_0\ ,$$
$$N = 0.4963 \times 4\pi\sigma_0^2/m^2\lambda^5 = 0.010\,96/\pi^{3/2}G^{5/2}m^2\sigma_0^3\ ,$$
$$M = 0.4861 \times 4\pi\sigma_0^2/m\lambda^4 = 0.030\,38/\pi G^2 m\sigma_0^2\ , \tag{3.91}$$
$$R = 1.0882/m\lambda^2 = 4.477GM\ ,$$
$$\bar{\rho}_{\text{in}} = 1.0882\ .$$

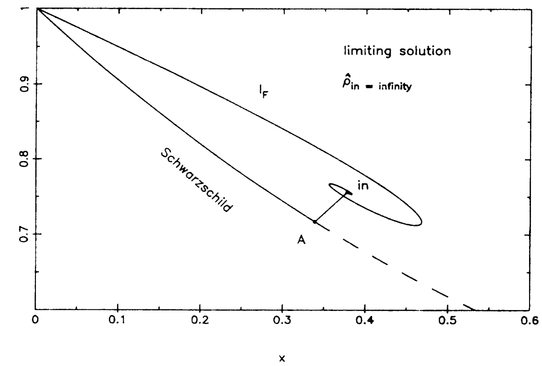

FIG. 5. (x,y) trajectory of the limiting case L for the soliton star when $e^{-\bar{u}(0)} \to \infty$ and $\hat{\rho}_{\text{in}} \to \infty$. The point "in" is at $x = y = 1/\sqrt{2}$; therefore the upper curve consists of the entire universal trajectory I.

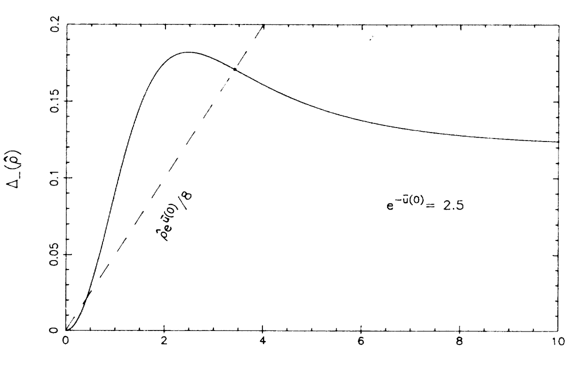

FIG. 4. The solid curve is $\Delta_-(\hat{\rho})$ defined by (3.85) and (3.86). The initial value $e^{-\bar{u}(0)}$ determines the slope of the dashed line, $\frac{1}{8}e^{\bar{u}(0)}\hat{\rho}$, whose intersection with $\Delta_-(\hat{\rho})$ gives a solution of the soliton star, in accordance with (3.87) and (3.88).

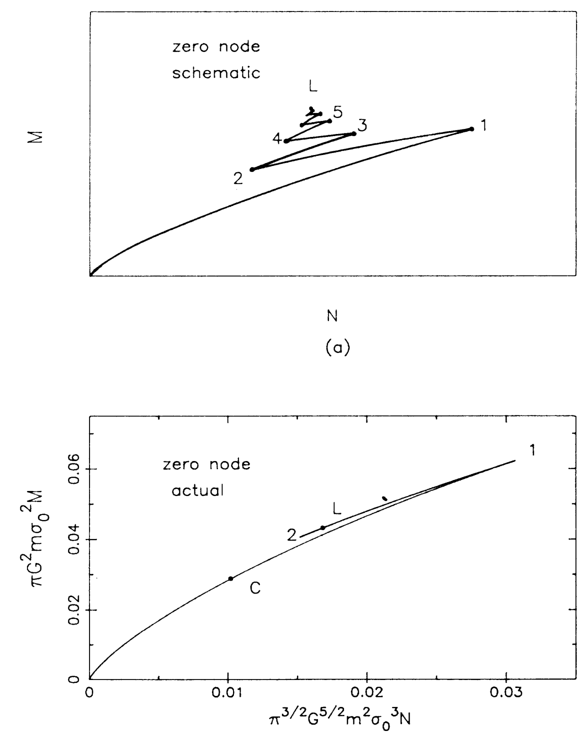

FIG. 6. Soliton star mass M vs the particle number N: a schematic drawing in (a) and the actual plot in (b). (The labels 1,2,... refer to the consecutive cusp number $n' = 1,2,\ldots$, with L standing for $n' = \infty$.)

TABLE I. Physical characteristics of the first six cusps for the zero-node solutions.

n'	$e^{-\bar{u}(0)}$	$e^{-u(0)}$	$\bar{\rho}_{\rm in}$	$\omega/(\pi G)^{1/2}m\sigma_0$	$\pi^{3/2}G^{5/2}m^2\sigma_0^3N$	$\pi G^2 m\sigma_0^2 M$	R/GM
1	2.029 04	4.406 05	1.430 10	1.302 52	0.030 668 3	0.062 303 1	2.869 25
2	16.578 4	28.847 5	0.981 914	1.625 47	0.015 183 9	0.040 597 4	3.023 33
3	96.909 8	175.162	1.047 40	1.564 85	0.017 119 9	0.043 696 1	2.996 25
4	599.072	1 076.18	1.036 53	1.574 49	0.016 791 0	0.043 180 1	3.000 61
5	3 669.86	6 599.18	1.038 30	1.632 94	0.016 844 2	0.043 263 9	2.999 90
6	22 514.5	40 479.2	1.038 01	1.573 17	0.016 835 5	0.043 250 3	3.000 02

The corresponding (x,y) trajectory is given in Fig. 2(a).

For $e^{-\bar{u}(0)} < e^{-\bar{u}_c}$, there is no solution; for $e^{-\bar{u}(0)} > e^{-\bar{u}_c}$, there are two solutions. When

$$e^{-\bar{u}(0)} \to \infty \ , \tag{3.92}$$

one of the solutions has $\hat{\rho}_{\rm in} \to \infty$, as can be inferred from Fig. 4; this solution will be referred to as L, the limiting solution. (The product $\bar{\rho}_{\rm in} = \hat{\rho}_{\rm in} e^{\bar{u}(0)}$ remains finite.) The (x,y) trajectory of L is shown in Fig. 5; in this case, the point "in" is at $x = y = 1/\sqrt{2}$. The various physical characteristics of L are

$$\begin{aligned}
&x_{\rm in} = y_{\rm in} = 1/\sqrt{2}, \quad x_A = y_A = 1/\sqrt{3} \ ,\\
&r_s = \tfrac{1}{2}(2+\sqrt{3})GM, \quad R = 3GM \ ,\\
&\omega = \tfrac{1}{2}(\sqrt{3}+\sqrt{2})(\pi G)^{1/2} m\sigma_0 \ ,\\
&N = \tfrac{1}{6}(\sqrt{3}-\sqrt{2})^2/\pi^{3/2}G^{5/2}m^2\sigma_0^3 \ ,\\
&M = \frac{1}{3}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right] \Big/ \pi G^2 m\sigma_0^2 \ .
\end{aligned} \tag{3.93}$$

Because of the symmetry under particle-antiparticle conjugation, there is a degeneracy under the sign change:

$$\omega \to -\omega \quad \text{and} \quad N \to -N \ . \tag{3.94}$$

In (3.91), (3.93), and also in all the tables and figures, only the $N>0$ solutions are given,

By systematically changing $e^{-\bar{u}(0)}$, we can survey all the zero-node solutions ($n=0$). In Figs. 6(a) and 6(b), M is plotted versus N, schematically in 6(a) and precisely in 6(b). As in the mini-soliton stars, it shows the typical pattern of cusps, which are labeled consecutively $n' = 1,2,3,\ldots$. The (x,y) trajectories of the first three cusps, $n' = 1$, 2, and 3, are given in Figs. 2(b)–2(d). The physical characteristics of the first six cusps are listed in Table I.

As representatives of the extensive analysis that has been made, we show in Fig. 7 the curves M vs ω, N vs ω, R vs ω, and R vs M, and in Fig. 8, the dependences of $e^{\bar{u}(0)}$, M, ω, and R on $\hat{\rho}_{\rm in}$.

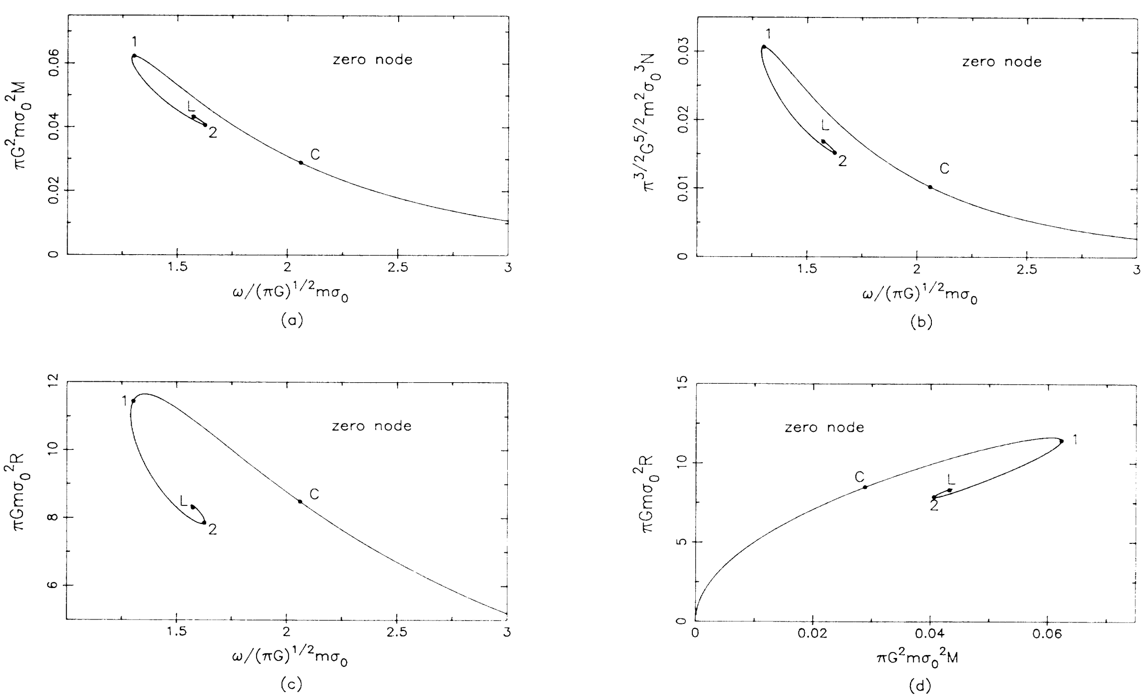

FIG. 7. M vs ω in (a), N vs ω in (b), R vs ω in (c), and R vs M in (d).

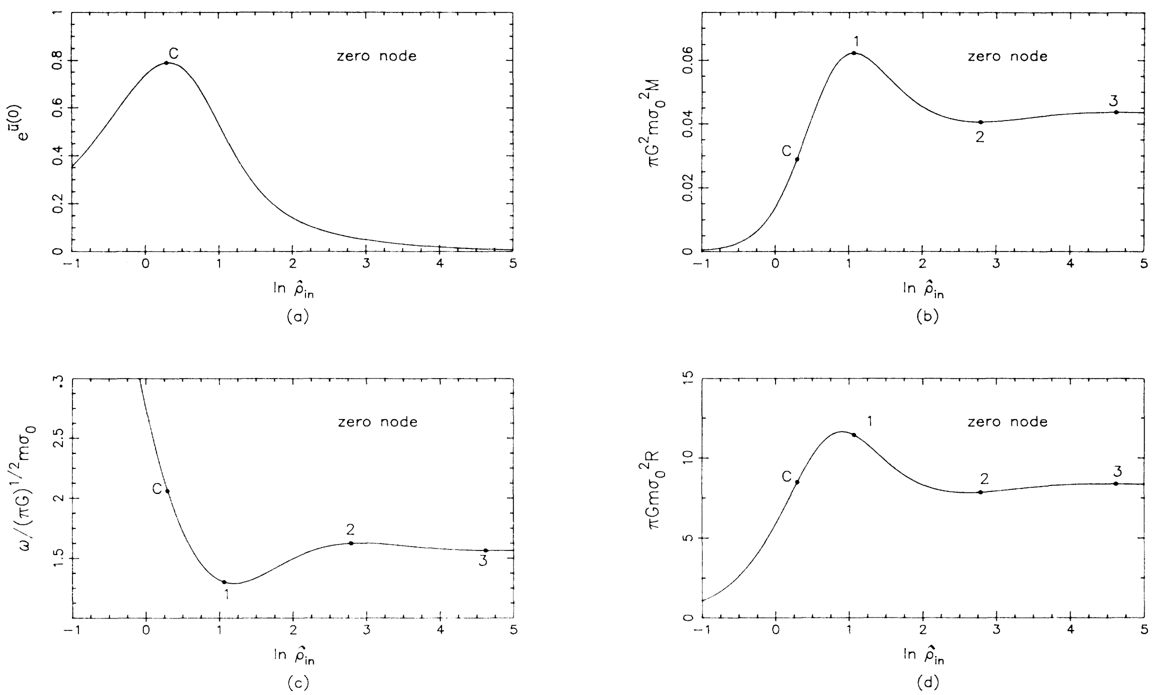

FIG. 8. $e^{\bar{u}(0)}$ vs $\hat{\rho}_{in}$ in (a), M vs $\hat{\rho}_{in}$ in (b), ω vs $\hat{\rho}_{in}$ in (c), and R vs $\hat{\rho}_{in}$ in (d), where $\hat{\rho}_{in}$ is defined by (3.88) and (3.33).

IV. BLACK HOLES

To obtain the black-hole solution, we extend the range of x and y, introduced in (2.30), to negative values.

A. The Schwarzschild solution

Without the matter field, we have the Schwarzschild solution, represented in the (x,y) plane by the hyperbola

$$2xy+y^2-1=0 , \tag{4.1}$$

which has two separate branches; each can in turn be divided into two regions depending on the sign of the product xy. Let $a=\frac{1}{2}GM$ always be kept *positive*. When $x>0$ and $y>0$, or $x<0$ and $y<0$, (4.1) is satisfied by having

$$x=\frac{2ar}{r^2-a^2} \text{ and } y=\frac{r-a}{r+a} ; \tag{4.2}$$

this will be referred to as the normal Schwarzschild region. When $x>0$ and $y<0$, or $x<0$ and $y>0$, (4.1) corresponds to

$$x=\frac{2ar}{a^2-r^2} \text{ and } y=\frac{r+a}{r-a} , \tag{4.3}$$

which is the abnormal (or anti-) Schwarzschild region. In this paper, we limit our interest to the normal Schwarzschild region only. (Note that, in our definition, the normal region includes both the $x>0,y>0$ part as well as the $x<0,y<0$ part.)

In the absence of matter, the two radii ρ and r are related by

$$\rho=\frac{(r+a)^2}{r}=a\left[\left[\frac{r}{a}\right]^{1/2}+\left[\frac{a}{r}\right]^{1/2}\right]^2 . \tag{4.4}$$

Therefore $2\pi\rho$, the circumference of a two-sphere, has a seemingly superficial degeneracy when r varies from less than a to greater than a, thereby giving rise to the two separate Schwarzschild regions. This situation changes drastically when there are matter fields, as will be analyzed below.

B. Matter field

Consider the special example discussed in Sec. III. Again, we shall adopt the approximation (3.28); i.e., treating $\lambda^2=8\pi G\sigma_0^2$ ($\sim 10^{-34}$ if $\sigma_0\sim m\sim 30$ GeV) as an infinitesimal. All discussions given in Secs. III B and III C are applicable here, except (3.42) and (3.48), as we shall see. As before, the (x,y) trajectory starts from $x=0,y=1$, when $r=0$ then it follows the trajectory I to the point "in." However, in the transition region, when r varies from r_s- to r_s+, the trajectory moves along the straight line (3.41), but instead of going to A given by (3.42), it now goes to

$$B:(x_B,y_B) \tag{4.5}$$

on the *other* branch of the Schwarzschild hyperbola (4.1) with x_B and y_B both <0. At B, the trajectory begins its exterior part where the matter density is zero. It follows the Schwarzschild curve to $x=-\infty,y=0$ which is the inner face of the horizon $r=a=\frac{1}{2}GM$, then switches to $x=\infty,y=0$ the outer face of the horizon, and back to $x=0,y=1$ (when $r=\infty$) along the $x>0,y>0$ branch of

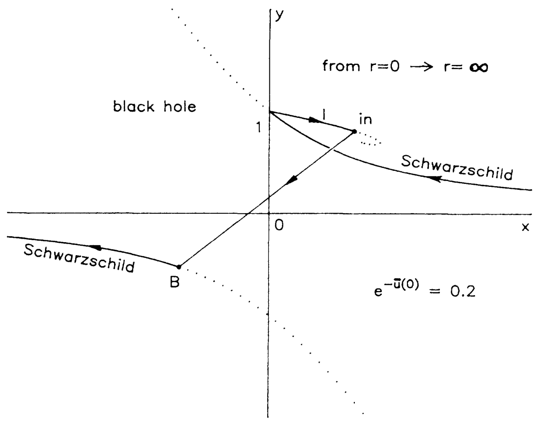

FIG. 9. Example of a (x,y) trajectory of a black-hole solution, with the arrow in the direction of increasing r (the radius in the isotropic coordinates).

the Schwarzschild hyperbola. This is illustrated in Fig. 9.

In the next section, we shall discuss how $(x_{\rm in}, y_{\rm in})$ and (x_B, y_B) are to be determined.

C. Basic equations

For the black-hole solutions, it is convenient to choose

$$\omega = -|\omega| < 0 , \tag{4.6}$$

where ω is the frequency introduced in (2.16). [As in (3.94), there is the particle-antiparticle symmetry. When $\omega \to -\omega$, we have $N \to -N$, but all other physical characteristics are unchanged.] As before, define

$$\lambda^2 = 8\pi G \sigma_0^2, \quad \bar{\rho} \equiv \lambda^2 m \rho$$

and (4.7)

$$e^{-\bar{u}} \equiv \frac{\omega}{\lambda m} e^{-u} .$$

[In the next section, it will be shown that inside the star e^{-u} is <0. The convention $\omega < 0$ makes $e^{-\bar{u}}$, thus defined, >0 and also $N = (2/\omega)\int e^{u+3v} 4\pi W r^2 dr > 0$.]

Let r_s be the star radius in the isotropic coordinates, and $a = \frac{1}{2}GM$ is the black-hole radius in the same coordinates. We have, for the black-hole solution,

$$r_s < a . \tag{4.8}$$

The basic equations inside the star, $r < r_s$, are the same (3.30) and (3.31) given before. Hence, by assigning an initial value $e^{-\bar{u}(0)} > 0$, we can integrate these equations and derive $\bar{u} = \bar{u}(\bar{\rho})$, $\bar{v} = \bar{v}(\bar{\rho})$, $y = e^{-\bar{v}(\bar{\rho})}$, and $x = y\bar{\rho}\, d\bar{u}/d\bar{\rho}$. Let

$$\Delta_+(x,y) \equiv \tfrac{1}{3}\{x + 2y + [(x-y)^2 + 3]^{1/2}\} , \tag{4.9}$$

and $\bar{\rho}_{\rm in}$ be the solution of

$$\tfrac{1}{8}\bar{\rho}_{\rm in} = \Delta_+(x_{\rm in}, y_{\rm in}) , \tag{4.10}$$

where

$$x_{\rm in} = x(\bar{\rho}_{\rm in}) \ \text{ and } \ y_{\rm in} = y(\bar{\rho}_{\rm in}) .$$

By following the arguments given in Sec. III E, one sees that

$$x_B \equiv x(\bar{\rho}_{\rm in}) - \tfrac{1}{8}\bar{\rho}_{\rm in} < 0$$

and (4.11)

$$y_B \equiv y(\bar{\rho}_{\rm in}) - \tfrac{1}{8}\bar{\rho}_{\rm in} < 0$$

describe the coordinates of a point on the Schwarzschild hyperbola. Except for the sign change between $\Delta_-(x,y)$ of (3.85) and $\Delta_+(x,y)$ of (4.9), the procedure for deriving a black-hole solution is essentially the same as that for the soliton star. As in (3.33), the star radius in the spherical coordinates is given by

$$\rho = R = \bar{\rho}_{\rm in}/m\lambda^2 . \tag{4.12}$$

From the Schwarzschild solution, we have

$$x_B = \frac{2ar_s}{r_s^2 - a^2} , \quad y_B = \frac{r_s - a}{r_s + a} , \quad R = \frac{(r_s + a)^2}{r_s} ; \tag{4.13}$$

consequently, r_s, a, and M can also be determined. Because x_B and y_B are negative, we have $r_s < a$; i.e., the radius of the star is smaller than the Schwarzschild radius (in the isotropic coordinates).

D. The sign of e^u and the mass of the black hole

In the exterior of the star $r > r_s$, there is no matter density; therefore, the Schwarzschild solution holds:

$$e^u = \frac{r-a}{r+a}, \quad e^v = \left[\frac{r+a}{r}\right]^2$$

and (4.14)

$$e^{\bar{v}} = e^{-u} .$$

Hence, while e^u is positive for $r > a$, it becomes negative for $r < a$, i.e., inside the black hole. Einstein's equation for free space, $u'' + v'' + u'^2 + r^{-1}(u' + v') = 0$, requires u' to be continuous; therefore e^u changes sign at $r = a\pm$. Since e^u is continuous at the surface of the star $r = r_s$, e^u is also <0 inside the star. In contrast, e^v is positive everywhere.

To further confirm the sign of e^u, let us consider the following calculation of M. From (2.13), (2.21), and (2.41), we have

$$M = \int_0^\infty \mathscr{E}\, dr ,$$

where (4.15)

$$\mathscr{E} = e^u r^2 [4\pi e^{3v}(U + V + W) - (2G)^{-1} e^v v'(2u' + v')] .$$

We may separate M into two parts:

$$M_{\text{space}} \equiv \int_{r_s+}^{\infty} \mathscr{E}\, dr$$

and (4.16)

$$M_{\text{star}} \equiv \int_0^{r_s+} \mathscr{E}\, dr ,$$

where M_{star} denotes the contribution due to the inside and the surface of the star, and M_{space} the contribution due to the (empty) space outside the star. For the integration of M_{space}, we have $U=V=W=0$ and, in accordance with (4.14) and (4.15)

$$\mathscr{E}=\frac{Ma}{r^2} , \tag{4.17}$$

which gives

$$M_{\text{space}}=M\frac{a}{r_s}>M . \tag{4.18}$$

Consequently, we must have

$$M_{\text{star}}<0 , \tag{4.19}$$

so that $M_{\text{space}}+M_{\text{star}}=M$. Since $\mathscr{E}\propto e^u$, this is in agreement with $e^u<0$ inside the star. The sign of e^u implies that under an infinitesimal time translation dt, the line element $e^u dt$ has opposite signs on the two sides of the horizon. Hence, one may regard the direction of time flow as also changing sign across the horizon (with respect to an appropriate overall frame).

It is of interest to note that the same mass can also be decomposed as

$$M=M_+ + M_- \tag{4.20}$$

where

$$M_+ \equiv \int_{a+}^{\infty} \mathscr{E}\, dr$$

and (4.21)

$$M_- \equiv \int_0^{a-} \mathscr{E}\, dr .$$

(The interval from $a-$ to $a+$ gives zero contribution.) From (4.17) it follows that

$$M_+=M ,$$

and therefore (4.22)

$$M_-=0 .$$

If one wishes, one may view M as entirely due to the gravitational energy outside the black hole. Of course, the localization of energy has no invariant significance since, for example, the same M can also be written as

$$M=4\pi \int_0^{\infty} (W+V+U)e^{5v/2}r^2 dr ,$$

in which the integrand is always positive and resides only in $r \le r_s \le a$.

E. Numerical results

As in previous sections, we study only the zero-node solution (of the matter field) in this paper. Replacing $\Delta_-(x,y)$ of (3.84) and A on the Schwarzschild hyperbola by $\Delta_+(x,y)$ of (4.9) and B on the other branch of the same hyperbola, the black-hole solution can be obtained by following the same steps as those given in Secs. III C and III H.

Define $\hat{\rho}$ by (3.82), and let $\hat{u}(\hat{\rho})$ and $\hat{v}(\hat{\rho})$ be the *same* solution of (3.79) with the boundary condition (3.80). By using $y(\hat{\rho})=\exp[-\hat{v}(\hat{\rho})]$ and $x(\hat{\rho})=y\hat{\rho}\, d\hat{u}/d\hat{\rho}$, we define, in terms of $\Delta_+(x,y)$ given by (4.9),

$$\Delta_+(\hat{\rho})\equiv\Delta_+(x(\hat{\rho}),y(\hat{\rho})) . \tag{4.23}$$

Clearly, $\Delta_+(\hat{\rho})$ is independent of $e^{-\bar{u}(0)}$. For each solution $\hat{\rho}=\hat{\rho}_{\text{in}}$ of

$$\tfrac{1}{8}e^{\bar{u}(0)}\hat{\rho}=\Delta_+(\hat{\rho}) , \tag{4.24}$$

a solution of the black hole can be derived, with

$$\begin{aligned}
&x_{\text{in}}=x(\hat{\rho}_{\text{in}}),\quad y_{\text{in}}=y(\hat{\rho}_{\text{in}}) ,\\
&x_B=x_{\text{in}}-\tfrac{1}{8}\bar{\rho}_{\text{in}}=\frac{2ar_s}{r_s^2-a^2}<0 ,\\
&y_B=y_{\text{in}}-\tfrac{1}{8}\bar{\rho}_{\text{in}}=\frac{r_s-a}{r_s+a}<0\\
&\bar{\rho}_{\text{in}}=\hat{\rho}_{\text{in}}e^{\bar{u}(0)},\quad R=r_s e^{v_s} ,\\
&e^{u_s}=\frac{r_s-a}{r_s+a},\quad e^{v_s}=\left[\frac{r_s+a}{r_s}\right]^2
\end{aligned} \tag{4.25}$$

and, as before, the Schwarzschild radius in the isotropic coordinates is

$$a=\tfrac{1}{2}GM .$$

The function $\Delta_+(\hat{\rho})$ is shown in Fig. 10. From its shape, one sees that for any given $e^{\bar{u}(0)}$, there is one and only one solution of (4.24). When $e^{-\bar{u}(0)}\to\infty$ we have the limiting solution L, whose physical characteristics are

$$\begin{aligned}
&x_{\text{in}}=y_{\text{in}}=1/\sqrt{2},\quad x_B=y_B=-1/\sqrt{3} ,\\
&r_s=\tfrac{1}{2}(2-\sqrt{3})GM,\quad R=3GM ,\\
&\omega=\tfrac{1}{2}(\sqrt{2}-\sqrt{3})(\pi G)^{1/2}m\sigma_0 ,\\
&N=\tfrac{1}{6}(\sqrt{3}+\sqrt{2})^2/\pi^{3/2}G^{5/2}m^2\sigma_0^2 ,\\
&M=\frac{1}{3}\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right]\Big/\pi G^2 m\sigma_0^2 .
\end{aligned} \tag{4.26}$$

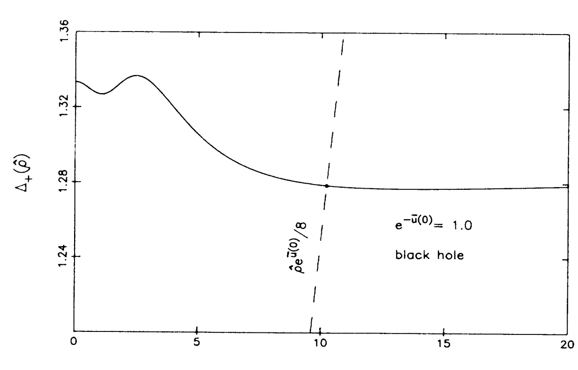

FIG. 10. Solid curve is $\Delta_+(\hat{\rho})$ defined by (4.23); its intersection with the dashed line $e^{\bar{u}(0)}\hat{\rho}/8$ gives the solution $\hat{\rho}=\hat{\rho}_{\text{in}}$, in accordance with (4.24).

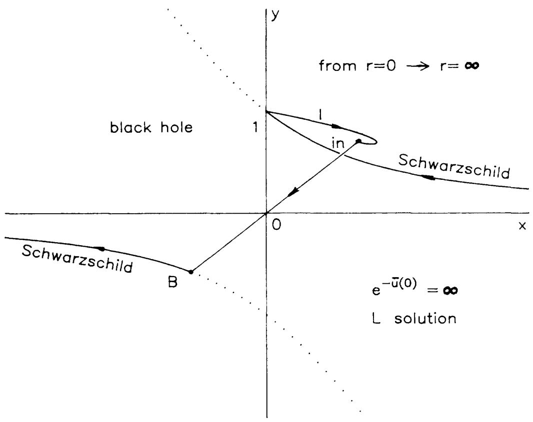

FIG. 11. The (x,y) trajectory of the (limiting) L solution for a black hole. See (4.26) for a description of the L solution.

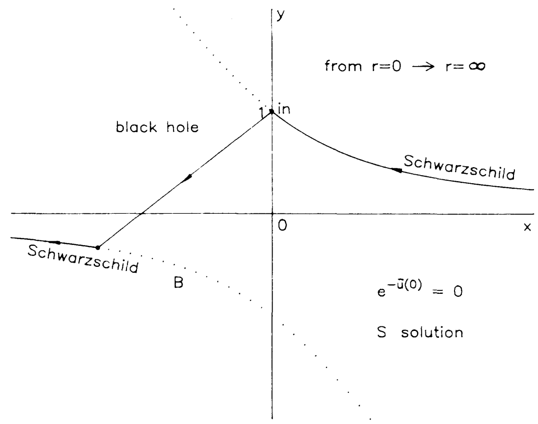

FIG. 12. The (x,y) trajectory of the (shell) S solution for a black hole. For $r<r_s$, the space is flat; at $r=r_s=\frac{1}{2}a$, there is a shell of matter, and for $r>r_s$ one has the Schwarzschild solution, with the horizon located at $r=a=\frac{1}{2}GM$, where $M=16/27\pi G^2 m\sigma_0^2$.

As before, because of particle-antiparticle symmetry, there is a degeneracy under $\omega\to-\omega$ and $N\to-N$, but leaving all other parameters unchanged. The (x,y) trajectory of the L solution is shown in Fig. 11.

We note that mathematically, when the initial value $e^{-\bar{u}(0)}\to 0$, there is another limiting solution, which may be called the S solution (or, shell solution, for reasons given below): the point "in" of the S solution is at $x=0$ and $y=1$; correspondingly, we have $x_B=-\frac{4}{3}$, $y_B=-\frac{1}{3}$, $M=16/27\pi G^2 m\sigma_0^2$, and $r_s=4/27\pi Gm\sigma_0^2$, but $N=\omega=0$. Consequently, $W=0$ in the interior of the star, and that invalidates our approximation (3.26) and (3.27). Physically it means that when ω/m is much less than $O(\lambda)$, say $O(\lambda^2)$, deviations from the S solution might occur. Nevertheless, when ω/m is $O(\lambda)\neq 0$, but $\omega/\lambda m$ small, the solution would resemble the overall characteristics of the S solution: inside the black hole there is a shell of matter located at $r=r_s=4/27\pi Gm\sigma_0^2=\frac{1}{2}a$ (with the horizon at $r=a=\frac{1}{2}GM$), and inside the shell, for $r<r_s$, we have a flat space with $e^{-u}=-3$, $e^{-v}=\frac{1}{9}$, $e^{-\bar{v}}=1$, and $e^{-\bar{u}}=\omega e^{-u}/\lambda m=0$. The (x,y) trajectory of the S solution is shown in Fig. 12.

In Fig. 13 we give the mass M of the black hole versus its particle number N. Again, it has an infinite number of cusps, which will be labeled consecutively by $n'=1,2,3,\ldots$. (Here, as well as in the other figures, $n\equiv$ number of nodes $=0$.) For N positive, the slope $dM/dN=\omega$ is negative, in agreement with (4.6); in contrast, for the soliton star solution of Sec. III, the corresponding slope dM/dN is positive. The physical characteristics of the first five cusp solutions are given in Table II. The limiting solution L refers to $n'\to\infty$ (and also $e^{-\bar{u}(0)}\to\infty$).

In Fig. 14 we plot ω vs N, ω vs r_s, M vs r_s, and M vs

TABLE II. Physical characteristics of the first five cusp solutions ($n'=1,2,\ldots,5$) of black holes (with $n=$ nodal number $=0$). Here, $n'=0$ and $n'=\infty$ refer to the S solution and the L solution.

n'	$e^{-\bar{u}(0)}$	$\omega/(\pi G)^{1/2}m\sigma_0$	$\pi^{3/2}G^{5/2}m^2\sigma_0^3N$	$\pi Gm\sigma_0^2 r_s$	$\pi G^2 m\sigma_0^2 M$
0	0	0	0	$\frac{4}{27}=0.148\,148$	$\frac{16}{27}=0.592\,593$
1	0.440 87	−0.185 508	1.855 80	0.044 636 7	0.394 495
2	2.795 47	−0.155 051	1.619 23	0.059 398 0	0.432 996
3	17.102 2	−0.159 560	1.654 88	0.057 028 7	0.427 349
4	104.949	−0.158 815	1.649 01	0.057 415 7	0.428 283
5	648.7	−0.158 653	1.649 96	0.057 352 7	0.428 131
∞	∞	$\frac{1}{2}(\sqrt{2}-\sqrt{3})$	$\frac{1}{6}(\sqrt{3}+\sqrt{2})^2$	$\frac{(2-\sqrt{3})(\sqrt{2}+\sqrt{3})}{6\sqrt{6}}$	$\frac{1}{3}\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right]$
		$=-0.158\,919$	$=1.649\,83$	$=0.057\,361\,5$	$=0.428\,152$

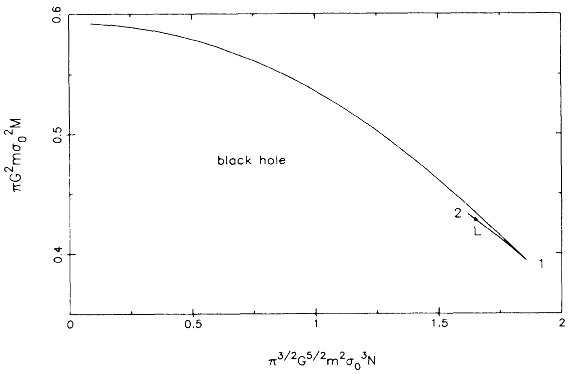

FIG. 13. M vs N for the zero-node black-hole solution, with $n'=1,2,\ldots$ denoting the successive cusps, and $n'=\infty$ the L solution.

r_s/a. We see that the ratio of the star radius r_s to the Schwarzschild radius $a=\frac{1}{2}GM$ (in the isotropic coordinates) is always <1, as expected.

F. Remarks

(1) As nomenclature, we have been calling the regions

$$\begin{aligned} r<r_s&: \text{ inside of a star },\\ r>r_s&: \text{ outside of a star },\\ r<a&: \text{ inside of a black hole },\\ r>a&: \text{ outside of a black hole }, \end{aligned} \tag{4.27}$$

where $r=a$ is the horizon and $a>r_s$. In this definition, the inside of a star refers to a region that is spacelike with respect to its outside (as in the usual practice). Likewise, the inside of a black hole also refers to a region that is *spacelike* with respect to its outside (which differs from some of the conventions in the literature).

Inside the star, when r increases from 0 to r_s-, we have $\rho=e^{v}r$ also increasing from 0 to $R-$. However, in contrast to $r_s<\frac{1}{2}GM$ (the Schwarzschild radius in the isotropic coordinates), R is $>2GM$ (the Schwarzschild radius in the spherical coordinates). In this region $r<r_s$, the function e^{u} is <0, but $e^{v}>0$, as mentioned before. However, since Einstein's equation depends only on e^{2u} and e^{2v} (which are both positive), there is nothing remarkable.

Outside the star, the Schwarzschild solution applies. Depending on whether one is inside or outside the black hole, the situation changes drastically.

In the region outside the star but inside the black hole, the circumference $2\pi\rho$ of a two-sphere decreases with increasing r, on account of

$$\rho=a\left[\left[\frac{r}{a}\right]^{1/2}+\left[\frac{a}{r}\right]^{1/2}\right]^2 .$$

Since $-g_{tt}=e^{2u}$, $g_{rr}=e^{2v}$, and $g_{\rho\rho}=e^{2\bar{v}}$ are all positive and are exactly the same as the usual ones in an empty space outside a black hole, an observer in this region, $a>r>r_s$, would feel a gravitational force in the direction *away* from the star (i.e., in the direction of increasing r, or decreasing ρ), as if there could be a gravitational repulsion from the matter distribution (inside the star). Hence, he

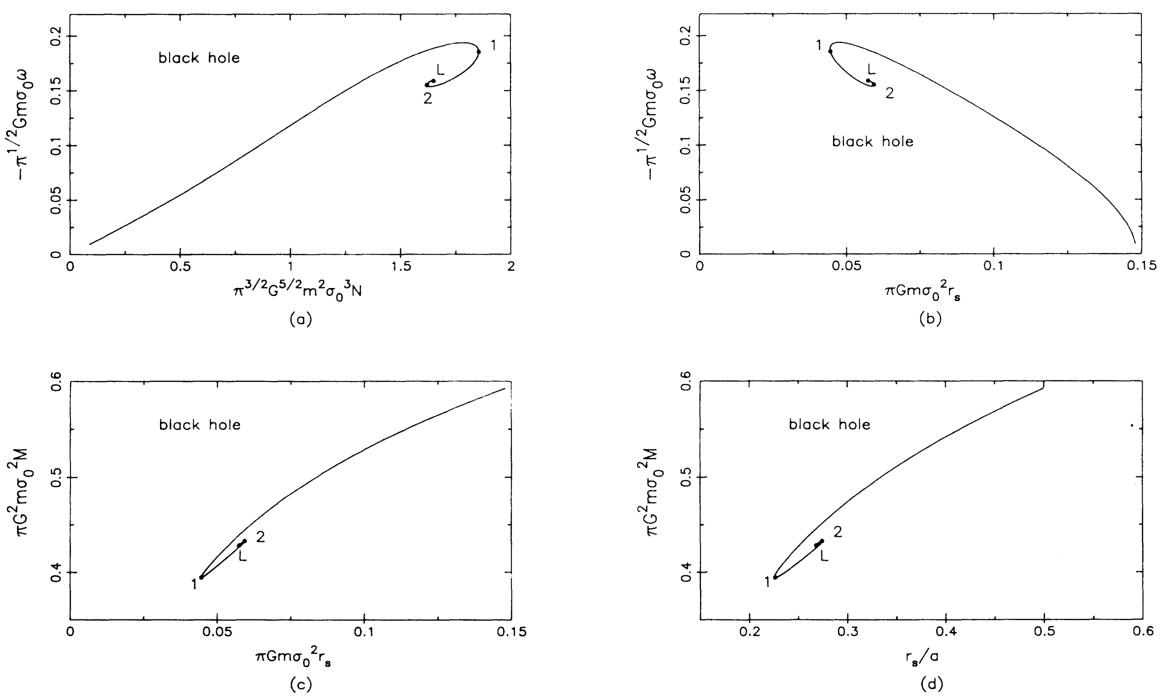

FIG. 14. ω vs N in (a), ω vs r_s in (b), M vs r_s in (c), and M vs r_s/a in (d), all for the zero-node black-hole solution.

might conclude that he is "outside" a black hole. Somehow, in spite of the presence of a star located at $r < r_s$ (which is within his horizon), to him the "real" black hole seems to lie in the opposite direction. In any case, he might be perplexed.

For $r > a$, the situation becomes normal again, and we have the usual description of an empty space outside a black hole.

The relation between ρ and r over all these regions is illustrated in Fig. 15.

(2) So far we have kept to the approximation

$$\lambda^2 = 0+ \ .$$

Note that outside the star, but inside the black hole, the amplitude of the matter field varies as

$$\sigma \sim \sigma_0 \exp[-m e^{v_s}(r - r_s)] \ .$$

For r near $a-$, and taking the difference $a - r_s$ to be of the same order as $a = \frac{1}{2}GM$, we have, for $\sigma_0 \sim m \sim 30$ GeV,

$$\frac{\sigma}{\sigma_0} \sim \exp(-\lambda^{-2}) \sim \exp(-10^{34}) \ . \qquad (4.28)$$

This number is so small that in all reasonableness one does not have to differentiate it from zero. Nevertheless, it is not zero. If one blindly substitutes this amplitude into the field equation of the scalar field, because $e^{-u} = \infty$ at $r = a$, difficulties might arise. In the following, we shall show one of the ways to avoid the complication and to indicate how to calculate the perturbation due to $\exp(-\lambda^{-2}) > 0$.

Recall that whenever the particle number N is $\neq 0$, the scalar field ϕ has to be t dependent; our solutions are all of the form $\sigma_\omega(r) e^{-i\omega t}$. Consider a superposition

$$\phi = \int C_\omega \sigma_\omega e^{-i\omega t} d\omega \ , \qquad (4.29)$$

so that

$$\phi = 0 \ \text{ at } r = a \text{ when } t = 0 \ . \qquad (4.30)$$

In the approximation

$$\exp(-\lambda^{-2}) = 0+ \ , \qquad (4.31)$$

we could have

$$C_\omega = \delta(\omega - \omega_0) \ .$$

Since $\exp(-\lambda^{-2})$ is extremely small, C_ω has only a tiny spread in frequencies. In time, ϕ has to develop dispersions. At $t = 0$, since ϕ is 0 at $r = a$, the space is empty (i.e., free of matter density) at the horizon. (Therefore, the singularity at $r = a$ is not due to any local distribution of matter, but is associated with Schwarzschild's coordinate system.)

Exclude from the Schwarzschild space a small region Ω that includes $r = a$. Inside Ω, we may express Einstein's equation in terms of, say, the Kruskal coordinates,[10] or other coordinates regular at $r = a$. (Hence, Ω may have to extend beyond the original Schwarzschild domain.) Outside Ω but inside the Schwarzschild space, we can use our solution as the zeroth approximation and treat the small parameter $\exp(-\lambda^{-2})$ as a perturbation; this provides a boundary condition for the region inside Ω. The consequences of $\exp(-\lambda^{-2}) \neq 0$ may then be analyzed.

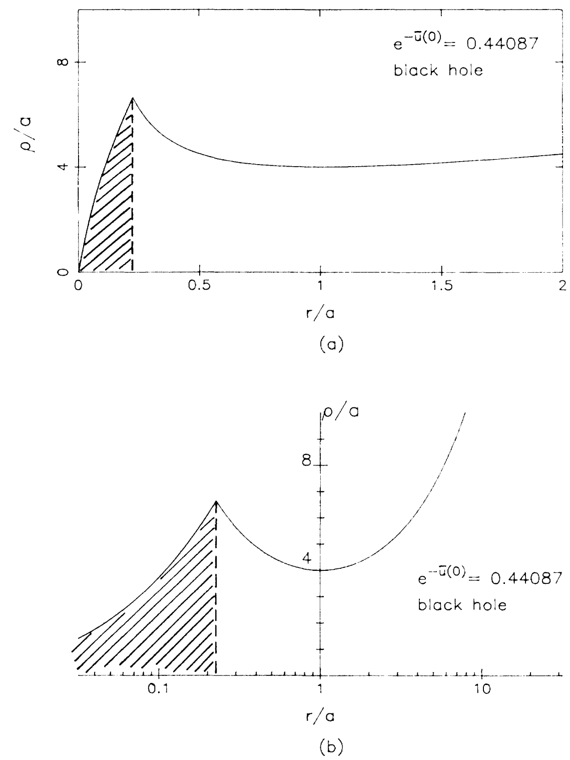

FIG. 15. ρ vs r for the $n'=1$ zero-node black-hole solution, (a) in a linear scale of r and (b) in a logarithmic scale. The matter distribution is located in the shaded region with $r \le r_s$, and the horizon is at $r/a = 1$. Outside the star $r > r_s$, we have the Schwarzschild solution $\rho/a = [(r/a)^{1/2} + (a/r)^{1/2}]^2$.

The solution for the scalar soliton stars derived in Sec. III covers a mass region

$$0 < (\pi G^2 m \sigma_0^2) M < 0.062\,303 \ ,$$

and the black-hole solution discussed in this section covers only the region

$$0.394\,495 < (\pi G^2 m \sigma_0^2) M < \tfrac{16}{27} \ .$$

For masses outside these ranges, the metric must have explicit nontrivial time dependence.

ACKNOWLEDGMENTS

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