Non-Abelian bosonization: Current correlation functions

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We demonstrate that, in $1+1$ dimensions, the effective action produced by massless fermions moving in external gauge potentials is identical with that produced by a suitably defined σ model which includes a Wess-Zumino term. Hence the current correlation functions of the fermion and boson theories are identical.

I. INTRODUCTION

The equivalence of a single Dirac spinor field and a real scalar field in $(1+1)$-dimensional space-time has been known for some time.[1] More recently, Witten discussed[2] a generalization of this Fermi-Bose equivalence to fields which form a representation of a non-Abelian group. Specifically, he considered on the one hand a system of N free Majorana fermions, and on the other, a certain $O(N)$ σ model with a Wess-Zumino term. Witten demonstrated that the chiral $O(N)\times O(N)$ currents of both models obey the same level-one Kac-Moody algebra, which implies that the two models have the same spectrum.

One might attempt to go further and ask whether or not this equivalence extends to more general groups of multiplets, and whether the correspondence can be sharpened by showing the identity of the effective actions—the generating functionals for the perturbations caused by externally introduced gauge potentials. These questions have been addressed by several authors.[3–8] Nonetheless, a number of issues remain unclear; in particular, it has been claimed[5,6] that the two theories do not have the same current correlation functions.

In this paper we provide a careful discussion of the generating functionals of the two models for a general gauge group. We emphasize that the two functionals are indeed equal, provided that the models are constructed in a manner that preserves (i) vector-current conservation and (ii) the duality of the vector and axial-vector currents, $\langle j_5^\mu \rangle = \epsilon^{\mu\nu}\langle j_\nu \rangle$. Alternatively, a left-right-symmetric scheme is also possible. From the equality of the Fermi and Bose generating functionals, it follows that the two models have the same current correlation functions. In this way we augment Witten's program of non-Abelian bosonization. We hasten to add that our analysis is based on many results that were previously derived by Polyakov and Wiegmann,[4,9] Gonzales and Redlich,[5] and by di Vecchia and collaborators.[3,6] However, we believe that our presentation clarifies much that has been done before and corrects some previous misconceptions.

It should be noted that in this paper we consider the Fermi-Bose equivalence of correlation functions of only Lie-algebra-valued currents. In particular we do not discuss correlation functions of energy-momentum tensors, preferring to treat this issue in a separate paper. It may happen that two models have the same current correlation functions, but different energy-momentum correlation functions. (For instance, this occurs for certain models with a Kac-Moody algebra with a level greater than one.) The sets of Fermi-Bose models which have the same current (Kac-Moody) algebra *and* the same Virasoro algebra have been classified by Goddard, Nahm, and Olive.[10]

II. FERMIONIC MODEL

We consider first the fermionic model defined by the action[11]

$$S[\psi,A_\mu^{(-)},A_\mu^{(+)}]=\int d^2x\frac{i}{2}\psi\alpha^\mu[\partial_\mu-A_\mu^{(-)}\tfrac{1}{2}(1-\gamma_5)-A_\mu^{(+)}\tfrac{1}{2}(1+\gamma_5)]\psi\ . \tag{2.1}$$

Here we take $\psi(x)$ to be a Hermitian, Majorana field so that $\alpha^0=1$ and $\alpha^1=\gamma^0\gamma^1=\gamma_5$ is a real, symmetrical 2×2 matrix (e.g., the Pauli matrix σ_3). The 2-spinor ψ also carries a group index that labels some real, orthogonal representation U of some compact, semisimple group G. The use of real fields is no restriction, rather it is a convenience that encompasses all situations since a complex field can be broken up into two real components. A unitary representation of the group can be accounted for by multiplying all real numbers by the replacement

$$1\to\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \tag{2.2a}$$

and replacing the imaginary unit by

$$i\to\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \tag{2.2b}$$

thereby doubling the number of components corresponding to writing the field out in real and imaginary parts.

The Fermi field can be decomposed into left and right components,

$$\psi_\pm = \tfrac{1}{2}(1 \mp \gamma_5)\psi , \tag{2.3}$$

in the sense that for free particles ψ_- describes a left- and ψ_+ a right-moving wave. In terms of these components, the action S is invariant under the local $G \times G$ transformation

$$\psi_\mp(x) \to U_\mp(x)\psi_\mp(x) , \tag{2.4}$$

provided that the non-Abelian, left, right, skew-Hermitian, vector potential matrices $A_\mu^{(\mp)}$ are also gauge transformed. For an infinitesimal operation

$$U_\mp(x) = 1 + a_\mp(x) , \tag{2.5}$$

this transformation is given by

$$\delta_\mp A_\mu^{(\mp)}(x) = \partial_\mu a_\mp(x) + [a_\mp(x), A_\mu^{(\mp)}(x)] . \tag{2.6}$$

Functional integration defines the effective action W_1, the generating functional of the connected current correlation functions:

$$\exp(iW_1[A_\mu^{(-)},A_\mu^{(+)}]) = \int [d\psi]\exp(iS[\psi,A_\mu^{(-)},A_\mu^{(+)}]) . \tag{2.7}$$

Here we have normalized the measure $[d\psi]$ so that the functional integral is unity in the absence of the external potentials.

The functional integral produces, of course, a functional, Fredholm determinant that must be regulated to obtain a well-defined result. As is well known, there is no regularization scheme that preserves full gauge invariance under the transformations of Eq. (2.6), but rather there must be an anomalous response. Left-right-symmetric regularization schemes yield

$$\delta_\mp W_1[A_\mu^{(-)},A_\mu^{(+)}] = \mp \frac{1}{8\pi} \int \omega_2^1(a_\mp, A^{(\mp)}) . \tag{2.8}$$

Here a differential form notation is used, as will often be done in the sequel: The vector potential is written as a one-form (matrix)

$$A = A_\mu dx^\mu , \tag{2.9}$$

in which the dx^μ are treated as completely anticommuting objects. Then, using the differential operator $d = dx^\nu \partial_\nu$, the anomaly (2.8) involves the two-form defined by[12]

$$\omega_2^1(a,A) = \mathrm{tr}\, a dA , \tag{2.10}$$

For reasons that will soon be clear, it is convenient to define a new effective action W_2 that is invariant under vector (left + right) gauge transformations. The new action W_2 yields a vector current which is covariantly conserved (with no vector anomaly). This is achieved by adding a finite local counterterm to W_1:

$$W_2[A_\mu^{(-)},A_\mu^{(+)}] = W_1[A_\mu^{(-)},A_\mu^{(+)}] + \frac{1}{8\pi}\int \mathrm{tr} A^{(-)}A^{(+)} . \tag{2.11}$$

Indeed, it is easy to check that, under the vector gauge transformation where $a_- = a_+ = v$,

$$\delta_v W_2[A_\mu^{(-)},A_\mu^{(+)}] = 0 , \tag{2.12a}$$

while, under an axial gauge transformation where $a_- = -a_+ = -a$,

$$\delta_a W_2[A_\mu^{(-)},A_\mu^{(+)}] = \frac{1}{4\pi}\int \mathrm{tr}\, a[dA^{(+)} + dA^{(-)} - A^{(+)}A^{(-)} - A^{(-)}A^{(+)}] . \tag{2.12b}$$

In terms of the vector and axial-vector potentials,

$$V = \tfrac{1}{2}(A^{(+)} + A^{(-)}) \tag{2.13a}$$

and

$$A = \tfrac{1}{2}(A^{(+)} - A^{(-)}) , \tag{2.13b}$$

the axial response (2.12b) reads

$$\delta_a W_2[V_\mu, A_\mu] = \frac{1}{2\pi}\int \mathrm{tr}\, a[dV - V^2 + A^2] . \tag{2.14}$$

The vector and axial-vector gauge variations (2.12a) and (2.12b) provide two functional-differential equations for the effective action W_2. If this action were a functional of only two external field components, it would be completely determined by these two equations. To examine the number of independent external fields that enter into the theory, we note that, in terms of vector and axial-vector potentials, the fermionic action (2.1) appears as

$$S[\psi,A_\mu^{(-)},A_\mu^{(+)}] = \int d^2x \frac{i}{2}\psi\alpha^\mu[\partial_\mu - V_\mu - A_\mu\gamma_5]\psi . \tag{2.15}$$

Now

$$\alpha^\mu\gamma_5 = \epsilon^{\mu\nu}\alpha_\nu , \tag{2.16}$$

and so the fermionic action is a functional only of

$$\mathscr{A}_\mu = V_\mu - \epsilon_{\mu\nu}A^\nu , \tag{2.17a}$$

or, in one-form notation,

$$\mathscr{A} = V - {}^*A . \tag{2.17b}$$

Thus, apparently, the effective action is a functional of only the two field components $\mathscr{A}^0(x), \mathscr{A}^1(x)$, and it is therefore completely determined by the two responses (2.12a) and (2.12b) to gauge variations. This, however, is contradicted by the anomaly given in Eq. (2.14), which is not simply a functional of $\mathscr{A}$ alone. The discrepancy arises because of the regularization needed to define the theory, with different schemes giving results that differ only by dimensionless, finite local counterterms. Therefore, if an additional counterterm can be found which, added to W_2, produces a third effective action W_3 whose gauge variation is only a functional of $\mathscr{A}$, then W_3 itself is only a functional of $\mathscr{A}$. And W_3 is thus completely determined by the two gauge variations (2.12a) and (2.12b).

To implement this program we first note that in form notation, the gauge variation (2.6) transcribes into the vector and axial-vector transformations

$$\delta_v V = dv + [v, V] \,, \tag{2.18a}$$

$$\delta_v A = [v, A] \,, \tag{2.18b}$$

and

$$\delta_a V = [a, A] \,, \tag{2.18c}$$

$$\delta_a A = da + [a, V] \,. \tag{2.18d}$$

It is now a simple matter to verify that the desired new effective action is given by[13]

$$W_3[V_\mu, A_\mu] = W_2[V_\mu, A_\mu] - \frac{1}{4\pi}\int \mathrm{tr}^* AA \,. \tag{2.19}$$

[The integration of the two-form $^*AA$ is, of course, just the ordinary d^2x integration[11] of $A^\mu A_\mu$. This notation, together with the relations between one-forms $^*ab = {}^*ba = -a\,{}^*b$ and $^{**}a = a$ (which imply that $ab = -{}^*a\,{}^*b$), simplify the subsequent calculations.] The additional counterterm is invariant under vector gauge transformations and so the vector gauge invariance is maintained:

$$\delta_v W_3[V_\mu, A_\mu] = 0 \,. \tag{2.20a}$$

On the other hand, a little calculation shows that under an axial-vector gauge transformation

$$\delta_a W_3[V_\mu, A_\mu] = \frac{1}{2\pi}\int \mathrm{tr}\, a(d\mathscr{A} - \mathscr{A}^2) \,, \tag{2.20b}$$

which is indeed a functional of $\mathscr{A}$ alone. In fact, this result is identical to the axial-vector gauge variation of the previous W_2 form of the action, Eq. (2.14), if the axial-vector potential is taken to vanish, $A = 0$, and the vector potential V is replaced by $\mathscr{A}$. Hence we conclude that

$$W_3[V_\mu, A_\mu] = W_2[\mathscr{A}_\mu, 0] \,, \tag{2.21}$$

with $W_2[\mathscr{A}_\mu, 0]$ completely determined by the two gauge variation conditions.

We should emphasize that the sequence of finite local counterterms that we have introduced to arrive at this result is a consequence of having started with a regularization that was left-right symmetric. Starting instead with a regularization that preserves the conservation of the vector current would yield immediately $W_2[V_\mu, 0]$, and no further counterterms would be needed for the construction of Eq. (2.21). Moreover we should also emphasize that the anomaly shown in Eq. (2.20b) follows from this type of regularization: it can, in fact, be inferred from the Abelian case[14] by simply replacing the Abelian field strength dV by the non-Abelian generalization $dV - V^2$.

Let T_a be Hermitian (imaginary, antisymmetric) generators, which satisfy

$$[T_a, T_b] = i f_{abc} T_c \,, \tag{2.22}$$

and are normalized by

$$\mathrm{tr} T_a T_b = 2\delta_{ab} \,. \tag{2.23}$$

Moreover, let us write the gauge potential matrices as $V_\mu = i V_{\mu a} T_a$, $A_\mu = i A_{\mu a} T_a$, and $\mathscr{A}_\mu = i \mathscr{A}_{\mu a} T_a$. The result (2.21) implies a simple relation between the vector-current expectation value

$$\langle j_a^\mu(x) \rangle = \frac{\delta W_3[V_\mu, A_\mu]}{\delta V_{\mu a}(x)} \tag{2.24a}$$

and the axial-vector expectation value

$$\langle j_{5a}^\mu(x) \rangle = \frac{\delta W_3[V_\mu, A_\mu]}{\delta A_{\mu a}(x)} \;; \tag{2.24b}$$

namely,

$$\langle j_{5a}^\mu(x) \rangle = \epsilon^{\mu\nu} \langle j_{\nu a}(x) \rangle \,. \tag{2.25}$$

This is a key result which was engineered by our choice of the counterterm in Eq. (2.11). The freedom of redefining the effective action by adding finite local counterterms is completely fixed by the two conditions: (i) vector-current conservation Eq. (2.20a), and (ii) the duality of the vector and axial-vector currents Eq. (2.25). This fact has been known for some time.[15] The invariance under vector gauge transformations (2.20a) can be reexpressed as

$$\delta_v W_3[V_\mu, A_\mu] = \int d^2x \left[\frac{\delta W_3}{\delta V_{\mu a}(x)} \delta_v V_{\mu a}(x) + \frac{\delta W_3}{\delta A_{\mu a}(x)} \delta_v A_{\mu a}(x) \right] = 0 \,. \tag{2.26}$$

Using Eqs. (2.18) and (2.25), this implies that

$$\mathscr{D}_\mu \langle j^\mu \rangle = \partial_\mu \langle j^\mu \rangle - [\mathscr{A}_\mu, \langle j^\mu \rangle] = 0 \,, \tag{2.27a}$$

where $j^\mu \equiv -(i/2) j_a^\mu T_a$. Similarly, the axial-vector gauge response (2.20b) implies that

$$\epsilon^{\mu\nu} \mathscr{D}_\mu \langle j_\nu \rangle = \frac{1}{4\pi} \epsilon^{\mu\nu} \mathscr{F}_{\mu\nu} \,, \tag{2.27b}$$

where

$$\mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu - [\mathscr{A}_\mu, \mathscr{A}_\nu] \,. \tag{2.28}$$

As observed long ago for the Abelian case,[14] these two divergence equations completely determine the current and thus the effective action. The extension to the non-Abelian case has been recently performed by Polyakov and Wiegmann.[9]

We shall solve the two divergence equations, or, equivalently, the variational statements (2.20a) and (2.20b) coupled with the condition that the effective action involve only the vector potential combination $\mathscr{A}$, by showing that they are obeyed by a certain σ model. But before doing this we should note that in terms of the left and right chiral potentials [see Eqs. (2.13)]

$$\mathscr{A} = \tfrac{1}{2}(A^{(-)} + {}^*A^{(-)}) + \tfrac{1}{2}(A^{(+)} - {}^*A^{(+)}) \,. \tag{2.29}$$

Thus the left-hand potential $A^{(-)}$ appears only in a self-dual combination, the right-hand potential $A^{(+)}$ only in an anti-self-dual combination, with these potentials therefore coupling only to anti-self-dual and self-dual currents, respectively. In $(1+1)$-dimensional space-time, these quantities have only one component. This is made explicit by going to light-cone coordinates $x^{\mp}=(1/\sqrt{2})(x^0 \pm x^1)$. In these coordinates, only $A_-^{(-)}$ and $A_+^{(+)}$ enter into the theory, and they couple to j_+ and j_-, respectively.

III. BOSONIC MODEL

In this section we construct a σ model which corresponds to the fermionic theory. For convenience we choose the σ model to contain the same representation of the group $G \times G$. Hence it is described by a matrix field $\mathscr{G}(x)$ of the same dimensionality as that of the $U_{\mp}(x)$ matrices, with $\mathscr{G}(x)$ obeying the chiral transformation laws

$$\delta_- \mathscr{G}(x) = a_-(x)\mathscr{G}(x) \tag{3.1a}$$

and

$$\delta_+ \mathscr{G}(x) = -\mathscr{G}(x) a_+(x) \; . \tag{3.1b}$$

As has been discussed in detail in Ref. 16, the fermionic anomalies are reproduced by the Wess-Zumino action

$$\Gamma[\mathscr{G}, A_\mu^{(-)}, A_\mu^{(+)}] = -\frac{1}{24\pi}\int_N \mathrm{tr}(\mathscr{G}^{-1} d\mathscr{G})^3 - \frac{1}{8\pi}\int \mathrm{tr}(A^{(-)} d\mathscr{G}\,\mathscr{G}^{-1} - \mathscr{G}^{-1} d\mathscr{G}\, A^{(+)} + A^{(-)}\mathscr{G} A^{(+)} \mathscr{G}^{-1}) \; . \tag{3.2}$$

Here N is a three-dimensional manifold whose boundary is the two-dimensional space-time. A variation of the three-form in Eq. (3.2) produces an exact form (a "total derivative") that results in a two-form integral over the two-dimensional space-time. Using this fact, it is a straightforward matter to verify that the $\mathscr{G}(x)$ transformations (3.1a) and (3.1b) together with the vector potential transformations (2.6) do indeed produce

$$\delta_{\mp}\Gamma[\mathscr{G}, A_\mu^{(-)}, A_\mu^{(+)}] = \mp\frac{1}{8\pi}\int \omega_2^1(a_{\mp}, A^{(\mp)}) \; . \tag{3.3}$$

These are exactly the chiral anomalies (2.8) of the fermionic model.

The fermionic theory has no parameters that carry dimensions and so the bosonic theory can contain only dimensionless constants. Thus, the most general bosonic action consists of the Wess-Zumino action, plus all possible chiral-invariant, dimensionless terms, of which there is only one:

$$S_1[\mathscr{G}, A_\mu^{(-)}, A_\mu^{(+)}; \alpha] = \frac{\alpha}{16\pi}\int \mathrm{tr}(\mathscr{G}^{-1}\, {}^*D\mathscr{G})(\mathscr{G}^{-1} D\mathscr{G}) + \Gamma[\mathscr{G}, A_\mu^{(-)}, A_\mu^{(+)}] \; . \tag{3.4}$$

Here

$$D\mathscr{G} = d\mathscr{G} - A^{(-)}\mathscr{G} + \mathscr{G} A^{(+)} \tag{3.5}$$

is the gauge-covariant derivative one-form. We shall soon see that the dimensionless parameter α is fixed by the Fermi-Bose equivalence.

We now follow the arguments of the previous section and add to S_1 a finite local counterterm, to obtain a new action

$$S_2[\mathscr{G}, A_\mu^{(-)}, A_\mu^{(+)}; \alpha] = S_1[\mathscr{G}, A_\mu^{(-)}, A_\mu^{(+)}; \alpha] + \frac{1}{8\pi}\int \mathrm{tr} A^{(-)} A^{(+)} \; , \tag{3.6}$$

which is invariant under vector gauge transformations. Again we reexpress the action in terms of vector (V_μ) and axial-vector (A_μ) potentials, and add a further counterterm, to obtain

$$S_3[\mathscr{G}, V_\mu, A_\mu; \alpha] = S_2[\mathscr{G}, A_\mu^{(-)}, A_\mu^{(+)}; \alpha] - \frac{1}{4\pi}\int \mathrm{tr}\, {}^*AA \; . \tag{3.7}$$

By the analysis of the previous section we see that the new action is invariant under vector gauge transformations,

$$\delta_v S_3[\mathscr{G}, V_\mu, A_\mu; \alpha] = 0 \; , \tag{3.8a}$$

while under an axial-vector transformation

$$\delta_a S_3[\mathscr{G}, V_\mu, A_\mu; \alpha] = \frac{1}{2\pi}\int \mathrm{tr}\, a[d\mathscr{A} - \mathscr{A}^2] \; . \tag{3.8b}$$

That is, S_3 correctly reproduces the variations of the fermionic effective action W_3 under both the vector (2.18a) and (2.18b) and axial-vector (2.18c) and (2.18d) transformations for all values of the parameter α.

To determine this parameter, we use the simple relations between one-forms noted before to write out the final version of the action as

$$\begin{aligned} S_3[\mathscr{G}, V_\mu, A_\mu; \alpha] = &-\frac{1}{24\pi}\int_N \mathrm{tr}(\mathscr{G}^{-1} d\mathscr{G})^3 + \frac{\alpha}{16\pi}\int \mathrm{tr}(\mathscr{G}^{-1}\, {}^*d\mathscr{G})(\mathscr{G}^{-1} d\mathscr{G}) \\ &-\frac{1}{8\pi}\int \mathrm{tr}[(A^{(-)} + \alpha\, {}^*A^{(-)}) d\mathscr{G}\,\mathscr{G}^{-1} - \mathscr{G}^{-1} d\mathscr{G}(A^{(+)} - \alpha\, {}^*A^{(+)}) \\ &+\tfrac{1}{2}(A^{(-)} + \alpha\, {}^*A^{(-)})\mathscr{G}(A^{(+)} - \alpha\, {}^*A^{(+)})\mathscr{G}^{-1} \\ &-\tfrac{1}{2}(\alpha^2 - 1) A^{(-)}\mathscr{G} A^{(+)}\mathscr{G}^{-1} - \tfrac{1}{2}(A^{(-)} + {}^*A^{(-)})(A^{(+)} - {}^*A^{(+)}) \\ &-\tfrac{1}{2}(\alpha - 1)({}^*A^{(-)} A^{(-)} + {}^*A^{(+)} A^{(+)})] \; . \end{aligned} \tag{3.9}$$

We see that the value $\alpha=1$ is uniquely selected by the requirement that the left-hand potential occurs only in the combination $A^{(-)}+{}^*A^{(-)}$ and the right-hand potential only in $A^{(+)}-{}^*A^{(+)}$. As discussed at the end of the previous section this is equivalent to having the action depend only on the field $\mathscr{A}_\mu$ given by Eq. (2.29), and we infer that

$$S_3[\mathscr{G},V_\mu,A_\mu;1]=S_2[\mathscr{G},\mathscr{A},0;1] . \tag{3.10}$$

The quantum theory is described by the bosonic effective action defined by

$$\exp(iW_3^B[V_\mu,A_\mu])=\int [d\mathscr{G}]\exp(iS_3[\mathscr{G},V_\mu,A_\mu;1]) . \tag{3.11}$$

Again, the measure is normalized so that the functional integral is equal to unity in the absence of the external potentials. Since the Haar measure $[d\mathscr{G}]$ is invariant under the chiral transformations (3.1) of $\mathscr{G}$, we find that, in view of Eqs. (3.8), the bosonic effective action $W_3^B[V_\mu,A_\mu]$ satisfies the same gauge variation equations that are obeyed by the fermionic action $W_3[V_\mu,A_\mu]$. Moreover, the bosonic action is a functional only of the combination $\mathscr{A}$ of V and A, just as is the fermionic action. We conclude, therefore, that the two actions are identical

$$W_3^B[V_\mu,A_\mu]=W_3[V_\mu,A_\mu] . \tag{3.12}$$

This is the Bose-Fermi equivalence. Similar results were first obtained [for the case of fermions in the fundamental representation of O(N) or U(N)] in Refs. 3–6.

It appears that our proof holds for any representation of any chiral $G\times G$ group. However, we must address the question of a possible topological obstruction. The Wess-Zumino term Γ involves an integration of a three-form over a three-dimensional manifold whose boundary is the two-space. The result must be the same for any such extension. The difference of two different extensions gives an integration over a closed three-dimensional manifold—a manifold with no boundary. Therefore, the Wess-Zumino term is unambiguously defined only if the integration over the closed three manifold—a topological invariant—is an integer multiple of 2π, which does not alter the value of the functional integral. The coefficient of this topological term is, of course, uniquely determined by the fermionic anomaly. It is a remarkable fact that this determination by the anomaly does indeed provide a Wess-Zumino term which is well defined. The proof of this fact is provided in the Appendix.

We should also note that the equivalence of the Bose theory to a free-fermion theory requires that the highly nonlinear Bose theory itself be free. This occurs, of course, only in the presence of the Wess-Zumino term, and only for the special value $\alpha=1$ which relates the strength of the usual "kinetic energy" term to the topologically quantized strength of the Wess-Zumino term. This is the phenomenon of "geometrostasis" which has been discussed recently.[17]

It is worthwhile to present our results in an explicit form, particularly for the work of the next section. Some algebraic effort can be employed to reexpress the bosonic action as

$$\begin{aligned}S_3[\mathscr{G},V_\mu,A_\mu;1]=-\frac{1}{8\pi}\int d^2x[&-\tfrac{1}{2}\mathrm{tr}\mathscr{G}^{-1}\partial_\mu\mathscr{G}\mathscr{G}^{-1}\partial^\mu\mathscr{G}\\&+(\eta^{\mu\nu}-\epsilon^{\mu\nu})\mathrm{tr}(\mathscr{A}_\mu\partial_\nu\mathscr{G}\mathscr{G}^{-1}-\mathscr{A}_\nu\mathscr{G}^{-1}\partial_\mu\mathscr{G}+\mathscr{A}_\mu\mathscr{G}\mathscr{A}_\nu\mathscr{G}^{-1}-\mathscr{A}_\mu\mathscr{A}_\nu)]\\&-\frac{1}{24\pi}\int_N \mathrm{tr}(\mathscr{G}^{-1}d\mathscr{G})^3 .\end{aligned} \tag{3.13}$$

This is the action for a σ model with a Wess-Zumino term, with only the left + right transformations gauged. The quantum expectation values of the current operators in the presence of the external potentials are just the functional-integral average of their classical counterparts. The classical vector current is defined by

$$j_a^\mu=\frac{\delta S_3}{\delta V_{\mu a}}[\mathscr{G},V_\mu,A_\mu;1] , \tag{3.14}$$

which, according to Eq. (3.13), gives

$$\begin{aligned}j^\mu=-\frac{1}{8\pi}[&(\eta^{\mu\nu}-\epsilon^{\mu\nu})\mathscr{D}_\nu\mathscr{G}\mathscr{G}^{-1}\\&-(\eta^{\mu\nu}+\epsilon^{\mu\nu})\mathscr{G}^{-1}\mathscr{D}_\nu\mathscr{G}] ,\end{aligned} \tag{3.15}$$

where

$$\mathscr{D}_\mu=\partial_\mu-[\mathscr{A}_\mu,\ ] , \tag{3.16}$$

and $j^\mu=-(i/2)j_a^\mu T_a$. It can be shown that the classical equation of motion for $\mathscr{G}$ is

$$(\eta^{\mu\nu}-\epsilon^{\mu\nu})\mathscr{D}_\mu[(\mathscr{D}_\nu\mathscr{G})\mathscr{G}^{-1}]=\epsilon^{\mu\nu}\mathscr{F}_{\mu\nu} , \tag{3.17a}$$

or, equivalently,

$$(\eta^{\mu\nu}+\epsilon^{\mu\nu})\mathscr{D}_\mu(\mathscr{G}^{-1}\mathscr{D}_\nu\mathscr{G})=\epsilon^{\mu\nu}\mathscr{F}_{\mu\nu} . \tag{3.17b}$$

The classical equations of motion hold under the functional integral—they are just the statement that the functional integral of a total functional derivative vanishes. Using (3.15) and (3.17), we immediately obtain

$$\mathscr{D}_\mu j^\mu=0 \tag{3.18a}$$

and

$$\epsilon^{\mu\nu}\mathscr{D}_\mu j_\nu=\frac{1}{4\pi}\epsilon^{\mu\nu}\mathscr{F}_{\mu\nu} , \tag{3.18b}$$

which, we see, also hold for the quantum expectation values. This confirms our previous results, Eqs. (3.8).

To conclude this section, we note that in Ref. 9, a closed if formal expression was obtained for the effective action $W_3^B[V_\mu, A_\mu]$. For the sake of completeness, let us show how this solution is obtained in the context of our methods. Since only the self- or anti-self-dual components of the vector potentials (which have only one component) enter into the theory, one may write them as effectively pure gauge terms:

$$A^{(-)} + {}^*A^{(-)} = -L(d + {}^*d)L^{-1} \tag{3.19a}$$

and

$$A^{(+)} - {}^*A^{(+)} = -R(d - {}^*d)R^{-1} . \tag{3.19b}$$

If the bosonic model were perfectly gauge invariant, there would be no effect of the gauge potentials since they would be completely removed by the change of variables

$$\mathscr{G} \to \mathscr{G}' = L\mathscr{G}R^{-1} . \tag{3.20}$$

This, however, is not the case since the response to gauge transformations is anomalous. One can, with some labor, explicitly compute the change in the bosonic action $S_3[\mathscr{G}, V_\mu, A_\mu; 1]$ under the transformation (3.20) to find the anomalous terms. But there is no need to do this. Indeed, from Eqs. (3.19) we see that the gauge transformations (2.6) of the chiral potentials $A^{(-)}$ and $A^{(+)}$ correspond to

$$\delta L(x) = a_-(x)L(x) \tag{3.21a}$$

and

$$\delta R^{-1}(x) = -R^{-1}(x)a_+(x) . \tag{3.21b}$$

Hence all we need do is find a functional that involves only $A^{(-)} + {}^*A^{(-)}$ and $A^{(+)} - {}^*A^{(+)}$ which has the correct response to the gauge transformations described by Eqs. (3.21). Clearly, the solution is simply the bosonic action $S_3[\mathscr{G}, V_\mu, A_\mu; 1]$, with $\mathscr{G}$ replaced by LR^{-1}:

$$W_3[V_\mu, A_\mu] = S_3[LR^{-1}, V_\mu, A_\mu; 1] . \tag{3.22}$$

IV. CURRENT COMMUTATION RELATIONS AND CORRELATION FUNCTIONS

We have demonstrated that a suitably quantized theory of fermions and a σ model with a Wess-Zumino term have the same effective action, and therefore, identical current commutation relations and current correlation functions. Here we provide some further details of this equivalence.

As a first step we derive the commutation relations of the ($\mathscr{A}$-dependent) currents. We shall perform our calculations using the bosonic formulation of the model; later we shall argue that the fermionic formulation yields the same results. One approach to finding these current commutators would be Witten's noncanonical light-cone quantization scheme.[2] Here we follow instead a more conventional procedure: we reexpress the model in terms of the "pion" field $\phi_a(x)$, which is related to the field $\mathscr{G}(x)$ according to

$$\mathscr{G} = \exp(2i\sqrt{\pi}\,\phi_a T_a) , \tag{4.1}$$

and then we proceed with a canonical quantization. Although we shall use the language of quantum mechanics in our discussion, because of the nonlinearities (operator-ordering ambiguities) we shall be really working at the classical level, with commutators denoting Poisson brackets. In the classical theory we may go to the north pole: We may perform a local group transformation so that the state is described by $\phi_a = 0$. Thus to derive the current algebra, it suffices to expand about $\phi_a = 0$. Examining Eq. (3.13), we find that the canonical momentum π_a conjugate to ϕ_a is related to $\partial_0\phi = \dot{\phi}$ by

$$\pi_a = \frac{\delta S_3}{\delta \dot{\phi}_a}[\mathscr{G}, V_\mu, A_\mu; 1] = \dot{\phi}_a - \frac{2\sqrt{\pi}}{3} f_{abc}\phi_b\partial_1\phi_c + \frac{1}{\sqrt{\pi}}\mathscr{A}^1_a + f_{abc}\phi_b\mathscr{A}^0_c + \cdots . \tag{4.2}$$

To derive commutation relations, the currents must be expressed in terms of the canonically conjugate variables ϕ_a and π_a. In terms of these variables, we reexpress the vector current $j^\mu = -(i/2)j^\mu_a T_a$, given by (3.15), as

$$j^0_a = -\frac{1}{\sqrt{\pi}}\partial_1\phi_a - f_{abc}\pi_b\phi_c + \cdots , \tag{4.3a}$$

$$j^1_a = \frac{1}{\sqrt{\pi}}\pi_a - \frac{1}{\pi}\mathscr{A}_{1a} + \tfrac{1}{3} f_{abc}\partial_1\phi_b\phi_c + \cdots . \tag{4.3b}$$

We observe that j^0_a is independent of $\mathscr{A}^\mu$, and that j^1_a is independent of $\mathscr{A}^0$, which is expected from general considerations.[18] That is,

$$\frac{\delta j^\nu_b(y)}{\delta \mathscr{A}_{a\mu}(x)} = \rho^{\mu\nu}_{ab}\delta^{(2)}(x-y) , \tag{4.4}$$

where

$$\rho^{11}_{ab} = -\frac{1}{\pi}\delta_{ab} , \tag{4.5a}$$

while

$$\rho^{0\mu}_{ab} = 0 = \rho^{\mu 0}_{ab} . \tag{4.5b}$$

Using the canonical equal-time commutation relations

$$[\phi_a(x), \pi_b(y)] = i\delta(x^1 - y^1)\delta_{ab} , \tag{4.6}$$

it is now a straightforward exercise to compute the equal-time current commutators, which can be conveniently expressed in the compact form

$$[j^0_a(x), j^\mu_b(y)] = i\delta(x^1 - y^1) f_{abc} j^\mu_c(x) + i\partial_1\delta(x^1 - y^1)\rho^{1\mu}_{ab} , \tag{4.7}$$

while, of course, the duality of the currents gives

$$[j^1_a(x), j^1_b(y)] = [j^0_a(x), j^0_b(y)] . \tag{4.8}$$

We see explicitly that the "Schwinger term" in the commutation relation, the term involving the gradient of the δ function, is directly related to the $\mathscr{A}$ dependence of the currents, which follows from general considerations.[18]

To elucidate the significance of the Schwinger term and the $\mathscr{A}$ dependence of the currents, let us examine the two-point current correlation function defined by

$$K^{\mu\nu}_{ab}(x,y)=\frac{1}{i}\frac{\delta}{\delta\mathscr{A}_{a\mu}(x)}\frac{\delta}{\delta\mathscr{A}_{b\nu}(y)}W_3 =\frac{\delta}{\delta\mathscr{A}_{a\mu}(x)}\langle j^\nu_b(y)\rangle \ . \tag{4.9}$$

It measures the correlation of the two currents in the presence of an arbitrary external potential $\mathscr{A}$. First we note that

$$[D_\mu\langle j^\mu(x)\rangle]_a=\partial_\mu\langle j^\mu_a(x)\rangle +f_{abc}\mathscr{A}_{b\mu}(x)\langle j^\mu_c(x)\rangle=0 \ . \tag{4.10}$$

Taking the functional derivative of this equation with respect to $\mathscr{A}_{b\nu}(y)$, we see that

$$D_\mu K^{\mu\nu}_{ab}(x,y)=-f_{abc}\langle j^\nu_c(x)\rangle\delta^{(2)}(x-y) \ . \tag{4.11}$$

On the other hand, taking account of the $\mathscr{A}$ dependence of the current, we see from Eqs. (4.4) and (4.9) that the two-point function has the explicit construction

$$K^{\mu\nu}_{ab}(x,y)=i\langle Tj^\mu_a(x)j^\nu_b(y)\rangle_{\text{conn}}+\delta^{(2)}(x-y)\rho^{\mu\nu}_{ab} \equiv i\langle T^*j^\mu_a(x)j^\nu_b(y)\rangle_{\text{conn}} \ . \tag{4.12}$$

It is necessary to work with canonical variables in order to obtain here the ordinary time-ordered product (which we denote by T), the product whose discontinuity at equal times is the commutator. We should also remark that the time-ordered product of the two currents is not Lorentz covariant; roughly speaking, it contains time derivatives inside rather than outside the time ordering. This lack of Lorentz covariance is canceled by the additional Schwinger term contribution that arises from the $\mathscr{A}$ dependence of the current operators. We may now see in detail how the current commutator is related to the divergence of the current correlation functions, for

$$D_\mu K^{\mu\nu}_{ab}(x,y)=\langle\delta(x^0-y^0)i[j^0_a(x),j^\nu_b(y)]\rangle +\partial_1\delta^{(2)}(x-y)\rho^{1\nu}_{ab} \tag{4.13}$$

which, with the current commutation relation (4.7), yields precisely the divergence condition (4.11).

It is very important to note that the functional derivative $\delta j/\delta\mathscr{A}$ which was needed to obtain the two-point function must be performed with care. It is incorrect to take the derivative naively on the expression given in Eq. (3.15) for the current. Rather, this differentiation must be performed holding the dynamical variables [in particular $\pi_a(x)$] fixed, as we have done. The same result is obtained in the fermionic formulation of the model. In that case, naively one concludes that $\delta j/\delta\mathscr{A}$ is zero. However, the current in the Fermi case is a singular operator. Regulating the current (by, say, point splitting in a gauge-invariant fashion) introduces an $\mathscr{A}$ dependence, and again one is led[18] to the results that we have just obtained for the Bose case. This, of course, must be the case. The effective action encapsulates all the information about the current correlation functions; and hence two models with the same effective action necessarily have the same correlation functions.

Finally, we observe that explicit expressions for the current correlation functions can be obtained by taking functional derivatives $\delta/\delta\mathscr{A}$ of the basic relations (2.27a) and (2.27b). In particular, the two-point current correlation function is given by

$$K^{\mu\nu}_{ab}(x,y)\bigg|_{\mathscr{A}=0}=\frac{\delta_{ab}}{\pi}\int\frac{d^2k}{(2\pi)^2}e^{ik(x-y)}\frac{k^\mu k^\nu-\eta^{\mu\nu}k^2}{k^2-i\epsilon} \ . \tag{4.14}$$

V. LEFT-RIGHT-SYMMETRIC SCHEME

The fundamental fact that allowed us to establish the equality of effective actions for the Bose and Fermi models is that for each model we had the same set of two conditions for two unknown current components. We chose to work in a scheme in which the vector current was conserved. Although at first it appeared that there were four unknown current components (j^μ and j^μ_5), the reduction in the number of unknowns to two was achieved by demanding duality of the vector and axial-vector currents.

Alternatively, one can work in a left-right-symmetric scheme. As before, there appear to be four unknown current components ($j^{(+)}_\mu$ and $j^{(-)}_\mu$). However, as we shall demonstrate, the number of unknowns can again be reduced to two by demanding that two light-cone components vanish, $j^{(+)}_+=j^{(-)}_-=0$. This approach has the advantage that the correlation function of a left-current with a right-current vanishes.

Consider again the fermionic model (2.1), with the effective action $W_1[A^{(-)}_\mu,A^{(+)}_\mu]$ given by (2.7). As already noted, regulating the theory in a left-right-symmetric fashion yields the anomalous response

$$\delta_\mp W_1[A^{(-)}_\mu,A^{(+)}_\mu]=\mp\frac{1}{8\pi}\int\omega^1_2(a_\mp,A^{(\mp)}) \tag{5.1}$$

under the gauge transformations (2.6).

Although the effective action W_1 depends on all four gauge potential components, we can exploit the freedom of adding finite local counterterms to define a new effective action W_F which depends only on the two light-cone components $A^{(-)}_-$ and $A^{(+)}_+$. Indeed, consider the new effective action given by

$$W_F=W_1[A^{(-)}_\mu,A^{(+)}_\mu]-\frac{1}{16\pi}\int\text{tr}({}^*A^{(+)}A^{(+)} +{}^*A^{(-)}A^{(-)}) \ . \tag{5.2}$$

It is easy to check that

$$\delta_-W_F=-\frac{1}{4\pi}\int d^2x\,\text{tr}a_-\partial_+A^{(-)}_- \ , \tag{5.3a}$$

$$\delta_+W_F=-\frac{1}{4\pi}\int d^2x\,\text{tr}a_+\partial_-A^{(+)}_+ \ . \tag{5.3b}$$

Hence W_F is indeed a functional of only $A^{(-)}_-$ and $A^{(+)}_+$, and the two conditions (5.3a) and (5.3b) completely determine W_F. As already remarked at the end of Sec. II, the fermion effective action W_3 also depends only on $A_-^{(-)}$ and $A_+^{(+)}$. However, one can check that W_F and W_3 are not equal. Although Eqs. (5.3) are written in a noncovariant form, they are Lorentz invariant since they are variations of the manifestly invariant Eq. (5.2). This Lorentz invariance can be made explicit by introducing the factors $\eta^{\mu\nu}\pm\epsilon^{\mu\nu}$.

Next, consider again the bosonic classical action $S_1[\mathcal{G},A_\mu^{(-)},A_\mu^{(+)};\alpha]$ given by (3.4). It has the same response as the fermion effective action W_1 under the gauge transformations (2.6) and (3.1). Consequently, the new bosonic action

$$S_B[\mathcal{G},A_\mu^{(-)},A_\mu^{(+)};\alpha]=S_1[\mathcal{G},A_\mu^{(-)},A_\mu^{(+)};\alpha] -\frac{1}{16\pi}\int \mathrm{tr}({}^*A^{(+)}A^{(+)} +{}^*A^{(-)}A^{(-)}) \tag{5.4}$$

satisfies

$$\delta_- S_B[\mathcal{G},A_\mu^{(-)},A_\mu^{(+)};\alpha]=-\frac{1}{4\pi}\int d^2x\, \mathrm{tr}a_-\partial_+A_-^{(-)} , \tag{5.5a}$$

$$\delta_+ S_B[\mathcal{G},A_\mu^{(-)},A_\mu^{(+)};\alpha]=-\frac{1}{4\pi}\int d^2x\, \mathrm{tr}a_+\partial_-A_+^{(+)} , \tag{5.5b}$$

for all values of the parameter α. By examining the explicit expression for S_B, we can see that the value $\alpha=1$ is uniquely determined by the requirement that the action depend only on $A_-^{(-)}$ and $A_+^{(+)}$.

It is possible to define a new gauge potential $\mathcal{A}_\mu$, whose light-cone components are given by $\mathcal{A}_+=A_+^{(+)}$ and $\mathcal{A}_-=A_-^{(-)}$. One can then show that the classical bosonic action (for the case $\alpha=1$) is given by

$$S_B=-\frac{1}{8\pi}\int d^2x[-\tfrac{1}{2}\mathrm{tr}\mathcal{G}^{-1}\partial_\mu\mathcal{G}\mathcal{G}^{-1}\partial^\mu\mathcal{G}+(\eta^{\mu\nu}-\epsilon^{\mu\nu})\mathrm{tr}(\mathcal{A}_\mu\partial_\nu\mathcal{G}\mathcal{G}^{-1}-\mathcal{A}_\nu\mathcal{G}^{-1}\partial_\mu\mathcal{G}+\mathcal{A}_\mu\mathcal{G}\mathcal{A}_\mu\mathcal{G}^{-1})] -\frac{1}{24\pi}\int_N \mathrm{tr}(\mathcal{G}^{-1}d\mathcal{G})^3 . \tag{5.6}$$

This expression does not include the term $\mathrm{tr}\mathcal{A}_\mu\mathcal{A}^\mu$ which is present in (3.13); and unlike the latter action, (5.6) is not invariant under vector gauge transformations.

The corresponding quantum theory is described by the bosonic effective action W_B defined by

$$\exp(iW_B[A_-^{(-)},A_+^{(+)}]) =\int [d\mathcal{G}]\exp(iS_B[\mathcal{G},A_-^{(-)},A_+^{(+)};1]) . \tag{5.7}$$

Since W_B obeys the same two conditions (5.3a) and (5.3b) as the fermion effective action W_F, we conclude that the two effective actions are equal

$$W_B[A_-^{(-)},A_+^{(+)}]=W_F[A_-^{(-)},A_+^{(+)}] . \tag{5.8}$$

This is the Fermi-Bose equivalence, in the left-right-symmetric scheme.

Writing $A_\mu^{(\pm)}=iA_{\mu a}^{(\pm)}T_a$, we can define the current expectation values

$$\langle j_a^{(-)}(x)\rangle=\frac{\delta W}{\delta A_{-a}^{(-)}(x)} , \tag{5.9a}$$

$$\langle j_a^{(+)}(x)\rangle=\frac{\delta W}{\delta A_{+a}^{(+)}(x)} . \tag{5.9b}$$

(We no longer make a distinction between W_F and W_B, since the two are equal.) From the basic relations (5.3a) and (5.3b), it can be shown that the two-point current correlation functions are given by

$$i\langle T^* j_a^{(-)}(x)j_b^{(-)}(y)\rangle^{(0)} =\frac{\delta_{ab}}{\pi}\int \frac{d^2k}{(2\pi)^2}e^{ik(x-y)}\frac{(k_+)^2}{k^2-i\epsilon} , \tag{5.10a}$$

$$\langle T^* j_a^{(+)}(x)j_b^{(+)}(y)\rangle^{(0)} =\frac{\delta_{ab}}{\pi}\int \frac{d^2k}{(2\pi)^2}e^{ik(x-y)}\frac{(k_-)^2}{k^2-i\epsilon} , \tag{5.10b}$$

and

$$i\langle T^* j_a^{(-)}(x)j_b^{(+)}(y)\rangle^{(0)}=0 . \tag{5.10c}$$

(Here the superscript (0) is used to indicate the absence of all sources.) That is, there is a symmetry between correlations of left currents and corresponding correlations of right currents, and correlations of left and right currents vanish. It is this left-right-symmetric scheme which is used by Witten[2] and by Knizhnik and Zamolodchikov.[7]

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APPENDIX

As discussed in the text, the coefficient of the Wess-Zumino term, given by

$$\Gamma=-\frac{1}{24\pi}\int_N \mathrm{tr}(\mathcal{G}^{-1}d\mathcal{G})^3+\cdots , \tag{A1}$$

is uniquely determined by the requirement that Γ reproduce the fermionic anomalies:

$$\delta_\mp\Gamma=\mp\frac{1}{8\pi}\int \omega_2^1 . \tag{A2}$$

Remarkably, this value for the coefficient also renders the Wess-Zumino term well defined. This fact is contained implicitly in Zumino's lectures.[12] For completeness we wish to present an explicit proof. To this end we continue to Euclidean space, and recall[19] that the index theorem gives

$$\frac{1}{16\pi^2}\int_{S^4} \mathrm{tr}\mathscr{F}^2 = \text{integer} . \qquad (A3)$$

Using the fact that locally $\mathrm{tr}\mathscr{F}^2$ is an exact form, with

$$\mathrm{tr}\mathscr{F}^2 = d\omega_3^0(\mathscr{A},\mathscr{F}) = d\,\mathrm{tr}(\mathscr{A}\mathscr{F}+\tfrac{1}{3}\mathscr{A}^3) , \qquad (A4)$$

we see that the integral over the four-sphere S^4 can be expressed as the sum of integrals over the upper (H_+) and lower (H_-) hemispheres:

$$\frac{1}{16\pi^2}\left[\int_{H_+} d\omega_3^0(\mathscr{A}_+,\mathscr{F}_+) + \int_{H_-} d\omega_3^0(\mathscr{A}_-,\mathscr{F}_-)\right] = \text{integer} . \qquad (A5)$$

Here $\mathscr{A}_+$ and $\mathscr{A}_-$ are the gauge potentials on the upper and lower hemispheres, respectively, which on the equator are related by a gauge transformation:

$$\mathscr{A}_+ = \mathscr{G}^{-1}\mathscr{A}_-\mathscr{G} - \mathscr{G}^{-1}d\mathscr{G} . \qquad (A6)$$

From Stokes' theorem, we now have

$$\frac{1}{16\pi^2}\int_{S^3}[\omega_3^0(\mathscr{A}_+,\mathscr{F}_+) - \omega_3^0(\mathscr{A}_-,\mathscr{F}_-)] = \text{integer} . \qquad (A7)$$

However, one can readily check that[12]

$$\omega_3^0(\mathscr{A}_+,\mathscr{F}_+) - \omega_3^0(\mathscr{A}_-,\mathscr{F}_-) = -\tfrac{1}{3}\mathrm{tr}(\mathscr{G}^{-1}d\mathscr{G})^3 + d\alpha_2 . \qquad (A8)$$

Hence we conclude that

$$\frac{1}{48\pi^2}\int_{S^3}\mathrm{tr}(\mathscr{G}^{-1}d\mathscr{G})^3 = \text{integer} . \qquad (A9)$$

This is precisely the condition which guarantees that the Wess-Zumino term is well defined. Clearly, this type of argument holds for any representation of any group G. [For a complex representation, such as the fundamental representation of SU(N), the coefficient of $\mathrm{tr}\mathscr{F}^2$ in the index theorem (A3) is larger by a factor of 2; correspondingly, the coefficient of $\mathrm{tr}(\mathscr{G}^{-1}d\mathscr{G})^3$ is also larger by this same factor.]

Finally we remark that these considerations can be generalized to higher dimensions. In particular it can be shown[12] that the Wess-Zumino term Γ which reproduces the fermionic anomalies

$$\delta_{\mp}\Gamma = \pm\frac{1}{2n!}\frac{i^n}{(2\pi)^{n-1}}\int \omega_{2n-2}^1 \qquad (A10)$$

is given by

$$\Gamma = \int_N \gamma , \qquad (A11)$$

where

$$\gamma = \frac{1}{2}\frac{i^n}{(2\pi)^{n-1}}\frac{(n-1)!}{(2n-1)!}\mathrm{tr}(\mathscr{G}^{-1}d\mathscr{G})^{2n-1} + \cdots , \qquad (A12)$$

and N is a $(2n-1)$-dimensional manifold, whose boundary is S^{2n-2}. The preceding work can be generalized in a straightforward way to prove that γ satisfies

$$\frac{1}{2\pi}\int_{S^{2n-1}}\gamma = \text{integer} ; \qquad (A13)$$

hence, the Wess-Zumino term Γ is well defined.

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