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Black-hole normal modes: A WKB approach. I. Foundations and application of a higher-order WKB analysis of potential-barrier scattering

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We present a semianalytic technique for determining the complex normal-mode frequencies of black holes. The method makes use of the WKB approximation, carried to third order beyond the eikonal approximation. Mathematically, the problem is similar to studying one-dimensional quantum-mechanical scattering near the peak of a potential barrier, and determining the scattering resonances. Under such conditions, a modification of the usual WKB approach must be used. We obtain the connection formulas that relate the amplitudes of incident, reflected, and transmitted waves, to the third WKB order. By imposing the normal-mode (resonance) boundary condition of a zero incident amplitude with nonzero transmitted and reflected amplitudes, we find a simple formula that determines the real and imaginary parts of the normal-mode frequency of perturbation (or of the quantum-mechanical energy of the resonance) in terms of the derivatives (up to and including sixth order) of the barrier function evaluated at the peak, and in terms of the quantity $(n+\frac{1}{2})$, where n is an integer and labels the fundamental mode (resonance), first overtone, and so on. This higher-order approach may find uses in barrier-tunneling problems in atomic and nuclear physics.

I. INTRODUCTION

For a number of years, the normal modes of oscillation of black holes have been of great interest both to gravitation theorists and to gravitational-wave experimentalists.[1] These modes are the resonant, nonradial perturbations of black holes, analogous to those of the Sun or Earth, that can be induced by external perturbations. They are characterized by a spectrum of discrete, complex frequencies, whose real parts determine the oscillation frequency, and whose imaginary parts determine the rate at which each mode is damped as a result of the emission of radiation. For a given kind of physical perturbation (scalar field, neutrino field, electromagnetic field, or graviational field), the complex frequencies are uniquely determined by the mass and angular momentum of the hole, the angular harmonic index (l,m) of the deformation, and the degree of the harmonic of the mode.

To the gravitational-wave astronomer, black-hole normal modes may be an important source of gravitational waves emitted at discrete frequencies by a deformed black hole left over following a supernova collapse. Recent numerical calculations of rotating collapse have found that for some collapse scenarios in which a black hole is formed, the bulk of gravitational radiation is emitted via normal-mode oscillations of the hole that continue after the matter has crossed the horizon.[2,3] The identification of the frequencies and damping times of such waves could aid in estimating the parameters of the black hole. Studies of perturbations of black holes by passing particles have also shown excitation of normal modes.[4] Normal modes are important in analyzing the stability of black holes against external perturbations. Although the nonrotating Schwarzschild black hole is known rigorously to be stable, the situation is not so certain in the case of the rotating Kerr black hole,[5] and a systematic study of normal modes could contribute to a resolution of this question.

Although the fundamental equations describing the perturbations of black holes reduce to a single second-order ordinary differential equation that is similar to the one-dimensional Schrödinger equation for a particle encountering a potential barrier on the infinite line, the nature of the potential precludes an exact, closed-form solution in terms of known functions. Thus there are several basic approaches to the study of black-hole normal modes: direct numerical integration of the differential equations, the use of infinite-series representations of solutions, and semianalytic methods based on an approximation. Mathematically, a normal mode is a solution to the differential equation with a complex frequency, satisfying the boundary condition of purely "outgoing" waves, that is waves propagating away from the barrier, at both $+\infty$ and $-\infty$, the latter boundary condition corresponding to waves traveling across the horizon to the interior of the black hole. The quantum-mechanical analogue of this is a scattering resonance with a complex energy. Because such a boundary condition cannot actually correspond to a stationary state, the energy or squared frequency must be complex, leading to a characteristic damping with time of wave packets constructed from the modes.

Studying black-hole normal modes numerically requires selecting a value for the complex frequency, integrating the differential equation, and checking whether the boundary conditions for a normal mode are satisfied. Since those conditions are *not* satisfied in general, the complex frequency plane must be surveyed for the discrete values that lead to normal modes. This technique is time consuming and therefore costly, and it makes difficult a systematic survey of normal modes for a wide range of parameter values. Following early work by Vishveshwara, Press, and Goebel,[6] Chandrasekhar and Detweiler[7] pioneered this method for the study of normal modes.

A few semianalytic analyses have been attempted. In one approach, employed by Mashhoon and co-workers,[8] the potential barrier in the effective one-dimensional Schrödinger equation is replaced by a parametrized analytic potential barrier function for which simple exact solutions are known. The overall shape approximates that of the true black-hole barrier, and the parameters of the barrier function are adjusted to fit the height and curvature of the true barrier at the peak. The resulting estimates for the normal-mode frequencies have been applied to the Schwarzschild, Reissner-Nordström, and Kerr black holes, with agreement within a few percent with the numerical results of Chandrasekhar and Detweiler[7(a)] in the Schwarzschild case, and with Gunter[9] in the Reissner-Nordström case. However, because this method relies upon a specialized barrier function, there is no systematic way to estimate the errors or to improve the accuracy. Another method by Leaver[10] which is a hybrid of the analytic and the numerical, successfully generates normal-mode frequencies by making use of an analytic infinite-series representation of the solutions, together with a numerical solution of an equation for the normal-mode frequencies which involves continued fractions.

We have developed an alternative technique for determining the normal-mode frequencies semianalytically, using the WKB approximation. Even though it is based on an approximation, we believe this approach will be powerful (a) because the WKB approximation is known in many cases to be more accurate than one has a right to expect *a priori* (see Ref. 11, pp. 487–492 for examples), (b) because the method can be carried to higher orders, either as a means to improve the accuracy or as a means to estimate the errors explicitly, and (c) because it will allow a more systematic study of normal modes than has been possible using outright numerical methods. In a previous paper, Schutz and Will[12] described the basic elements of this method at lowest WKB order, and applied it to a simple test case: the fundamental normal-mode frequencies of the Schwarzchild black hole. The result was a simple analytic expression for the complex frequency, that, for the fundamental quadrupole mode agreed with the numerical results of Chandrasekhar and Detweiler[7(a)] within 7% for the real part and 0.8% for the imaginary part.

The motivation for using the WKB approximation is the similarity alluded to above between the equations of black-hole perturbation theory and the one-dimensional Schrödinger equation for a potential barrier. This similarity has been emphasized and exploited by Chandrasekhar (Ref. 13, Secs. 27 and 28), for example. In both cases the central equation has the form

$$d^2\psi/dx^2+Q(x)\psi(x)=0 . \qquad (1.1)$$

In the black-hole case, ψ represents the radial part of the perturbation variable, assumed to have time dependence $e^{-i\omega t}$, and angular dependence appropriate to the particular perturbation and black hole under study. The coordinate x is linearly related to the "tortoise radial coordinate" r_*, which ranges from $-\infty$ at the event horizon to $+\infty$ at spatial infinity. The "potential" $-Q(x)$ is constant at $x=\pm\infty$, although not necessarily the same at both ends, and it rises to a maximum at $x=x_0$ (Fig. 1).

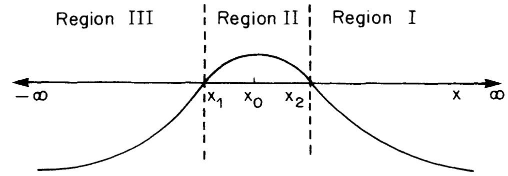

FIG. 1. The function $-Q(x)$.

It depends on the nature of the perturbing field, the mass, and angular momentum of the black hole, the angular harmonic indices, and the frequency. This includes perturbations of Schwarzschild, Kerr, and Reissner-Nordström black holes.

For example, for gravitational perturbations of Schwarzschild, $x=r_*$, and the central equation can be expressed in the "Regge-Wheeler" form

$$d^2\psi/dr_*{}^2+\{\omega^2-(1-2M/r)[l(l+1)/r^2 -6M/r^3]\}\psi(r_*)=0 , \qquad (1.2)$$

where M is the mass of the black hole, and the radial coordinate r and the tortoise coordinate r_* are related by

$$dr/dr_*=1-2M/r . \qquad (1.3)$$

For formulas for r_* and Q for Kerr black holes, for example, see Ref. 7(a), Ref. 14, pp. 90 and 91, and Ref. 15.

In quantum mechanics, $-Q(x)=(2m/\hbar^2)[V(x)-E]$, where E is the energy of the particle of mass m and $V(x)$ is the potential barrier.

The field ψ in regions I and III of Fig. 1 is approximated by linear combinations of incoming-wave and outgoing-wave WKB functions with real frequencies, carried to third order in the WKB expansion. These functions must be matched through region II. However, for reasons to be described in more detail below, the lowest-lying normal modes or resonances are expected to occur for values of the frequency such that the "classical turning points" x_1 and x_2 are close together, near the peak of the barrier. In this case, the usual WKB approach involving a matching of the solution in region III across x_1 to a WKB solution in the interior region II, and a similar matching at x_2, is no longer valid; the two turning points are too close together to permit a valid WKB approximation to the interior solution. Our method, which is based on the technique described by Bender and Orszag (Ref. 11, pp. 531–533), circumvents this difficulty by matching the two exterior WKB solutions across both turning points simultaneously. In region II we expand the function $Q(x)$ about its maximum at $x=x_0$ in a Taylor expansion, up to and including terms of order $(x-x_0)^6$ (in Ref. 12, only the constant and quadratic terms were kept). The quantity x_0 will depend explicitly on the frequency. We then obtain an asymptotic approximation to the general interior solution and use it to connect the two WKB solutions.

The result is a pair of connection formulas, relating the amplitudes of the incoming and outgoing solutions on either side of the barrier. For a black-hole normal mode, the boundary condition of only "outgoing" waves leads to a constraint (to be derived in the next section) on the values of the coefficients in the Taylor expansion of $Q(x)$ about x_0, given by

$$iQ_0/(2Q_0'')^{1/2}-\Lambda(n)-\Omega(n)=n+\tfrac{1}{2}\,,\qquad n=\begin{cases}0,1,2,\ldots, & \mathrm{Re}\omega>0\,,\\ -1,-2,\ldots, & \mathrm{Re}\omega<0\,,\end{cases} \tag{1.4}$$

where

$$\Lambda(n)=\frac{1}{(2Q_0'')^{1/2}}\left[\frac{1}{8}\left[\frac{Q_0^{(4)}}{Q_0''}\right]\left[\frac{1}{4}+\alpha^2\right]-\frac{1}{288}\left[\frac{Q_0'''}{Q_0''}\right]^2(7+60\alpha^2)\right], \tag{1.5a}$$

$$\Omega(n)=\frac{n+\frac{1}{2}}{2Q_0''}\left[\frac{5}{6912}\left[\frac{Q_0'''}{Q_0''}\right]^4(77+188\alpha^2)-\frac{1}{384}\left[\frac{Q_0'''^2Q_0^{(4)}}{Q_0''^3}\right](51+100\alpha^2)+\frac{1}{2304}\left[\frac{Q_0^{(4)}}{Q_0''}\right]^2(67+68\alpha^2)\right.$$
$$\left.+\frac{1}{288}\left[\frac{Q_0'''Q_0^{(5)}}{Q_0''^2}\right](19+28\alpha^2)-\frac{1}{288}\left[\frac{Q_0^{(6)}}{Q_0''}\right](5+4\alpha^2)\right]. \tag{1.5b}$$

Here the primes and the superscript (n) denote the appropriate numbers of derivatives of Q, evaluated at x_0 ($Q_0''\neq 0$), and $\alpha\equiv n+\frac{1}{2}$. Since $Q(x)$ in general depends on the frequency ω, Eqs. (1.4) and (1.5) can be solved for the normal-mode frequencies by analytically continuing Eq. (1.4) into the complex frequency plane.

In the special case in which $Q(x)$ has the form $Q(x)=\omega^2-V(x)$, where V is independent of frequency, Eq. (1.4) for the normal-mode frequencies simplifies to

$$\omega^2=[V_0+(-2V_0'')^{1/2}\tilde{\Lambda}(n)]-i(n+\tfrac{1}{2})(-2V_0'')^{1/2}[1+\tilde{\Omega}(n)]\,, \tag{1.6a}$$

where

$$\tilde{\Lambda}\equiv\Lambda/i,\quad \tilde{\Omega}\equiv\Omega/(n+\tfrac{1}{2})\,, \tag{1.6b}$$

where Q and its derivatives are replaced by $-V$ and its derivatives in Λ and Ω. As we noted in Eq. (1.2), this is the case in perturbations of Schwarzschild black holes.

The lowest-order contributions to the real and imaginary parts of ω^2 in Eq. (1.6a) correspond to the first-order WKB results of Ref. 12, with Q truncated at quadratic order. Contributions through second WKB order, and through $(x-x_0)^4$ in Q yield the correction term Λ (Ref. 16) while contributions through third WKB order, and through $(x-x_0)^6$ in Q yield the term Ω.

Equation (1.4) applies to any physical problem governed by Eq. (1.1) and the normal-mode boundary conditions. In particular, it applies to the determination of quantum-mechanical resonances near the tops of one-dimensional potential barriers (with the restriction $\mathrm{Re}\omega>0$ corresponding to $\mathrm{Re}E>0$). Furthermore, the connection formulas derived below (Sec. III) can be used to determine reflection and transmission coefficients in quantum-mechanical tunneling near barrier peaks. Application of this higher-order WKB approach to other areas of physics than black holes is currently under investigation.

The structure of this paper is as follows. In Sec. II we review the manner in which the master equation (1.1) arises from black-hole perturbation theory and establish the proper normal-mode boundary conditions. Black holes will rarely be mentioned from that point on. Section III applies the WKB approximation to the master equation and obtains the connection formulas. In Sec. IV, we impose the normal-mode boundary condition and obtain the equation for the normal-mode frequency. Section V presents concluding remarks. In an appendix we outline a method for extending the results to higher order without explicitly obtaining connection formulas, and give the result to fourth WKB order.

II. BLACK-HOLE PERTURBATION THEORY

A. The master radial equation for black-hole perturbations

The stationary, axisymmetric solutions of Einstein's equations corresponding to isolated black holes (Schwarzschild, Kerr, and Reissner-Nordström) can be perturbed in a variety of ways, by adding dynamical nonvacuum test fields to the black-hole spacetime, such as scalar fields, neutrino fields, electromagnetic fields, or by perturbing the spacetime directly with gravitational perturbations (for reviews, see Refs. 13 and 14). The variables describing these perturbations can be complicated and numerous (six for electromagnetic perturbations, ten for gravitational), and lead to coupled differential equations in general. However, it turns out that for each case, appropriate linear combinations of the variables can be found for which the equations decouple. The result is a single hyperbolic partial differential equation for a variable $\Psi^{(s)}$, where s is a parameter whose value depends on the field under study. The parameter s, known as the spin weight of the field, has the value $s=0$ for a scalar field,

$s=\pm\frac{1}{2}$ for a neutrino field $s=\pm 1$ for an electromagnetic field, and $s=\pm 2$ for gravitational perturbations. For a given value of s, the field $\Psi^{(s)}$ is a function of particular components of the field under study on a tetrad of null vectors in the background black-hole spacetime: for electromagnetic perturbations, they are components of the Maxwell tensor, for gravitational perturbations they are components of the perturbed Weyl tensor, and so on. The decoupled equations take the general form

$$\mathscr{L}\Psi^{(s)}=T \ , \tag{2.1}$$

where $\mathscr{L}$ is a second-order linear differential operator that depends on s, the mass, angular momentum, and charge of the black hole, on the coordinate system [such as (t,r,θ,ϕ)], and on the particular choice of null-tetrad basis. The quantity T is a function of the source of the perturbing field. In a vacuum, $T=0$.

Equation (2.1) can be solved by separation of variables. Setting

$$\Psi^{(s)}(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty} d\omega\, R^s_{lm\omega}(r)S^s_{lm\omega}(\theta)e^{im\phi}e^{-i\omega t} \ , \tag{2.2}$$

leads to ordinary differential equations for each variable. The eigenfunctions of the angular differential equation $S^s_{lm\omega}(\theta)\ e^{im\phi}$, depend upon the black hole under study, and upon the spin weight of the perturbing field. They may also depend upon the frequency. For example, for Kerr, they are known as "spin-weighted spheroidal harmonics" (generalizations of tensor spherical harmonics) and are complete and orthonormal on the unit oblate spheroid. The radial differential equation for R can be transformed by defining a new radial function $\psi^s_{lm\omega}=f(r)R$ and by converting from the usual radial coordinate r to a so-called "tortoise coordinate" r_* related to r by the equation

$$dr_*/dr=h(r) \ , \tag{2.3}$$

where $h(r)$ is a positive function that tends to unity as r tends to infinity, and becomes singular as r tends to its value at the event horizon [see Eq. (1.3) for the case of Schwarzschild]. Thus r_* runs from $-\infty$ to $+\infty$ as r runs from the horizon to infinity. As a consequence of these transformations, the equation for $\psi^s_{lm\omega}$ can be written

$$d^2\psi^s_{lm\omega}/dr_*^2+Q^s_{lm\omega}(r_*)\psi^s_{lm\omega}=T^s_{lm\omega} \ , \tag{2.4}$$

where $T^s_{lm\omega}$ is defined from T as in Eq. (2.2). The function $Q^s_{lm\omega}(r_*)$ tends to constant values as $r_*\to\pm\infty$, though not necessarily the same at each end. This equation is the "master radial equation" for black-hole perturbations. Equation (1.2) gives an example for Schwarzschild; for master radial equations for Kerr, see Ref. 7(a), Ref. 14, pp. 90 and 91, and Ref. 15; for Reissner-Nordström, see Ref. 13, Sec. 42.

For vacuum perturbations ($T=0$) of Schwarzschild [Eq. (1.2)], the master radial equation is exactly of the form of the one-dimensional Schrödinger equation of nonrelativistic quantum mechanics, for a particle of energy E and mass m in a potential $V(r_*)$ given by

$$V(r_*)=E-(\hbar^2/2m)Q^s_{lm\omega}(r_*) \ , \tag{2.5}$$

where $\hbar$ is Planck's constant. Thus the function $-Q(r_*)$ plays the role of $V-E$. Because $-Q$ for Schwarzschild typically attains a maximum value for some finite r_* before decreasing to its values at $r_*=\pm\infty$, the problem is equivalent to quantum-mechanical scattering off a potential barrier. In other black-hole situations, the effective "potential" V may be energy or frequency dependent (and in some representations of the equations may be explicitly complex), so the quantum-mechanical analogy may not be so clear-cut. Nevertheless, since the WKB method we develop in this paper is nothing but a problem in matched asymptotic expansions, the results should apply to all "potentials."

B. Normal-mode boundary conditions

The boundary conditions to be applied to the function $\psi^s_{lm\omega}$ at $r_*=\pm\infty$ can be determined by studying the flux of radiation detected by physical observers at infinity and near the event horizon. At infinity, observers can detect incoming and outgoing radiation, while at the horizon, only radiation propagating *into* the black hole is present. These physical conditions can be translated into conditions on the radial functions $\psi^s_{lm\omega}$ at $r_*=\pm\infty$, whose specific form depends on the field or spin weight and on the coordinate system being used (Ref. 14, Sec. 6.10). Throughout this discussion we will use a timelike coordinate system, in which t is the time associated with the stationary nature of the background spacetime, as opposed to a null coordinate system.

For this discussion we will focus on frequencies whose real part is positive ($\mathrm{Re}\omega>0$), corresponding to the positive half of the Fourier integral in Eq. (2.2). In the limit $r_*\to\infty$, spacetime becomes asymptotically flat, and

$$(Q^s_{lm\omega})^{1/2}\to\omega \ , \tag{2.6}$$

where we consistently choose the positive square root. Then incoming and outgoing radiation at infinity correspond, respectively, to the radial solutions proportional to $e^{-i\omega r_*}$ and $e^{+i\omega r_*}$. If we represent solutions with these asymptotic behaviors by the notation $\psi_{\rm in}$ and $\psi_{\rm out}$, then the general solution valid as $r_*\to\infty$ can be written

$$\psi^s_{lm\omega}=Z_{\rm in}\psi_{\rm in}+Z_{\rm out}\psi_{\rm out}, \quad r_*\to\infty \ , \tag{2.7}$$

where $Z_{\rm in}$ and $Z_{\rm out}$ are complex amplitudes for incoming and outgoing waves, respectively.

Near the event horizon, in the limit $r_*\to-\infty$,

$$(Q^s_{lm\omega})^{1/2}\to\tilde{k} \ , \tag{2.8}$$

where $\tilde{k}$ is a constant that depends on ω, s, and the parameters of the hole. The physically correct boundary condition at the horizon corresponds to the solution proportional to $e^{-i\tilde{k}r_*}$. If we denote the solution with this behavior by $\psi_{\rm hole}$, then the general solution valid as $r_*\to-\infty$ is

$$\psi^s_{lm\omega}=Z_{\rm hole}\psi_{\rm hole}, \quad r_*\to-\infty \ . \tag{2.9}$$

It is important to note that it is possible for the real part of $\tilde{k}$ to be negative. This occurs in perturbations of Kerr, for example, where $\mathrm{Re}\tilde{k}=\omega-m\omega_+$ where ω_+ is the angular velocity of the horizon, and m is the azimuthal quantum number. When $\mathrm{Re}\tilde{k}<0$, the energy flux down the hole, while positive according to local observers, is actually negative as seen from infinity, and the outgoing energy flux exceeds the incident flux. This phenomenon is called super-radiance. Nevertheless, with our convention for choosing the square root of Q, the correct boundary condition in all cases is that of Eq. (2.9).

For frequencies whose real part is negative ($\mathrm{Re}\omega<0$), the identification of the incoming and outgoing parts of the radial functions is reversed. However, the overall analysis is unchanged, and this case can be handled at the end by suitable interchanges of the incoming and outgoing variables.

Given an amplitude $Z_{\rm in}$ of incoming radiation, it is possible to solve Eq. (2.4) (with $T=0$) for $\psi^s_{lm\omega}$ to determine $Z_{\rm out}$ and $Z_{\rm hole}$. Generally speaking, the ratio $|Z_{\rm out}|^2/|Z_{\rm in}|^2$ will be a smooth function of the frequency ω, except in the neighborhood of a set of discrete values ω_0, where it may have the behavior

$$\frac{|Z_{\rm out}|^2}{|Z_{\rm in}|^2}=\frac{(\Gamma/2)^2}{(\omega-\omega_0)^2+(\Gamma/2)^2} . \tag{2.10}$$

This is the usual Lorentzian response appropriate to a resonance in a damped harmonic oscillator, where ω_0 is the resonant frequency, and $\Gamma/2$, the half width at half maximum of the Lorentzian curve, determines the damping rate of the oscillation. Such an occurrence corresponds to a resonant normal mode of the black hole, a mode whose response to an external perturbation is a maximum. An alternative way to characterize the resonance is to analytically continue Eq. (2.10) into the complex frequency plane, and to write

$$\frac{Z_{\rm out}}{Z_{\rm in}}=\frac{\Gamma/2}{\omega-\omega_0+i\Gamma/2} . \tag{2.11}$$

The resonance then corresponds to a pole in the function $Z_{\rm out}/Z_{\rm in}$, at the complex resonant frequency $\omega_R=\omega_0-i\Gamma/2$. In the vacuum differential equation for $\psi^s_{lm\omega}$, we therefore seek solutions with complex ω subject to the boundary conditions $Z_{\rm in}=0$, with $Z_{\rm out}\neq 0$ and $Z_{\rm hole}\neq 0$. [For a discussion of normal-mode boundary conditions from a different point of view, see Ref. 8(b).]

From this point on, the discussion need no longer refer to black holes; it will be equally relevant for quantum-mechanical tunneling, or for any other physics governed by the above mathematics. Therefore it will be most convenient to discuss the problem in terms of tunneling and resonances near the peaks of potential barriers.

III. TUNNELING NEAR THE PEAKS OF POTENTIAL BARRIERS: A THIRD-ORDER WKB ANALYSIS

A. The WKB approximation

Our goal is to use the WKB approximation to find normal-mode or resonant solutions to the master equation (1.1). Initially, the procedure follows the standard textbook approach.[11] We first rewrite the master equation (1.1) in the generic form

$$\epsilon^2 d^2\psi/dx^2+Q(x)\psi(x)=0 , \tag{3.1}$$

where we introduce the perturbation parameter ϵ to keep track of orders in the WKB approximation. We then define the asymptotic approximation

$$\psi\sim\exp[S(x)/\epsilon] , \tag{3.2}$$

where S is expanded in powers of the parameter ϵ:

$$S(x)=\sum_{n=0}^{\infty}\epsilon^n S_n(x) . \tag{3.3}$$

By substituting into Eq. (3.1), and equating like powers of ϵ, we find, through the third nontrivial order (see Ref. 11, p. 487),

$$S_0(x)=\pm i\int^x[Q(\eta)]^{1/2}d\eta , \tag{3.4a}$$

$$S_1(x)=-\tfrac{1}{4}\ln Q(x) , \tag{3.4b}$$

$$S_2(x)=\mp\frac{i}{8}\int^x\left[\frac{Q''}{Q^{3/2}}-\frac{5}{4}\frac{Q'^2}{Q^{5/2}}\right]d\eta , \tag{3.4c}$$

$$S_3(x)=\frac{1}{16}\left[\frac{Q''}{Q^2}-\frac{5}{4}\frac{Q'^2}{Q^3}\right] . \tag{3.4d}$$

The boundary conditions on ψ can be translated into particular choices of the sign of the exponent in Eq. (3.4a). As $x\to\infty$, $Q(x)\to\omega^2$, so $S_0\to\pm i\omega x$, thus the positive or upper sign in Eq. (3.4a) corresponds to outgoing waves, while the negative or lower sign corresponds to incoming waves. As $x\to-\infty$, $Q(x)\to\tilde{k}^2$, so $S_0\to\pm i\tilde{k}x$, thus the positive or upper sign corresponds to waves coming from $-\infty$ (incoming), while the negative or lower sign corresponds to waves going toward $-\infty$ (outgoing). For the present we will retain all four solutions. Using subscripts $(+)$ and $(-)$ to denote the appropriate signs of the WKB solutions, we can write the general solution in regions I and III of Fig. 1 in the form

$$\begin{aligned}&\psi\sim Z^{\rm I}_{\rm in}\psi^{\rm I}_-+Z^{\rm I}_{\rm out}\psi^{\rm I}_+ \quad \text{region I} ,\\&\psi\sim Z^{\rm III}_{\rm in}\psi^{\rm III}_++Z^{\rm III}_{\rm out}\psi^{\rm III}_- \quad \text{region III} .\end{aligned} \tag{3.5}$$

Note that, for the black-hole case, $Z^{\rm III}_{\rm out}=Z_{\rm hole}$, and $Z^{\rm III}_{\rm in}=0$.

The object now is to determine formulas that connect the amplitudes near $+\infty$ with those near $-\infty$, in other words to determine the coefficients in the linear relationship given by

$$\begin{bmatrix}Z^{\rm III}_{\rm out}\\ Z^{\rm III}_{\rm in}\end{bmatrix}\equiv\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix}Z^{\rm I}_{\rm out}\\ Z^{\rm I}_{\rm in}\end{bmatrix} . \tag{3.6}$$

To do so, we must consider the solution in region II. The WKB approximation is valid except near the "classical turning points" the points where $Q(x)=0$. Nevertheless, it is possible to connect WKB solutions that are valid on either side of such turning points, using, for example, asymptotic matching techniques. The idea would be to match a WKB solution in region III across the turning point x_2 to a WKB solution in region II, and then match that solution across the turning point x_1 to the WKB solution in region I (see Sec. 10.5 of Ref. 11 for details). This would give the required coefficients in Eq. (3.6). However, a consideration of the boundary conditions for normal modes shows that this standard method will not work here. To see this, let us compare and contrast the standard quantum-mechanical barrier problem with the resonance or normal-mode problem.

For a wave incident on a potential barrier from $x=\infty$ with a given amplitude, it is a standard calculation in quantum mechanics to determine the amplitude of the wave reflected back to $+\infty$ and that transmitted to $-\infty$. If the function $-Q(x)$ is positive anywhere, i.e., if the energy is below the peak of the potential, the reflected amplitude is generally comparable to the incident amplitude, while the transmitted amplitude is much smaller. When the WKB approximation is applied to such problems and the standard matching across the two turning points described above is carried out, it leads to an estimate for the transmitted amplitude of e^{-B} where B is a "barrier penetration factor," given by an integral of $[-Q(x)]^{1/2}$ between the turning points (see Sec. 10.6 of Ref. 11).

Normal-mode or resonance problems, on the other hand, involve a rather different set of boundary conditions, namely, reflected and transmitted amplitudes that are of the same order of magnitude, with vanishing incident amplitude. At first glance, one might expect the WKB approximation to be useless for normal modes, because it always seems to lead to the exponentially small factor e^{-B} relating the transmitted to the reflected amplitude, rather than a factor of order unity. However, there is at least one case in which this is not true, namely, that in which the maximum value of $-Q(x)$ is precisely zero. In quantum mechanics this occurs when the energy coincides with the peak of the potential $V(x)$. In this "second-order turning point" problem, the WKB approximation leads to equal magnitudes for the two outgoing waves, each a factor $2^{-1/2}$ times the incident amplitude (see, for example, Ref. 11, pp. 531–533). This suggests that, if normal modes or resonances exist for a given potential, then some of them exist "nearby," in other words for complex frequencies such that $[-Q(x)]_{\max}=0$. However, if $[-Q(x)]_{\max}\gtrsim 0$, the classical turning points will in general be too close together to allow application of the standard WKB matching approach. Nevertheless, a simple modification of the matching procedure allows a complete solution of the normal-mode problem.

B. Asymptotic approximation to the interior solution to third order

The modification of the usual WKB procedure involves matching the two WKB solutions of Eq. (3.1) across both of the turning points simultaneously. Outside the turning points (regions I and III of Fig. 1), the WKB functions will be given by Eq. (3.2), expanded to third order. In region II we approximate $-Q(x)$ by a Taylor expansion up to and including terms in the sixth derivative of $-Q$. An asymptotic approximation to the interior solution will be obtained, and used to match the two WKB solutions. This approach was suggested by the discussions of scattering off the peak of a potential barrier and of higher-order WKB approximations in Ref. 11 (Secs. 10.6 and 10.7); see also Refs. 17 and 18.

We expand $Q(x)$ in a Taylor series about the point x_0 at which $-Q$ reaches a maximum:

$$Q(x)=Q_0+\tfrac{1}{2}Q_0''z^2+\tfrac{1}{6}Q_0'''z^3+\tfrac{1}{24}Q_0^{(4)}z^4+\tfrac{1}{120}Q_0^{(5)}z^5+\tfrac{1}{720}Q_0^{(6)}z^6\,, \tag{3.7}$$

where $z\equiv x-x_0$. Equation (3.1) can then be written

$$\epsilon^2 d^2\psi/dz^2+k(-z_0{}^2+z^2+bz^3+cz^4+dz^5+fz^6)\psi=0\,, \tag{3.8}$$

where

$$\begin{aligned} &k\equiv\tfrac{1}{2}Q_0'',\quad z_0{}^2\equiv-2Q_0/Q_0''\,,\\ &b\equiv\tfrac{1}{3}Q_0'''/Q_0'',\quad c=\tfrac{1}{12}Q_0^{(4)}/Q_0''\,,\\ &d\equiv\tfrac{1}{60}Q_0^{(5)}/Q_0'',\quad f=\tfrac{1}{360}Q_0^{(6)}/Q_0''\,. \end{aligned} \tag{3.9}$$

Now, since region II corresponds to $|z|<z_0=(-2Q_0/Q_0'')^{1/2}\approx\epsilon^{1/2}$, we define a new variable $t\propto z/\epsilon^{1/2}$ in which to express the asymptotic solution of the interior problem. The limit $t\to\infty$ as $\epsilon\to 0$ will correspond to finite overlap regions outside the turning points where the solution may be matched asymptotically to the WKB solutions. We also define constants ν, Λ, and Ω, related to z_0, and rescale the parameters b, c, d, and f, as

$$t\equiv(4k)^{1/4}e^{-i\pi/4}z/\epsilon^{1/2}\,, \tag{3.10a}$$

$$\nu+\tfrac{1}{2}\equiv-ik^{1/2}z_0{}^2/2\epsilon-\epsilon\Lambda-\epsilon^2\Omega\,, \tag{3.10b}$$

$$\bar{b}\equiv\tfrac{1}{4}b(4k)^{-1/4}e^{i\pi/4},\quad \bar{c}=\tfrac{1}{4}c(4k)^{-1/2}e^{i\pi/2}\,, \tag{3.10c}$$

$$\bar{d}=\tfrac{1}{4}d(4k)^{-3/4}e^{3i\pi/4},\quad \bar{f}=\tfrac{1}{4}f(4k)^{-1}e^{i\pi}\,. \tag{3.10d}$$

Equation (3.8) then takes the form

$$\ddot{\psi}+[\nu+\tfrac{1}{2}-\tfrac{1}{4}t^2-\epsilon^{1/2}\bar{b}t^3+\epsilon(\Lambda-\bar{c}t^4)-\epsilon^{3/2}\bar{d}t^5+\epsilon^2(\Omega-\bar{f}t^6)]\psi=0\,, \tag{3.11}$$

where an overdot denotes d/dt. Without the terms proportional to powers of ϵ, the solutions to Eq. (3.11) would be parabolic cylinder functions $D_\nu(t)$ and $D_{-\nu-1}(it)$ (Ref. 11, pp. 573 and 574). Including the ϵ terms, we look for a solution of the form $f(t)D_\nu[g(t)]$ (Ref. 18). Substituting into Eq. (3.11), and defining $Q(t)$ to be the polynomial in square brackets, we obtain

$$[-f\dot{g}^2(\nu+\tfrac{1}{2}-\tfrac{1}{4}g^2)+\ddot{f}+Qf]D_\nu+(2\dot{f}\dot{g}+f\ddot{g})\dot{D}_\nu=0\,. \tag{3.12}$$

The first derivative of D_ν can be eliminated by choosing

$$f=\dot{g}^{-1/2}\,. \tag{3.13}$$

Substituting for f and equating the coefficient of D_ν to zero, we get a differential equation for $g(t)$:

$$\dot{g}^2(\nu+\tfrac{1}{2}-\tfrac{1}{4}g^2)+\tfrac{1}{2}\dddot{g}/\dot{g}-\tfrac{3}{4}\ddot{g}^2/\dot{g}^2-Q(t)=0 . \tag{3.14}$$

Now, $Q(t)$ is an expansion in ϵ of the form

$$Q(t)=\nu+\tfrac{1}{2}-\tfrac{1}{4}t^2+\sum_{n=1}^{\infty}\epsilon^{n/2}Q_n(t) , \tag{3.15}$$

where $Q_n(t)$ is a polynomial, so we try a similar expansion for $g(t)$. At lowest order, $g(t)=t$, so we try

$$g(t)=t+\sum_{n=1}^{\infty}\epsilon^{n/2}A_n(t) , \tag{3.16}$$

where $A_n(t)$ is a polynomial in t. Substituting into Eq. (3.14), equating to zero the coefficient of each order of $\epsilon^{1/2}$, and defining the operator L_t by

$$L_t=d^3/dt^3+4(\nu+\tfrac{1}{2}-\tfrac{1}{4}t^2)d/dt-t , \tag{3.17}$$

we obtain a sequence of equations for the A_n:

$$L_tA_1+2\bar{b}t^3=0 , \tag{3.18a}$$

$$L_tA_2-\tfrac{1}{2}A_1^2-2tA_1\dot{A}_1+2\dot{A}_1^2(\nu+\tfrac{1}{2}-\tfrac{1}{4}t^2)-\dddot{A}_1\dot{A}_1-\tfrac{3}{2}\ddot{A}_1^2-2\Lambda+2\bar{c}t^4=0 , \tag{3.18b}$$

$$L_tA_3-A_1A_2-\dot{A}_1(A_1^2+2tA_2)-tA_1(\dot{A}_1^2+2\dot{A}_2)+4\dot{A}_1\dot{A}_2(\nu+\tfrac{1}{2}-\tfrac{1}{4}t^2)+\dddot{A}_1(\dot{A}_1^2-\dot{A}_2)-\dot{A}_1\dddot{A}_2-3\ddot{A}_1\ddot{A}_2$$
$$+3\ddot{A}_1^2\dot{A}_1+2\bar{d}t^5=0 , \tag{3.18c}$$

$$L_tA_4-\tfrac{1}{2}(2A_1A_3+A_2^2)-2\dot{A}_1(tA_3+A_1A_2)-\tfrac{1}{2}(\dot{A}_1^2+2\dot{A}_2)(A_1^2+2tA_2)-2tA_1(\dot{A}_3+\dot{A}_1\dot{A}_2)$$
$$+2(2\dot{A}_1\dot{A}_3+\dot{A}_2^2)(\nu+\tfrac{1}{2}-\tfrac{1}{4}t^2)-\dddot{A}_1(\dot{A}_3-2\dot{A}_1\dot{A}_2+\dot{A}_1^3)+(\dot{A}_1^2-\dot{A}_2)\dddot{A}_2-\dot{A}_1\dddot{A}_3+\tfrac{3}{2}\ddot{A}_1^2(2\dot{A}_2-3\dot{A}_1^2)+6\ddot{A}_1\ddot{A}_2\dot{A}_1$$
$$-\tfrac{3}{2}(\ddot{A}_2^2+2\ddot{A}_1\ddot{A}_3)-2\Omega+2\bar{f}t^6=0 . \tag{3.18d}$$

From the nature of L_t and the inhomogeneous terms in Eqs. (3.18), it is easy to see that each polynomial A_n should have the form

$$A_n(t)=\sum_{i=0}a_{2i}^{n}t^{n+1-2i} \quad (n+1-2i\geq 0) . \tag{3.19}$$

When these forms are substituted into Eqs. (3.18), we find that, for n odd, the number of distinct powers of t in each equation exactly matches the number of unknown coefficients a_{2i}^{n}, so we can solve for these coefficients by setting the coefficient of each power of t to zero. But for n even, the number of distinct powers of t exceeds by one the number of unknown coefficients, so the system is overdetermined. However, in these cases, the constants Λ and Ω, appear in the equations, and these can then be chosen to achieve a consistent solution. This of course was the reason for introducing them in Eqs. (3.10). The results for Λ and Ω are

$$\Lambda=\tfrac{1}{2}(3\bar{c}-7\bar{b}^2)+(\nu+\tfrac{1}{2})^2(6\bar{c}-30\bar{b}^2) , \tag{3.20a}$$

$$\Omega=-(\nu+\tfrac{1}{2})(1155\bar{b}^4-918\bar{b}^2\bar{c}+67\bar{c}^2+190\overline{bd}-25\bar{f})-(\nu+\tfrac{1}{2})^3(2820\bar{b}^4-1800\bar{b}^2\bar{c}+68\bar{c}^2+280\overline{bd}-20\bar{f}) . \tag{3.20b}$$

The resulting polynomials are given by

$$A_1(t)=\tfrac{16}{3}\bar{b}(\nu+\tfrac{1}{2}+\tfrac{1}{8}t^2) , \tag{3.21a}$$

$$A_2(t)=(\nu+\tfrac{1}{2})(3\bar{c}-\tfrac{103}{9}\bar{b}^2)t+\tfrac{1}{2}(\bar{c}-\tfrac{13}{9}\bar{b}^2)t^3 , \tag{3.21b}$$

$$A_3(t)=\tfrac{1}{5}(\tfrac{2744}{9}\bar{b}^3-216\bar{b}\bar{c}+48\bar{d})+\tfrac{1}{5}(\nu+\tfrac{1}{2})^2(\tfrac{126\,544}{81}\bar{b}^3-784\bar{b}\bar{c}+\tfrac{256}{3}\bar{d})$$
$$+\tfrac{1}{15}(\nu+\tfrac{1}{2})(\tfrac{11\,558}{27}\bar{b}^3-234\bar{b}\bar{c}+32\bar{d})t^2+\tfrac{1}{5}(\tfrac{173}{27}\bar{b}^3-\tfrac{17}{3}\bar{b}\bar{c}+2\bar{d})t^4 , \tag{3.21c}$$

$$A_4(t)=[(-\tfrac{10\,619}{36}\bar{b}^4+\tfrac{519}{2}\bar{b}^2\bar{c}-\tfrac{91}{4}\bar{c}^2-60\bar{b}\,\bar{d}+10\bar{f})$$
$$+(\nu+\tfrac{1}{2})^2(-\tfrac{2\,770\,843}{2430}\bar{b}^4+\tfrac{3733}{5}\bar{b}^2\bar{c}-\tfrac{77}{2}\bar{c}^2-\tfrac{5276}{45}\bar{b}\,\bar{d}+10\bar{f})]t$$
$$+(\nu+\tfrac{1}{2})(-\tfrac{94\,357}{1215}\bar{b}^4+\tfrac{2846}{45}\bar{b}^2\bar{c}-\tfrac{17}{3}\bar{c}^2-\tfrac{538}{45}\bar{b}\,\bar{d}+\tfrac{5}{3}\bar{f})t^3+(-\tfrac{9013}{3240}\bar{b}^4+\tfrac{187}{60}\bar{b}^2\bar{c}-\tfrac{11}{24}\bar{c}^2-\tfrac{14}{15}\bar{b}\,\bar{d}+\tfrac{1}{3}\bar{f})t^5 . \tag{3.21d}$$

The general solution to the interior problem then has the form

$$\psi\sim\dot{g}^{-1/2}[AD_\nu(g(t))+BD_{-\nu-1}(ig(t))] . \tag{3.22}$$

The overlap region with the WKB functions corresponds to large values of $|t|$, so using the appropriate asymptotic forms of the parabolic cylinder functions (Ref. 11, p. 132), we obtain

$$\psi \sim \dot{g}^{-1/2}\left[A g^{\nu} e^{-g^2/4}\left[1-\frac{\nu(\nu-1)}{2g^2}+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{8g^4}-\cdots\right] + B(ig)^{-\nu-1}e^{g^2/4}\left[1+\frac{(\nu+1)(\nu+2)}{2g^2}+\frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)}{8g^4}+\cdots\right]\right],$$

$$x/\epsilon^{1/2}\to\infty,\quad |t|\to\infty,\quad \arg t=-\pi/4, \tag{3.23a}$$

$$\psi \sim \dot{g}^{-1/2}\left[\left[A+B\frac{(2\pi)^{1/2}e^{-i\pi\nu/2}}{\Gamma(\nu+1)}\right]g^{\nu}e^{-g^2/4}\left[1-\frac{\nu(\nu-1)}{2g^2}+\cdots\right] + \left[Be^{3i\pi(\nu+1)/2}-A\frac{(2\pi)^{1/2}e^{i\pi\nu}}{\Gamma(-\nu)}\right]g^{-\nu-1}e^{g^2/4}\left[1+\frac{(\nu+1)(\nu+2)}{2g^2}+\cdots\right]\right],$$

$$x/\epsilon^{1/2}\to-\infty,\quad |t|\to\infty,\quad \arg t=3\pi/4, \tag{3.23b}$$

where $\Gamma(x)$ is the Γ function.

C. Asymptotic matching and the connection coefficients

The third-order WKB solutions given by Eqs. (3.2)—(3.4) can be matched to the interior solution in overlap regions $z\sim\pm z_0$ following standard methods. To the necessary order, the zeros $\bar{z}$ of $Q(z)$ are given by

$$\bar{z}=\pm z_0-\tfrac{1}{2}bz_0^2\pm\tfrac{1}{8}(5b^2-4c)z_0^3-\tfrac{1}{2}(2b^3-3bc+d)z_0^4\pm\tfrac{1}{128}(231b^4-504b^2c+224bd+112c^2-64f)z_0^5, \tag{3.24}$$

using which, we can write Q in the form

$$Q(z)=k[-\bar{z}^2+z^2+b(z^3-\bar{z}^3)+c(z^4-\bar{z}^4)+d(z^5-\bar{z}^5)+f(z^6-\bar{z}^6)]. \tag{3.25}$$

We then substitute this into Eqs. (3.4) for the WKB functions, where the integrals extend from $\bar{z}$ to values of z in regions I and III such that $z>\bar{z}_+$ and $z<\bar{z}_-$. We expand the functions in powers of z consistent with the order of expansion of Q, and simultaneously in powers of $\bar{z}=O(\epsilon^{1/2})$, with S_0 expanded to $O(\bar{z}^6)=O(\epsilon^3)$, S_1 to $O(\bar{z}^4)=O(\epsilon^2)$, and so on, consistent with our expansion to third nontrivial order. The results are

$$\begin{aligned}S_0\sim \pm ik^{1/2}\{&\tfrac{1}{2}z^2+\tfrac{1}{6}bz^3+\tfrac{1}{32}(4c-b^2)z^4+\tfrac{1}{80}(b^3-4bc+8d)z^5\\&-\tfrac{1}{768}(5b^4-24b^2c+32bd+16c^2-64f)z^6+\cdots\\&+\bar{z}^2[-\tfrac{1}{4}+\tfrac{1}{4}bz-\tfrac{1}{32}(3b^2-4c)z^2+\tfrac{1}{96}(5b^3-12bc+8d)z^3+\cdots]\\&+\bar{z}^3[\tfrac{1}{4}b^2z-\tfrac{1}{32}(3b^3-4bc)z^2+\cdots]\\&+\bar{z}^4[\tfrac{1}{16}z^{-2}-\tfrac{3}{16}bz^{-1}+\tfrac{1}{256}(79b^2-28c)+\tfrac{1}{128}(35b^3-28bc+24d)z+\cdots]\\&+\bar{z}^5(\tfrac{1}{8}bz^{-2}-\tfrac{3}{8}b^2z^{-1}+\cdots)+\bar{z}^6[\tfrac{1}{64}z^{-4}-\tfrac{5}{96}bz^{-3}+\tfrac{3}{256}(17b^2+4c)z^{-2}+\cdots]\\&-[\tfrac{1}{2}\bar{z}^2+\tfrac{1}{2}b\bar{z}^3+\tfrac{5}{64}(3b^2+4c)\bar{z}^4+\tfrac{1}{32}(15b^3-12bc+16d)\bar{z}^5+\cdots]\ln(2z/\bar{z})\},\end{aligned} \tag{3.26a}$$

$$\begin{aligned}e^{S_1}\sim Q^{-1/4}\sim k^{-1/4}z^{-1/2}\{&1-\tfrac{1}{4}bz+\tfrac{1}{32}(5b^2-8c)z^2-\tfrac{1}{128}(15b^3-4bc+32d)z^3+\cdots\\&+\bar{z}^2[\tfrac{1}{4}z^{-2}-\tfrac{5}{16}bz^{-1}+\tfrac{5}{128}(9b^2-8c)+\cdots]+\bar{z}^3(\tfrac{1}{4}bz^{-2}-\tfrac{5}{16}b^2z^{-1}+\cdots)\\&+\bar{z}^4[\tfrac{5}{32}z^{-4}-\tfrac{45}{128}bz^{-3}+\tfrac{13}{1024}(45b^2-8c)z^{-2}+\cdots]\},\end{aligned} \tag{3.26b}$$

$$\begin{aligned}S_2\sim \mp\tfrac{1}{8}ik^{-1/2}\{&\tfrac{3}{2}z^{-2}-\tfrac{1}{2}bz^{-1}+\tfrac{1}{12}(b^2-2c)+\tfrac{1}{16}(35b^3-92bc+72d)z-\tfrac{1}{3}\bar{z}^{-2}+\tfrac{1}{3}b\bar{z}^{-1}\\&+\bar{z}^2[\tfrac{19}{8}z^{-4}-\tfrac{31}{12}bz^{-3}+\tfrac{1}{32}(75b^2-68c)z^{-2}+\cdots]\\&-[\tfrac{1}{8}(7b^2-12c)+\cdots]\ln(2z/\bar{z})\},\end{aligned} \tag{3.26c}$$

$$S_3\sim k^{-1}[-\tfrac{3}{16}z^{-4}+\tfrac{1}{8}bz^{-3}-\tfrac{3}{64}(3b^2-4c)z^{-2}+\cdots]. \tag{3.26d}$$

In order to compare most simply the WKB functions in the overlap region $|z|\gtrsim z_0$ with the asymptotic expansions Eqs. (3.23), in the region $|t|\to\infty$, it is useful to reexpress $\bar{z}$ in terms of $\nu+\frac{1}{2}$ by inverting Eq. (3.10b):

$$z_0=(2\epsilon)^{1/2}e^{i\pi/4}k^{-1/4}[(\nu+\tfrac{1}{2})+\epsilon\Lambda+\epsilon^2\Omega]^{1/2} \tag{3.27}$$

and combining with Eq. (3.24). In the asymptotic expansions Eqs. (3.23), we substitute Eqs. (3.10a), (3.16), and (3.21).

For simplicity we illustrate the explicit matching using only the first-order solutions. The extension to higher orders is straightforward but tedious, and we will argue in an appendix that it is not entirely necessary.

To lowest order near $z=\pm z_0$, we can approximate $\bar{z}=z_0$ and $Q(z)=k(z^2-z_0{}^2)$; then, in the overlap region,

$$S_0/\epsilon\sim \pm ik^{1/2}z^2/2\epsilon\pm\tfrac{1}{2}(\nu+\tfrac{1}{2})\pm(\nu+\tfrac{1}{2})\ln(2z/z_0),\quad S_1\sim-\tfrac{1}{4}\ln(kz^2),\quad z\gtrsim z_0\,, \tag{3.28}$$

$$S_0/\epsilon\sim \mp ik^{1/2}z^2/2\epsilon\mp\tfrac{1}{2}(\nu+\tfrac{1}{2})\mp(\nu+\tfrac{1}{2})\ln(-2z/z_0),\quad S_1\sim-\tfrac{1}{4}\ln(kz^2),\quad z\lesssim -z_0\,. \tag{3.29}$$

Substituting into Eqs. (3.2)—(3.5) gives

$$\psi\sim[Z^{\rm I}_{\rm in}e^{-ik^{1/2}z^2/2\epsilon}(z/\epsilon^{1/2})^{-(\nu+1)}(4k)^{-(\nu+1)/4}e^{i\pi(\nu+1)/4}R$$
$$+Z^{\rm I}_{\rm out}e^{ik^{1/2}z^2/2\epsilon}(z/\epsilon^{1/2})^{\nu}(4k)^{\nu/4}e^{-i\pi\nu/4}R^{-1}](4/\epsilon^2ke^{i\pi})^{1/8},\quad z\gtrsim z_0\,, \tag{3.30a}$$

$$\psi\sim[Z^{\rm III}_{\rm in}e^{-ik^{1/2}z^2/2\epsilon}(-z/\epsilon^{1/2})^{-(\nu+1)}(4k)^{-(\nu+1)/4}e^{i\pi(\nu+1)/4}R$$
$$+Z^{\rm III}_{\rm out}e^{ik^{1/2}z^2/2\epsilon}(-z/\epsilon^{1/2})^{\nu}(4k)^{\nu/4}e^{-i\pi\nu/4}R^{-1}](4/\epsilon^2ke^{i\pi})^{1/8},\quad z\lesssim -z_0\,, \tag{3.30b}$$

where

$$R=(\nu+\tfrac{1}{2})^{(\nu+1/2)/2}e^{-(\nu+1/2)/2}\,. \tag{3.31}$$

By substituting for $g(t)$ to lowest order in Eqs. (3.23) and keeping only the leading terms, we obtain

$$\psi\sim Ae^{ik^{1/2}z^2/2\epsilon}(z/\epsilon^{1/2})^{\nu}(4k)^{\nu/4}e^{-i\pi\nu/4}$$
$$+Be^{-ik^{1/2}z^2/2\epsilon}(z/\epsilon^{1/2})^{-(\nu+1)}(4k)^{-(\nu+1)/4}e^{-i\pi(\nu+1)/4},\quad z\gtrsim z_0\,, \tag{3.32a}$$

$$\psi\sim\left[A+B\frac{(2\pi)^{1/2}e^{-i\pi\nu/2}}{\Gamma(\nu+1)}\right]e^{ik^{1/2}z^2/2\epsilon}(-z/\epsilon^{1/2})^{\nu}(4k)^{\nu/4}e^{3i\pi\nu/4}$$
$$+\left[Be^{3i\pi(\nu+1)/2}-A\frac{(2\pi)^{1/2}e^{i\pi\nu}}{\Gamma(-\nu)}\right]e^{-ik^{1/2}z^2/2\epsilon}(-z/\epsilon^{1/2})^{-(\nu+1)}(4k)^{-(\nu+1)/4}e^{-3i\pi(\nu+1)/4},\quad z\lesssim -z_0\,. \tag{3.32b}$$

Equating the coefficients of the corresponding functions, and eliminating the coefficients A and B, we obtain the equations

$$\begin{bmatrix} Z^{\rm III}_{\rm out}\\ Z^{\rm III}_{\rm in}\end{bmatrix}=\begin{bmatrix} e^{i\pi\nu} & iR^2e^{i\pi\nu}(2\pi)^{1/2}/\Gamma(\nu+1)\\ R^{-2}(2\pi)^{1/2}/\Gamma(-\nu) & -e^{i\pi\nu}\end{bmatrix}\begin{bmatrix} Z^{\rm I}_{\rm out}\\ Z^{\rm I}_{\rm in}\end{bmatrix}. \tag{3.33}$$

In the third-order match, the result is identical, except that R now has the form

$$R=(\nu+\tfrac{1}{2})^{(\nu+1/2)/2}\exp[-\tfrac{1}{2}(\nu+\tfrac{1}{2})-\tfrac{1}{48}(\nu+\tfrac{1}{2})^{-1}]\,. \tag{3.34}$$

Equation (3.33) connects the amplitudes of the WKB solutions on either side of the barrier, correctly to third order in the expansion. The result is applicable to quantum-mechanical tunneling near the peak of a potential barrier, as well as to black-hole oscillations. Notice from Eqs. (3.10) and (3.20), that if $Q(x)$ is real, as in quantum mechanics with real energy and potential, then $\nu+\frac{1}{2}$ is imaginary. As a consequence,

$$(e^{i\pi\nu})^*=-e^{i\pi\nu},\quad R^*=e^{-i\pi(\nu+1/2)/2}R^{-1}\,, \tag{3.35}$$

and the connection coefficients M_{ij} [Eq. (3.6)] satisfy the constraints

$$M^*_{11}=M_{22},\quad M^*_{12}=M_{21},\quad |M_{21}|^2-|M_{11}|^2=1\,, \tag{3.36}$$

which express the reality of the potential and unitarity of the S matrix (Ref. 13, p. 168). For potential-barrier tunneling from region I to region III, $Z^{\rm III}_{\rm in}=0$, $Z^{\rm I}_{\rm in}/Z^{\rm I}_{\rm out}=-M_{21}/M_{22}$, and therefore $Z^{\rm III}_{\rm out}/Z^{\rm I}_{\rm in}=(M_{21})^{-1}$. As a consequence of Eq. (3.35), the transmission and reflection coefficients $T\equiv|Z^{\rm III}_{\rm out}|^2/|Z^I_{\rm in}|^2$ and $R\equiv 1-T$ are then given by

$$T=(1+e^{2i\pi(\nu+1/2)})^{-1}\,, \tag{3.37a}$$

$$R=(1+e^{-2i\pi(\nu+1/2)})^{-1}\,. \tag{3.37b}$$

This result may be of formal and practical interest in the general question of WKB potential-barrier tunneling.[19]

IV. THE EQUATION FOR THE NORMAL-MODE FREQUENCIES

The connection formula, Eq. (3.33), can now be used to determine a condition leading to a formula for the frequencies of black-hole normal modes, provided we assume that the formula can be analytically continued into the complex frequency plane. For black holes, the amplitude $Z^{\rm III}_{\rm in}=0$. For a normal mode, $Z^{\rm I}_{\rm in}=0$. From Eq. (3.33) the only nontrivial way to satisfy these conditions is if $\Gamma(-\nu)=\infty$. This condition implies that ν must be a non-negative integer. This conclusion applies to frequencies with a positive real part. For frequencies with a negative real part, the identifications "in" and "out" in Eq. (3.33) must be interchanged. The boundary condition of no incoming waves now corresponds in Eq. (3.33) to $Z^{\rm III}_{\rm out}=Z^{\rm I}_{\rm out}=0$, which can only be satisfied if $\Gamma(\nu+1)=\infty$, which implies that $\nu=-1,-2,-3,\ldots$. Together these lead to the simple condition for a normal mode from Eq. (3.10b):

$$n+\tfrac{1}{2}=-ik^{1/2}z_0{}^2/2\epsilon-\epsilon\Lambda-\epsilon^2\Omega\ , \tag{4.1}$$

where

$$n=\begin{cases}0,1,2,\ldots, & {\rm Re}\,\omega>0\ ,\\ -1,-2,\ldots, & {\rm Re}\,\omega<0\ .\end{cases} \tag{4.2}$$

Alternatively, this condition can be obtained by demanding that the transmission coefficient [Eq. (3.37a)] be infinite.

By setting $\epsilon=1$ and substituting Eqs. (3.9), (3.10), and (3.20) into Eq. (4.1), we obtain Eqs. (1.4) and (1.5). Since Q depends in general on the frequency, these equations will lead to a discrete set of complex values for ω, for $n=0,\pm1,\pm2,\ldots$. If, for example, Q is of the Schrödinger form $Q=\omega^2-V$, where V is independent of ω, then the frequencies can be determined directly from Eqs. (1.6). This will be the case, for example, in the Schwarzschild normal-mode problem. If Q is a more complicated function of frequency, Eqs. (1.4) and (1.5) can be solved numerically for ω, once the $Q_0^{(n)}$ are known.

V. CONCLUDING REMARKS

We have used the WKB approximation to obtain a formula that will determine the normal-mode frequencies of black holes. The resulting expansion procedure can in principle be carried to higher orders to achieve more accuracy. In the second paper in this series,[20] we apply these results to the normal modes of the Schwarzschild black hole, and find that the agreement with other methods is excellent (fractions of a percent) for all the low-lying modes (small values of n). Future papers in this series will apply them to Reissner-Nordström and to Kerr black holes.

At a given order in ϵ, the second-order and third-order correction terms represented by Λ and Ω [Eqs. (1.5)] increase with increasing n, and we thus expect the accuracy to decrease with increasing n. This is because the value of $-Q_0$ changes as a consequence of the changing resonant frequency, and a polynomial approximation to the interior potential that is cut off at a given power of z may no longer be adequate. We hope to be able to adapt the WKB approximation to the case of large n as well.

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APPENDIX: EXTENSION TO HIGHER ORDERS

The fact that the connection coefficients M_{ij} depend only on ν suggests that they may be determined to higher order simply by solving the interior problem (region II) to higher order in ϵ, without performing an explicit match of the solutions to WKB solutions in regions I and III to the same order. By finding a value of ν such that the interior solution has the form of Eq. (3.22), we guarantee in some sense that the matching to WKB solutions will be straightforward. Although the coefficient R in Eq. (3.34) will change as a result of a higher-order match, its property given in Eq. (3.35) will not; hence the transmission coefficient will still be given by Eq. (3.37a).

With this in mind we used the symbolic algebra computer program MACSYMA[21] to extend the procedure of Sec. III B to one more order in ϵ, corresponding to fourth WKB order.

In place of Eq. (3.11) we write

$$\ddot{\psi}+[\nu+\tfrac{1}{2}-\tfrac{1}{4}t^2-\epsilon^{1/2}\bar{b}t^3+\epsilon(\Lambda-\bar{c}t^4)-\epsilon^{3/2}\bar{d}t^5+\epsilon^2(\Omega-\bar{f}t^6)-\epsilon^{5/2}\bar{g}t^7+\epsilon^3(\Phi-\bar{h}t^8)]\psi=0\ , \tag{A1}$$

where

$$\nu+\tfrac{1}{2}=-ik^{1/2}z_0{}^2/2\epsilon-\epsilon\Lambda-\epsilon^2\Omega-\epsilon^3\Phi\ , \tag{A2a}$$

$$\bar{g}=\tfrac{1}{4}g(4k)^{-5/4}e^{5i\pi/4},\quad g=\tfrac{1}{2520}Q_0^{(7)}/Q_0''\ , \tag{A2b}$$

$$\bar{h}=\tfrac{1}{4}h(4k)^{-3/2}e^{3i\pi/2},\quad h=\tfrac{1}{20\,160}Q_0^{(8)}/Q_0''\ . \tag{A2c}$$

In Eq. (3.16), we add the polynomials A_5 and A_6, as defined by Eq. (3.19), and generate differential equations for them, analogous to Eqs. (3.18). In the equation for A_5, we find a solution for the coefficients by setting the coefficient of each power of t equal to zero. But as before, the equation for A_6 is overdetermined, leading to a constraint on the value of the constant Φ. Here we quote only the final result for Φ:

$$\Phi = -\tfrac{1}{8}(202\,958\bar{b}^{\,6} - 263\,634\bar{b}^{\,4}\bar{c} + 80\,522\bar{b}^{\,2}\bar{c}^{\,2} + 59\,108\bar{b}^{\,3}\bar{d} - 22\,668\bar{b}\bar{c}\bar{d}$$
$$- 12\,110\bar{b}^{\,2}\bar{f} + 2310\bar{b}\bar{g} - 3078\bar{c}^{\,3} + 1890\bar{c}\bar{f} + 1107\bar{d}^{\,2} - 315\bar{h})$$
$$-(\nu+\tfrac{1}{2})^2(418\,110\bar{b}^{\,6} - 479\,970\bar{b}^{\,4}\bar{c} + 124\,026\bar{b}^{\,2}\bar{c}^{\,2} + 95\,460\bar{b}^{\,3}\bar{d} - 29\,340\bar{b}\bar{c}\bar{d}$$
$$- 17\,070\bar{b}^{\,2}\bar{f} + 2730\bar{b}\bar{g} - 3414\bar{c}^{\,3} + 1770\bar{c}\bar{f} + 1085\bar{d}^{\,2} - 245\bar{h})$$
$$-(\nu+\tfrac{1}{2})^4(463\,020\bar{b}^{\,6} - 465\,300\bar{b}^{\,4}\bar{c} + 99\,780\bar{b}^{\,2}\bar{c}^{\,2} + 78\,120\bar{b}^{\,3}\bar{d} - 19\,320\bar{b}\bar{c}\bar{d}$$
$$- 10\,860\bar{b}^{\,2}\bar{f} + 1260\bar{b}\bar{g} - 1500\bar{c}^{\,3} + 660\bar{c}\bar{f} + 630\bar{d}^{\,2} - 70\bar{h})\ . \tag{A3}$$

This result may be useful in higher-order WKB analyses of tunneling or of normal modes.[19]

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