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Black-hole normal modes: A WKB approach. II. Schwarzschild black holes

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We employ a semianalytic technique, based on a modified WKB approach, to determine the complex normal-mode frequencies of Schwarzschild black holes. It yields a simple analytic formula that gives the real and imaginary parts of the frequency in terms of the parameters of the black hole and of the field whose perturbation is under study, and in terms of the quantity $(n+\frac{1}{2})$, where $n=0,\pm 1,\pm 2,\ldots$, and labels the fundamental mode, first overtone mode, and so on. In the case of the fundamental gravitational normal modes of the Schwarzschild black hole, the WKB estimates agree with numerical results to better than 0.13% in the real part of the frequency and 0.22% in the imaginary part. The agreement for both the real and imaginary parts of the low overtones is better than 0.5%. The relative agreement improves with increasing angular harmonic.

I. INTRODUCTION AND SUMMARY

The fundamental equations describing the perturbations of black holes reduce, in many cases, to a single second-order ordinary differential equation that is similar to the one-dimensional Schrödinger equation for a particle encountering a potential barrier on the infinite line. (See Ref. 1, Secs. 27 and 28.) A normal mode is a solution to the differential equation with a complex frequency, satisfying the boundary condition of purely "outgoing" waves, that is, waves propagating away from the barrier, at both $+\infty$ and $-\infty$, the latter boundary condition corresponding to waves traveling across the horizon to the interior of the black hole.

We have developed a new semianalytic technique to determine the normal-mode frequencies, using the WKB approximation. In a previous paper, Schutz and Will[2] described the basic elements of this method at lowest WKB order, and applied it to the Schwarzschild black hole. For the fundamental gravitational modes, their results agreed with the numerical results of Chandrasekhar and Detweiler[3] within 7% for the real part and 0.8% for the imaginary part, but disagreed with the numerical results for higher overtone modes. In another paper (hereafter referred to as paper I), Iyer and Will[4] laid the foundations of the method in detail, and carried it to the third WKB order, with the goal of obtaining higher accuracy. The method resulted in a simple analytic formula that determines the normal-mode frequencies [Eqs. (1.4) and (1.5) of paper I].

The master equation for perturbations of a Schwarzschild black hole has the form

$$d^2\Psi/dr_*{}^2+\{\omega^2-[1-(2M/r)][\lambda/r^2+(2\beta M/r^3)]\}\Psi(r_*)=0\;, \quad (1.1)$$

where $\lambda=l(l+1)$, l being the angular harmonic index; $\beta=1,0,-3$ for scalar, electromagnetic, and gravitational perturbations, respectively, and M is the mass of the black hole. The "tortoise coordinate" r_* is related to r by $dr/dr_*=1-2M/r$. The quantity Ψ represents the radial part of the perturbation variable, assumed to have time dependence $e^{-i\omega t}$, and appropriate angular dependence. (See Ref. 1, pp. 143 and 144 for further details.)

Let $\{\omega^2-V(r_*)\}$ be identified as the quantity in the curly brackets in Eq. (1.1). Figure 1 depicts the "potential" $V-\omega^2$. The field Ψ in regions I and III of Fig. 1 is approximated by linear combinations of incoming-wave and outgoing-wave WKB functions, carried to third order in the WKB expansion. These functions are to be matched through region II. For the low-lying normal modes, the expected value of ω is such that the two turning points $(r_*)_1$ and $(r_*)_2$ are too close together to permit a valid WKB approximation to the solution in region II; so we match the two exterior WKB solutions across both turning points simultaneously. In region II we expand the function $V(r_*)$ about $(r_*)_0$, the location of the peak of V, in a Taylor expansion, retaining terms up to and including the sixth order. We then obtain an asymptotic approximation to the general interior solution, and use it to connect the two WKB solutions (see paper I for details).

The result is a pair of connection formulas [Eq. (3.33) of paper I], relating the amplitudes of the incoming and outgoing solutions on either side of the barrier. The boundary condition of only "outgoing" waves leads to a for-

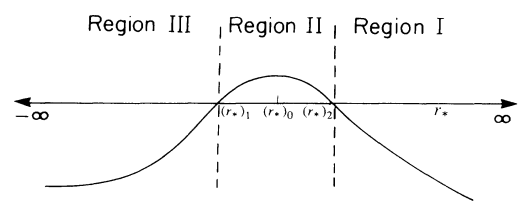

FIG. 1. The function $V(r_*)-\omega^2$.

mula for the normal-mode frequencies, given by

$$\omega^2=[V_0+(-2V_0'')^{1/2}\tilde{\Lambda}] -i(n+\tfrac{1}{2})(-2V_0'')^{1/2}(1+\tilde{\Omega}) , \tag{1.2}$$

$$n=\begin{cases}0,1,2,\ldots, & \mathrm{Re}(\omega)>0 ,\\ -1,-2,-3,\ldots, & \mathrm{Re}(\omega)<0 ,\end{cases}$$

where

$$\tilde{\Lambda}(n)=\frac{1}{(-2V_0'')^{1/2}}\left[\frac{1}{8}\left[\frac{V_0^{(4)}}{V_0''}\right]\left[\frac{1}{4}+\alpha^2\right]-\frac{1}{288}\left[\frac{V_0'''}{V_0''}\right]^2(7+60\alpha^2)\right] , \tag{1.3a}$$

$$\tilde{\Omega}(n)=\frac{1}{(-2V_0'')}\left[\frac{5}{6912}\left[\frac{V_0'''}{V_0''}\right]^4(77+188\alpha^2)-\frac{1}{384}\left[\frac{V_0'''^2V_0^{(4)}}{V_0''^3}\right](51+100\alpha^2)+\frac{1}{2304}\left[\frac{V_0^{(4)}}{V_0''}\right]^2(67+68\alpha^2) +\frac{1}{288}\left[\frac{V_0'''V_0^{(5)}}{V_0''^2}\right](19+28\alpha^2)-\frac{1}{288}\left[\frac{V_0^{(6)}}{V_0''}\right](5+4\alpha^2)\right] . \tag{1.3b}$$

Here, the primes and the superscript (n) denote differentiation with respect to r_*. The subscript 0 on a variable denotes the value of the variable at $(r_*)_0$ ($V_0''\neq 0$), and $\alpha\equiv n+1/2$.

The frequencies of the first few gravitational normal modes are shown in Fig. 2. The values for the fundamental modes agree with numerical results to within 0.13% in the real part of the frequency, and to within 0.22% in the imaginary part. The agreement in both real and imaginary parts is within 0.5% for the lower overtones. The relative agreement improves with increasing angular harmonic. (When $l=4$ the WKB estimate for the fundamental frequency is practically identical to the numerical estimate.)

The structure of this paper is as follows. In Sec. II we discuss the master equation (1.1), and derive the normal-mode frequencies. In Sec. III we derive certain properties of the normal modes: (1) the symmetry between modes with $\mathrm{Re}(\omega)>0$ and modes with $\mathrm{Re}(\omega)<0$, (2) the large-l limit, (3) stability of the lower modes, and (4) stability in the large-l limit. Section IV presents concluding remarks.

II. NORMAL-MODE FREQUENCIES

The odd-parity perturbations of the Schwarzschild black hole are described by the Regge-Wheeler equation[5]

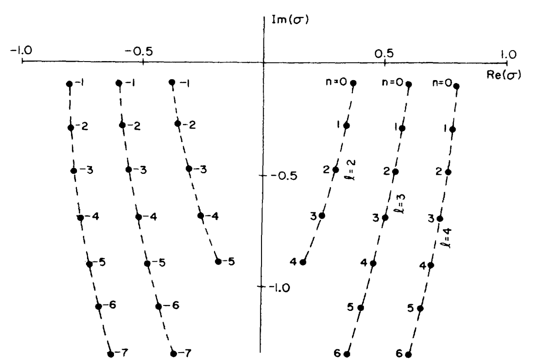

FIG. 2. Gravitational normal modes. Here $\sigma=M\omega$, where M is the mass of the black hole.

$$\frac{d^2\Psi}{dr_*^2}+\left\{\sigma^2-\left[1-\frac{2}{r}\right]\left[\frac{\lambda}{r^2}+\frac{2\beta}{r^3}\right]\right\}\Psi=0 , \tag{2.1}$$

where we have defined the dimensionless frequency $\sigma=M\omega$, where M is the mass of the black hole, and have expressed the radial coordinates in units of M, with r_* now related to r by

$$\frac{dr}{dr_*}=1-\frac{2}{r} . \tag{2.2}$$

The even-parity perturbations of the Schwarzschild black hole are described by the Zerilli equation.[6] However, since both equations yield the same normal-mode frequencies,[3] we shall deal only with Regge-Wheeler equation.

With $\{\sigma^2-V(r_*)\}$ identified as the quantity in the curly brackets in Eq. (2.1), we obtain

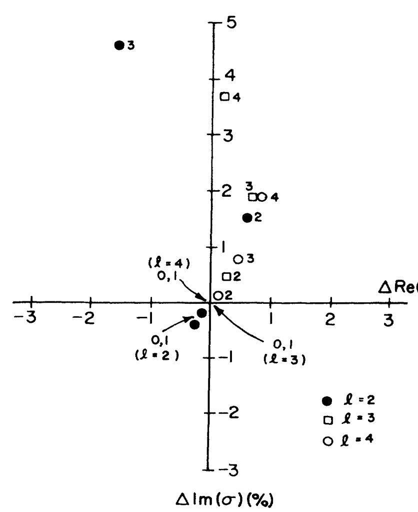

FIG. 3. Percentage deviation of WKB results for low-lying gravitational modes from Leaver's (Ref. 7) results. Accuracy decreases with increasing n because of the increasing inadequacy of a polynomial approximation to the potential as the turning points migrate away from the peak.

$$V'(r_*)=-\left[1-\frac{2}{r}\right]\left[\frac{\lambda}{r^3}\left[2-\frac{6}{r}\right]+\frac{2\beta}{r^4}\left[3-\frac{8}{r}\right]\right], \tag{2.3a}$$

$$V''(r_*)=-\left[1-\frac{2}{r}\right]\left[-\frac{\lambda}{r^4}\left[6-\frac{40}{r}+\frac{60}{r^2}\right]-\frac{2\beta}{r^5}\left[12-\frac{70}{r}+\frac{96}{r^2}\right]\right], \tag{2.3b}$$

$$V'''(r_*)=-\left[1-\frac{2}{r}\right]\left[\frac{2\lambda}{r^5}\left[12-\frac{130}{r}+\frac{420}{r^2}-\frac{420}{r^3}\right]+\frac{4\beta}{r^6}\left[30-\frac{282}{r}+\frac{826}{r^2}-\frac{768}{r^3}\right]\right], \tag{2.3c}$$

$$V^{(4)}(r_*)=-\left[1-\frac{2}{r}\right]\left[\frac{-4\lambda}{r^6}\left[30-\frac{462}{r}+\frac{2380}{r^2}-\frac{5040}{r^3}+\frac{3780}{r^4}\right]-\frac{8\beta}{r^7}\left[90-\frac{1197}{r}+\frac{5560}{r^2}-\frac{10\,890}{r^3}+\frac{7680}{r^4}\right]\right], \tag{2.3d}$$

$$V^{(5)}(r_*)=-\left[1-\frac{2}{r}\right]\left[\frac{8\lambda}{r^7}\left[90-\frac{1827}{r}+\frac{13\,216}{r^2}-\frac{44\,100}{r^3}+\frac{69\,300}{r^4}-\frac{41\,580}{r^5}\right]+\frac{16\beta}{r^8}\left[315-\frac{5508}{r}+\frac{35\,793}{r^2}-\frac{110\,050}{r^3}+\frac{162\,030}{r^4}-\frac{92\,160}{r^5}\right]\right], \tag{2.3e}$$

$$V^{(6)}(r_*)=-\left[1-\frac{2}{r}\right]\left[\frac{-16\lambda}{r^8}\left[315-\frac{8028}{r}+\frac{75\,915}{r^2}-\frac{352\,660}{r^3}+\frac{866\,250}{r^4}-\frac{1\,081\,080}{r^5}+\frac{540\,540}{r^6}\right]-\frac{32\beta}{r^9}\left[1260-\frac{27\,621}{r}+\frac{234\,045}{r^2}-\frac{998\,998}{r^3}+\frac{2\,292\,780}{r^4}-\frac{2\,705\,430}{r^5}+\frac{1\,290\,240}{r^6}\right]\right], \tag{2.3f}$$

where the primes and the superscript (n) denote differentiation with respect to r_*. The peak of V is determined by $V'=0$, and occurs at

$$r=r_0\equiv\tfrac{3}{2}\lambda^{-1}[\lambda-\beta+(\lambda^2+\tfrac{14}{9}\lambda\beta+\beta^2)^{1/2}], \tag{2.4}$$

TABLE I. Normal modes for scalar perturbations ($\beta=1$). The percentage deviation of the WKB results from Leaver's results is given in parentheses.

l	n	σ_{Leaver}	σ_{WKB}
0	0	0.1105−0.1049i	0.1046−0.1152i (−5.3%)(−9.8%)
	1	0.0861−0.3481i	0.0892−0.3550i (3.6%)(−2.0%)
1	0	0.2929−0.0977i	0.2911−0.0980i (−0.61%)(−0.31%)
	1	0.2645−0.3063i	0.2622−0.3074i (−0.87%)(−0.36%)
	2	0.2295−0.5401i	0.2235−0.5268i (−2.6%)(2.5%)
	3	0.2033−0.7883i	0.1737−0.7486i (−15%)(5.0%)
2	0	0.4836−0.0968i	0.4832−0.0968i (−0.08%)(0.0%)
	1	0.4639−0.2956i	0.4632−0.2958i (−0.15%)(−0.07%)
	2	0.4305−0.5086i	0.4317−0.5034i (0.28%)(1.0%)
	3	0.3939−0.7381i	0.3926−0.7159i (−0.33%)(3.0%)

TABLE II. Normal modes for electromagnetic perturbations ($\beta=0$).

l	n	σ_{Leaver}	σ_{WKB}
1	0	0.2483−0.0925i	0.2459−0.0931i (−0.97%)(−0.65%)
	1	0.2145−0.2937i	0.2113−0.2958i (−1.5%)(−0.72%)
	2	0.1748−0.5252i	0.1643−0.5091i (−6.0%)(3.1%)
	3	0.1462−0.7719i	0.1019−0.7256i (−30%)(6.0%)
2	0	0.4576−0.0950i	0.4571−0.0951i (−0.11%)(−0.11%)
	1	0.4365−0.2907i	0.4358−0.2910i (−0.16%)(−0.10%)
	2	0.4012−0.5016i	0.4023−0.4959i (0.27%)(1.1%)
	3	0.3626−0.7302i [a]	0.3605−0.7056i (−0.58%)(−3.4%)
3	0	0.6569−0.0956i	0.6567−0.0956i (−0.03%)(0.0%)
	1	0.6417−0.2897i	0.6415−0.2898i (−0.03%)(−0.03%)
	2	0.6138−0.4921i	0.6151−0.4901i (0.21%)(0.41%)
	3	0.5779−0.7063i	0.5814−0.6955i (0.61%)(1.5%)

[a]The $l=2$, $n=3$ mode was inadvertently omitted from the tabulation of modes in Ref. 7(c). We are grateful to E. Leaver for supplying us with this value.

TABLE III. Normal modes for gravitational perturbations ($\beta=-3$).

l	n	$\sigma_{\rm CD}$	$\sigma_{\rm Leaver}$	$\sigma_{\rm WKB}$
2	0	$0.3737-0.0889i$	$0.3737-0.0890i$	$0.3732-0.0892i$ $(-0.13\%)(-0.22\%)$
	1	$0.3484-0.2747i$	$0.3467-0.2739i$	$0.3460-0.2749i$ $(-0.20\%)(-0.36\%)$
	2		$0.3011-0.4783i$	$0.3029-0.4711i$ $(0.60\%)(1.5\%)$
	3		$0.2515-0.7051i$	$0.2475-0.6730i$ $(-1.6\%)(4.6\%)$
3	0	$0.5994-0.0927i$	$0.5994-0.0927i$	$0.5993-0.0927i$ $(-0.02\%)(0.0\%)$
	1	$0.5820-0.2812i$	$0.5826-0.2813i$	$0.5824-0.2814i$ $(-0.03\%)(-0.04\%)$
	2		$0.5517-0.4791i$	$0.5532-0.4767i$ $(0.27\%)(0.50\%)$
	3		$0.5120-0.6903i$	$0.5157-0.6774i$ $(0.72\%)(1.9\%)$
	4		$0.4702-0.9156i$	$0.4711-0.8815i$ $(0.19\%)(3.7\%)$
	5		$0.4314-1.152i$	$0.4189-1.088i$ $(-2.9\%)(5.6\%)$
4	0	$0.8092-0.0941i$	$0.8092-0.0942i$	$0.8091-0.0942$ $(-0.01\%)(0.0\%)$
	1	$0.7965-0.2844i$	$0.7966-0.2843i$	$0.7965-0.2844i$ $(-0.01\%)(-0.04\%)$
	2	$0.5061-0.4232i$	$0.7727-0.4799i$	$0.7736-0.4790i$ $(0.12\%)(0.19\%)$
	3		$0.7398-0.6839i$	$0.7433-0.6783i$ $(0.47\%)(0.82\%)$
	4		$0.7015-0.8982i$	$0.7072-0.8813i$ $(0.81\%)(1.9\%)$

with r_* being equal to $(r_*)_0$, when $r=r_0$. For a given λ and β we determine r_0, substitute into Eqs. (2.3), and substitute the resulting values into Eqs. (1.2) and (1.3). The resulting frequencies are listed in Tables I, II, and III, which also include a comparison with the results of Leaver,[7] and those of Chandrasekhar and Detweiler[3] (CD). Notice that Chandrasekhar and Detweiler quote a gravitational mode with $l=4$ that has no counterpart in either our results or Leaver's results. This anomalous value was presumably a result of numerical instabilities in their computational method that appeared when $\mathrm{Im}(\sigma)\sim\mathrm{Re}(\sigma)$. (For further discussion of these instabilities, see Ref. 3.) Our results for the electromagnetic modes agree well also with the results of Cunningham, Price, and Moncrief.[8] Figure 3 depicts the percentage deviation of our results from Leaver's results for the low-lying gravitational modes. It is instructive to compare Table III with Table I of Ref. 2. For the fundamental $l=2$, $n=0$ mode, the percentage deviation from Leaver's results in the real and imaginary parts decreases from (6.7%, 0.79%) in the first WKB order[2] to (0.13%, 0.22%) at third WKB order; for the $l=3$, $n=0$ mode it decreases from (2.9%, 0.43%) to (0.02%, 0.0%). In the first WKB order, the excited mode $l=2$, $n=1$ was not even close, with a percentage deviation of (30%, 15%), while at third WKB order, the percentage deviation is (0.20%, 0.36%). The accuracy of the third-order WKB approximation is, in fact, excellent for any $n<l$, and for a given n, the relative agreement improves with increasing l. For a given l, the accuracy decreases with increasing n because the polynomial approximation to the interior potential, truncated at a given order, is no longer adequate.

Mashhoon and co-workers[9] have employed another approach, based on a connection between the normal modes and the bound states of inverted black-hole effective potentials. Use of the "Pöschl-Teller" potential, which has two free parameters available for fitting to the black-hole potential yields, for the fundamental gravitational modes, values of $\mathrm{Re}(\omega)$ that are within 1.3% of Leaver's results,[7] and values of $\mathrm{Im}(\omega)$ within 1.7%. However, in this case, $\mathrm{Re}(\omega)$ is independent of n, contrary to the numerical estimates. The "Eckart" potential, which has three parameters for fitting gives $\mathrm{Re}(\omega)$ decreasing with increasing n, in agreement with the numerical estimates, but yields fewer normal modes than the Pöschl-Teller potential as it has fewer bound states. For further discussion of the results obtained, the characteristics of the various potentials, and the limitations of this method, see Ref. 9.

III. PROPERTIES OF THE NORMAL MODES

It is apparent from Eq. (1.2) that modes with $\mathrm{Re}(\omega)>0$ and modes with $\mathrm{Re}(\omega)<0$ are in one-to-one correspondence, the corresponding modes being characterized by

$+\alpha$ and $-\alpha$, respectively, where $\alpha = n+\frac{1}{2}$. Further, we see from Eqs. (1.3) that these modes have the same $\tilde{\Lambda}$ and $\tilde{\Omega}$, and hence that the real part of ω^2 is identical for both, with the imaginary parts being of equal magnitude but of opposite sign. This leads to the conclusion that $\mathrm{Im}(\omega)$ is identical for the two modes, and $\mathrm{Re}(\omega)$ is of the same magnitude for both, but of opposite sign. (See also Ref. 9.) This symmetry is reflected in Fig. 2.

Using Eqs. (1.2), (1.3), (2.3), and (2.4) in the limit of large l, we obtain, for the modes with $\mathrm{Re}(\omega)>0$,

$$\sigma = \frac{1}{3\sqrt{3}}\left[l+\frac{1}{2}\right] - \frac{1}{9\sqrt{3}}\left[\frac{5\alpha^2}{12}-\beta+\frac{115}{144}\right]l^{-1} + \frac{1}{18\sqrt{3}}\left[\frac{5\alpha^2}{12}-\beta+\frac{115}{144}\right]l^{-2} - i\alpha\left[\frac{1}{3\sqrt{3}}+\frac{1}{27\sqrt{3}}\left[\frac{235\alpha^2}{432}+\beta-\frac{1415}{1728}\right]l^{-2}\right]. \tag{3.1}$$

This is in agreement with the analytical results of Press[10] and Detweiler,[11] and it extends Eqs. (28) and (29) of Ref. 9(c).

It is known rigorously that the Schwarzschild black hole is stable against external perturbations (Ref. 1, pp. 199—201). Not surprisingly, the results for the lower modes support this conclusion, as shown by the negative $\mathrm{Im}(\sigma)$. Equation (3.1) indicates that in the large-l limit $\mathrm{Re}(\sigma)$ increases linearly with l, and $\mathrm{Im}(\sigma)$ tends to a negative constant $(-\alpha/3\sqrt{3})$ for a given n, consistent with stability. As shown in Fig. 2 and Tables I, II, and III, for fixed l the magnitude of $\mathrm{Im}(\sigma)$ increases steadily with n, and the sign remains negative. $\mathrm{Im}(\sigma)$ exhibits the same behavior in the large-l limit, as seen from Eq. (3.1). This argument is supported by Leaver's results for $n \gg l$.[7(a)]

IV. CONCLUSIONS

We have used the WKB approximation to obtain a simple analytic formula that determines the normal-mode frequencies of black holes. In the case of the Schwarzschild black hole, the agreement with other methods is excellent for the low-lying modes. In future papers in this series the WKB approximation will be applied to Reissner-Nordström and Kerr black holes.

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