Induced toroid structures and toroid polarizabilities

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The frequency-dependent toroid dipole polarizability $\gamma(\omega)$ of a (nonrelativistic, spinless) hydrogenlike atom in its ground state is calculated analytically in terms of two Gauss hypergeometric functions. The static result reads $\gamma(\omega=0)=(23/60)\alpha^2Z^{-4}a_0{}^5$($\alpha$=fine-structure constant, Z = nucleus charge number, a_0=Bohr radius). Comparing the present evaluations for atoms with previous ones for pions, one sees that the role of the induced toroid moments (as against that of the usual electric ones) increases considerably towards smaller distances (or higher characteristic excitation energies). It might become dramatic at the subhadronic level.

There is now interest in toroid moments[1,2] which are being studied in a variety of domains.[1-4] In Ref. 5 a new type of polarizability, hereafter called "toroid" (different from the usual electric and magnetic ones) has been introduced (and some effects related to it considered) in connection with the toroid multipole moments induced in a system by an external magnetic field $\mathbf{H}^{\text{ext}}$ of nonvanishing $\nabla\times\mathbf{H}^{\text{ext}}$ (or, alternatively, by an external conduction or displacement current). While the *intrinsic* toroid moments of elementary quantum systems are ruled out either by parity conservation or by invariance under time reversal, the corresponding *induced* ones (as emphasized in Ref. 5) are not forbidden by these discrete symmetries and the toroid polarizability is just measuring their size. In the multipole decomposition of the Hamiltonian (time dependent, in general) describing the system's interaction with the external electromagnetic fields, alongside with the usual electric and magnetic dipole (and higher multipole) pieces, there are also contributions expressing the interaction of the toroid moments with the external fields;[1,2] the latter begin with the toroid dipole term[1,2]

$$H_{(\text{toroid dipole})}(t)=-\mathbf{T}(t)\cdot(\nabla\times\mathbf{H}^{\text{ext}})_{\mathbf{x}=0,t}=-\mathbf{T}(t)\cdot\left[\frac{4\pi}{c}\mathbf{J}^{\text{ext}}+\frac{1}{c}\frac{d\mathbf{D}^{\text{ext}}}{dt}\right]_{\mathbf{x}=0,t}, \quad (1)$$

where $\mathbf{J}^{\text{ext}}$ and $(4\pi)^{-1}d\mathbf{D}^{\text{ext}}/dt$ are the external conduction and displacement currents, while[2]

$$\mathbf{T}(t)=\frac{1}{10c}\int\{\mathbf{x}[\mathbf{x}\cdot\mathbf{j}(\mathbf{x},t)]-2\mathbf{x}^2\mathbf{j}(\mathbf{x},t)\}d^3x \quad (2)$$

is the toroid dipole moment [$\mathbf{j}(\mathbf{x},t)$ denotes the system's current density]. According to well-known nonstationary perturbation rules, the response of a quantum system to the particular interaction of Eq. (1) is described by the following dynamic [i.e., frequency- (ω-) dependent] toroid dipole polarizability:[5]

$$\gamma_{ij}(\omega)=i\int e^{i\omega t}\theta(t)\langle 0\,|\,[T_i(t),T_j(0)]\,|\,0\rangle dt=\sum_n\left[\frac{\langle 0\,|\,T_i\,|\,n\rangle\langle n\,|\,T_j\,|\,0\rangle}{E_n-E_0-\omega-i\epsilon}+\frac{\langle 0\,|\,T_j\,|\,n\rangle\langle n\,|\,T_i\,|\,0\rangle}{E_n-E_0+\omega+i\epsilon}\right]. \quad (3)$$

The ground-state ($|0\rangle$) contribution, as usual, is to be taken from Eqs. (3); E_0,E_n denote the energies of the ground and excited states (in the discrete and continuous spectrum). The toroid dipole moment induced in the system (irrespective of whether or not it does possess a nonvanishing intrinsic one) has the Fourier components[6]

$$T_i^{\text{induced}}(\omega)=\sum_j\gamma_{ij}(\omega)[\nabla\times\mathbf{H}^{\text{ext}}(\omega)]_j\ . \quad (4)$$

As shown in Ref. 5, unlike the (static) electric and magnetic dipole polarizabilities $\alpha_{l=1}(\omega=0)$, $\beta_{l=1}(\omega=0)$ [the subscript l indicates the multipole (2^l-pole) order] which establish the angular structure of the amplitude for elastic scattering of low-energy photons on the considered system (Compton scattering) in the *second* (photon) energy order, the (static) toroid dipole polarizability $\gamma_{l=1}$ ($\omega=0$) enters only beginning with the *fourth* energy order, together with the usual (static) electric and magnetic quadrupole polarizabilities $\alpha_{l=2}(\omega=0)$, $\beta_{l=2}(\omega=0)$, and some derivatives of the usual (dynamic) dipole polarizabilities, such as $\alpha'_{l=1}(\omega=0)\equiv[d\alpha_{l=1}(\omega)/d\omega^2]\,|_{\omega=0}$ (Ref. 7).

The purpose of this note is to look into the relative importance of the induced toroid moments [measured by $\gamma(\omega)$] as against the (induced) usual electric and magnetic ones [measured by the usual multipole polarizabilities $\alpha_l(\omega),\beta_l(\omega)$], first for atoms and then for hadrons, in order to try getting some guesses on what might happen at even smaller distances (or larger characteristic excitation energies), at the subhadronic level. As an example for the atomic-physics case we shall consider a (nonrelativistic, spinless) hydrogenlike atom in its ground state [we are then able to compute $\gamma_{l=1}^{(H)}(\omega)$ exactly and use for comparison the available (also exact) results for $\alpha_l^{(H)}(\omega)$ derived in Ref. 8]. As a typical example for hadrons we shall consider the (charged) pion and use some numerical estimates given in Ref. 5 for $\gamma_{l=1}^{(\pi)}$ (in conjunction with previous ones for $\alpha_l^{(\pi)}$ found in Ref. 9) to perform an analogous comparison. Looking then at what the situation is in each of the two cases, at the length scale 10^{-8} cm on one side, and 10^{-13} cm on the other, we shall put forward a speculative idea about what might happen, say, at yet 5 orders of magnitude lower, down to 10^{-18} cm, i.e., at such distances which are expected to be explored, for instance, by the HERA electron-proton collider at DESY.

To calculate the toroid dipole polarizability of (ground-state) H-like atoms, one starts with the definition Eq. (3) in which the one-particle operator for the toroid dipole moment, by Eq. (2), is[2]

$$T_i=\frac{e}{10mc}\sum_k(-2\mathbf{r}^2\delta_{ik}P_k+x_ix_kP_k)\ , \tag{5}$$

where $P_k=-i\hbar\partial/\partial x_k$ (e and m are the charge and mass of the electron). The (exact) calculation is nonstandard, long and tedious, and will be presented in detail elsewhere.[10] It is essentially based on the use of the integral representation for the nonrelativistic Coulomb Green's function[11] in the form obtained by Schwinger[11] and the fact that a certain "basic" momentum-space integral (which is at the root of many exact calculations in studies concerning the interaction of the nonrelativistic H-like atoms with the radiation) can be taken exactly.[12] Next we shall give only the result. Because of the spherical symmetry of the ground s state, one has $\gamma_{ij}^{(H)}(\omega)=\delta_{ij}\gamma_{l=1}^{(H)}(\omega)$. We have obtained the formula

$$\gamma_{l=1}^{(H)}(\omega)=\frac{\alpha^2}{20}\frac{a_0^5}{Z^4}\sum_{i=1,2}\frac{\tau_i^2}{(\tau_i+1)^4}\frac{1}{2-\tau_i}\frac{1}{3-\tau_i}\left[\frac{8\tau_i^2(\tau_i^2+1)^2}{(\tau_i+1)^2(4-\tau_i)}F(1,-1-\tau_i,5-\tau_i;\zeta_i)-15\tau_i^4+7\tau_i^3+53\tau_i^2+57\tau_i+18\right]\ , \tag{6}$$

where $F(a,b,c;z)$ is the usual Gauss hypergeometric function with the series expansion

$$F(a,b,c;z)=1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2}+\cdots$$

and

$$\begin{aligned}&\tau_1=(1-\omega_0)^{-1/2},\quad \tau_2=(1+\omega_0)^{-1/2}\ ,\\ &\zeta_i=\left[\frac{\tau_i-1}{\tau_i+1}\right]^2,\\ &\omega_0=\frac{\hbar\omega}{|E_0|}=\frac{2\hbar\omega}{\alpha^2Z^2mc^2}=\frac{2}{\alpha Z}\frac{a_0}{Z}\frac{\omega}{c}\ ,\\ &a_0=\frac{\hbar}{\alpha mc}=0.53\times10^{-8}\text{ cm},\quad \alpha=\frac{e^2}{\hbar c}=\frac{1}{137\ldots}\ .\end{aligned} \tag{7}$$

$\gamma_{l=1}^{(H)}(\omega)$ as given by Eq. (6) is an even analytic function of ω having (in the complex ω plane) the right singularities at the right place: simple poles at $\omega=(E_n-E_0)/\hbar$, $n=2,3,4,\ldots$ ($E_n=E_0/n^2$ represents the discrete spectrum of the H-like atom) and a branch cut along the real ω axis above ionization threshold $\omega>|E_0|$. For the static (i.e., $\omega=0$) toroid dipole polarizability of a (nonrelativistic, spinless) H-like atom in its ground state we find then from Eq. (6) the very simple result

$$\gamma_{l=1}^{(H)}(\omega=0)=\frac{23}{60}\alpha^2\frac{a_0^5}{Z^4}\simeq Z^{-4}\times0.86\times10^{-46}\text{ cm}^5\ . \tag{8}$$

This is the toroid analog of the well-known static electric dipole polarizability

$$\alpha_{l=1}^{(H)}(\omega=0)=\frac{9}{2}\frac{a_0^3}{Z^4} \tag{9}$$

found in 1926 by Epstein and by Waller[13].

Now we have to assess the relative importance of the toroid effects with respect to the usual electric ones. To that aim, $\gamma_{l=1}^{(H)}$ ($\omega=0$) is to be compared with $\alpha'^{(H)}_{l=1}$ ($\omega=0$) and $\alpha_{l=2}^{(H)}$ ($\omega=0$). Using Eq. (8) above and the results of Ref. 8 (rewritten in the conventions used in this paper), we get the (exact) formulas

$$\frac{\gamma_{l=1}^{(H)}(\omega=0)}{\alpha'_{l=1}(\omega=0)}=\frac{23(\alpha Z)^4}{1595},\quad \frac{\gamma_{l=1}^{(H)}(\omega=0)}{\alpha_{l=2}^{(H)}(\omega=0)}=\frac{23(\alpha Z)^2}{900}\ . \tag{10}$$

Thus one sees that for H-like atoms (and this holds also in atomic physics, in general) the effects of the induced toroid moments appear very small indeed with respect to those of the corresponding usual electric ones. Perhaps it is also for this reason that the induced toroid moments have been so far disregarded in atomic physics. In a larger perspective, what is important to us is that they are there, however small. Moreover, as seen from Eqs. (10), the toroid effects are increasing with Z and this allows for possible applications even in atomic-physics problems (for instance, in what concerns the neutral component of plasma), but this subject is outside the scope of this paper.

Now we shall turn to see what a similar comparative analysis will say when instead of an H-like atom one is considering a typical hadron, the (charged) pion, for example. In Ref. 5 an order of magnitude estimate of the static toroid dipole polarizability of $\pi^\pm$, $\gamma_{l=1}^{(\pi)}(\omega=0)$ has been obtained by evaluating the contribution of the A_1 (1270 MeV) resonance [in terms of the experimentally known radiative width $\Gamma(A_1\to\pi\gamma)\simeq0.6$ MeV; see also Ref. 9 and the literature cited therein], with the result $\gamma_{l=1}^{(\pi)}(\omega=0)\simeq1.2\times10^{-5}$ fm^5. Under the same approximations in Ref. 5 it has been found that $\alpha'^{(\pi)}_{l=1}(\omega=0)\simeq0.8\times10^{-5}$ fm.[5] From the results obtained in the first of Ref. 9 it is known that the static electric quadrupole polarizability of $\pi^\pm$ is expected to be of the order $\alpha_{l=2}^{(\pi)}(\omega=0)\sim10^{-5}$ fm^5. So the picture which then emerges for (charged) pions looks in sharp contrast to the corresponding one for H-like atoms:

$$\frac{\gamma_{l=1}^{(\pi)}(\omega=0)}{\alpha'^{(\pi)}_{l=1}(\omega=0)} \sim \frac{\gamma_{l=1}^{(\pi)}(\omega=0)}{\alpha_{l=2}^{(\pi)}(\omega=0)} \sim 1 \; . \tag{11}$$

Comparing Eqs. (10) and (11), one sees that unlike the case of atoms, for hadrons not only the toroid polarizabilities and effects related to them can no longer be neglected but, on the contrary, they are expected to be of the same order of magnitude as the usual electric (and magnetic[9]) ones (of one order of multipolarity higher, of course, because it is with them that the comparison has to be made).

Equations (10) and (11) are here to stay and they must be taken seriously. In a context in which there is nowadays such an intense activity in supersymmetric, string, superstring theories, we cannot refrain from putting forward the following speculative idea: Eqs. (10) and (11) seem to tell us that the more "elementary" the object is (or, otherwise, the higher are the characteristic excitation energies of the system), the better might it respond to an external current ($\nabla \times \mathbf{H}^{\text{ext}}$) rather than to the external fields $\mathbf{E}^{\text{ext}}$, $\mathbf{H}^{\text{ext}}$ themselves; toward yet smaller distances (at 10^{-18} cm, say), the role of the (induced) toroid moments might increase further and become as predominant over the usual (induced) electric and magnetic ones as the latter were dominating over the toroid moments in atomic physics. Such a "linear" extrapolation from things reliably known in atomic and hadronic physics down to the "next substructure" comes in line with at least two features of the topical theories mentioned above. First, stringlike objects are likely to provide good candidates for systems having large toroid polarizabilities but small electric and magnetic ones. Indeed, while all types of polarizabilities are more or less extensive quantities (i.e., more or less proportional with the volume of the body), for the toroid ones (if the material properties are properly chosen), one may expect comparatively large values in the case of (closed) filiform structures (strings) on account of large numbers of turns of winding. We recall (see Refs. 2 and 5) that for a classical toroidal current, the toroid dipole moment [calculated by means of Eq. (2)] is

$$\mathbf{T}_{\text{torus}} = \mathbf{n}\frac{NIV_T}{4\pi c} \tag{12}$$

[I = the intensity of the current, N = the number of turns of winding (N = even), V_T = the volume of the toroid, $\mathbf{n}$ = unit vector along the axis of the toroid]. When we deal with induced toroid moments, some kind of external current flowing through the system induces in it closed toroidal secondary currents; this is, in a sense, a usual transformational effect and the toroid polarizability measures its intensity. In a quantum-field-theoretical picture, while the usual electric and magnetic polarizabilities express the ability of the cloud of virtual particles to get deformed in electric and magnetic fields, the toroid polarizabilities represent analogously a measure of the cloud's deformations which are topologically nontrivial and nonstationary.

Second, if Majorana fermions (currently recurring in grand unified and supersymmetric theories as well as in connection with neutrinos) are really to play a role both as the next "ultimate" constituents of matter and as terribly massive gauge fields, it could not be only a mere coincidence that the only possible electromagnetic structure they might possess is just represented by toroid moments and distributions (all other usual electric and magnetic multipole moments and distributions are for them forbidden[4]). Of course, we are now speaking about *intrinsic* toroid moments, but bound systems of such Majorana fermions with nonvanishing, intrinsic toroidal electromagnetic structure would rather have large toroid polarizabilities in much the same way as an usual macroscopic piece of matter composed of polar molecules would have all chances to possess a large electric polarizability. Anyway, the fact that Majorana particles single them out in a clear-cut manner by choosing to possess only toroid electromagnetic structures, might have other consequences too, if supersymmetry has something to do with facts; the boson-fermion symmetry is very special indeed and sometimes leads to strange surprises (see, for instance, Ref. 14).

We end with the remark that a certain experimental information on the (static, dipole) toroid polarizability of a system may be obtained by determining the fourth-order frequency part of the low-energy Compton scattering amplitude; for direct extraction, one has to go, however, to charge scattering, or virtual Compton scattering, or other process in which a current is flowed through the system.[15] Details will be presented elsewhere.[16]

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circular currents (equal, but opposite to each other), or a coaxial collection of such pairs of currents (a toroidal current). These latter current configurations, topologically nontrivial, have no resultant magnetic moment but still represent a certain kind of "dipole" characteristic, a "toroid" dipole.

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