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Spinning fluids in general relativity. II. Self-consistent formulation

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Methods used earlier to derive the equations of motion for a spinning fluid in the Einstein-Cartan theory are specialized to the case of general relativity. The main idea is to include the spin as a thermodynamic variable in the theory.

In an earlier paper we formulated a Lagrangian-based theory of a spinning fluid in general relativity (GR). (Let us refer to this work as I.[1]) The basis of this work is a Lagrangian which contains the usual scalar curvature density plus a generalization of a special relativistic spinning-fluid part due to Halbwachs[2] and Unal and Vigier[3] and a perfect-fluid energy-density term patterned after the method used in general relativity by Ray.[4] In addition, the spin density is introduced via a set of tetrads which are introduced through the method of constraints in the Lagrangian. (The interested reader is referred to Ref. 1 for details.)

Bedran and Vasconcellos-Vaidya have used the energy-momentum tensor found in I to obtain a Raychaudhuri equation in GR with spin density.[5] Although this work stands on its own merits, we do not feel that it is the most general treatment for a spinning fluid in GR.

In a later set of papers, we extended I to Riemann-Cartan (RC) spacetime, generalizing the Einstein-Cartan theory to spinning fluids;[6,7] however, in this work we self-consistently treated the spin density by including it in the thermodynamics of the fluid. The extension of including the spin as a thermodynamic degree of freedom can also be carried out in general relativity. This was mentioned but not elaborated on in Refs. 6 and 7. The purpose of this paper is to present some of the details of the self-consistent theory for the case when the gravitational theory under study is general relativity.

In order to obtain the self-consistent theory for general relativity, the fluid's specific internal energy ϵ becomes an explicit function of the spin tensor s_{ij}:

$$\epsilon=\epsilon(\rho,s,s_{ij}) \ , \tag{1}$$

where ρ is the mass density and s is the specific entropy. The thermodynamic law for the fluid now is assumed to have the form

$$d\epsilon=Tds+Pd(1/\rho)+\tfrac{1}{2}\omega_{ij}ds^{ij} \ , \tag{2}$$

where T is the temperature, P is the pressure, and ω_{ij} is the angular velocity tensor associated with the tetrad used to describe the spin. With these changes to the Lagrangian density of I, the variations with respect to the same variables as in I yield the equations of motion for the self-consistent theory of a spinning fluid in general relativity.

Since the results parallel those in Refs. 1 and 6, we only quote a few of them. For the source term in the Einstein equations we obtain the energy-momentum tensor

$$T^{ik}=T_F^{ik}+T_s^{ik} \ , \tag{3}$$

where the fluid part has the form

$$T_F^{ik}=\rho(1+\epsilon+P/\rho)u^iu^k+Pg^{ik} \ , \tag{4}$$

and the spin part is

$$T_s^{ik}=\rho u^{(i}s^{k)l}\dot{u}_l+\nabla_j(\rho u^{(k}s^{i)l}) -\rho\omega^{l(i}s^{k)}{}_l+\rho u^{(i}s^{k)l}\omega_{lj}u^j \ . \tag{5}$$

The last two terms in Eq. (5) come from the inclusion of the spin as a thermodynamic variable in Eqs. (1) and (2). To pass back to the case of I, we need only omit these two terms in Eq. (5). The form of Eq. (5) is nearly identical to Eq. (3.10) in Ref. 6 as was pointed out there. It is, however, important to note that Eq. (5), which represents the self-consistent GR theory, is not obtained from the self-consistent EC theory by setting the torsion equal to zero. In the EC theory, there is, due to the torsion field equation, a one-to-one correspondence between the proper torsion and the spin.[8] Thus, setting the torsion equal to zero also means setting the spin, as well, to zero. Thus the proper limit of the self-consistent EC theory when torsion vanishes is GR itself.

Thus, for studies of a self-consistent fluid in general relativity we should use the energy-momentum tensor defined by Eqs. (3)–(5). There are two immediate differences between I and the self-consistent theory: Because of the identity $\omega_{jk}u^k=\dot{u}_j$ the last term in Eq. (5) can be combined with the first so that the self-consistent version is larger by a factor of 2, and a new term [the third term in Eq. (5)] occurs which represents the interaction between spin density and the spin angular velocity. This latter term will lead to additional effects in the Raychaudhuri equation for spinning fluids in GR. In addition, we will be able to directly compare the self-consistent theories for spinning fluids in both GR and RC spacetime.

We mention that this self-consistent theory implies a consistency equation among the field variables given in the present case by[9,10]

$$u_i \nabla_j (\rho \omega^i_p s^{pj}) = 0 \ . \tag{6}$$

This consistency condition arises due to the inclusion of spin as a thermodynamic variable. The meaning of this consistency equation in the theory is not clear. In special solutions to date Eq. (6) has been found to be satisfied identically.

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