Extended solutions of an SU(2) gauge theory with fermions

Dennis Sivers
High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439
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This paper explores the idea that there can exist interesting classical solutions to non-Abelian gauge theories which represent extended systems. These solutions provide possible alternative starting points for perturbative quantum field theories. For the particular case of an SU(2) gauge theory with spherical symmetry, we find a simple example of a "vacuum" solution with a nonvanishing chiral order parameter: $\chi=\bar{\psi}\psi$. We also find a simple set of spinors representing systems of massive SU(2) fermions in the field of a Wu-Yang monopole.

I. INTRODUCTION

One of the tools for extending the range of quantum field theory is a semiclassical approximation based on a special solution to the dynamical field equations.[1] Of course, to use this method, it is necessary to find a solution to the classical field equations which provides an approximate description of the system whose properties are to be calculated. This preliminary task can be formidable. For non-Abelian gauge theories, only a small number of solutions are known[2] and most of these involve gauge fields only—without regard for the fermions which couple to them.

The range of possible "interesting" solutions is limited only by our imagination. For example, it would be useful to have a simple solution of the field equations which could describe an extended system with a nonvanishing chiral density

$$\chi=\bar{\psi}\psi \tag{1.1}$$

and/or a nonvanishing field-strength density

$$G^2=G^a_{\mu\nu}G^{\mu\nu a} . \tag{1.2}$$

Such solutions could serve as starting points for quantum calculations involving the effects on nonvanishing vacuum condensates;

$$\langle \text{vac} \mid \bar{\psi}\psi \mid \text{vac} \rangle \neq 0 , \tag{1.3}$$

$$\left\langle \text{vac} \left| \frac{\alpha_s}{\pi} G^a_{\mu\nu}G^{\mu\nu a} \right| \text{vac} \right\rangle \neq 0 . \tag{1.4}$$

The necessity for an alternative to the usual perturbation-theory approach to deal with (1.3) and (1.4) has long been appreciated. It is not yet known whether semiclassical techniques can be of any use in describing physical condensates but there are some promising indications.[3,4]

This paper builds on the work of Refs. 3 and 4 in order to discuss some simple solutions to the SU(2) Dirac equation in the presence of gauge fields with spherical symmetry. The Dirac fermions can, in turn, be considered as sources for the components of the non-Abelian field-strength tensor. The solutions we study start with particularly simple gauge field configurations. However, the solutions can serve as illustrative examples and can serve to test the general technique for the construction of solutions of more physical interest. We consider first the case of a single doublet of Dirac fermions in the "background" of a vacuum characterized by $A^a_\mu=0$. We also look at solutions representing fermions in the field of a Wu-Yang monopole.[5]

To fix notation for the purpose of this paper, we will define the Lagrangian density for the SU(2) gauge theory to be

$$\mathscr{L}=-\tfrac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a+\bar{\psi}\left[i\partial\!\!\!/-gA\!\!\!/^a\left[\frac{\tau_a}{2}\right]-m\right]\psi , \tag{1.5}$$

with a single flavor of fermions and will work in Minkowski space. The specific conventions we use are those specified in Refs. 3 and 4. The remainder of this paper can be outlined as follows. Section II presents a self-contained introduction to the requirements of spherical symmetry on SU(2) gauge fields and fermions. It also contains a discussion of the SU(2) Dirac equation. Section III discusses the form of the chiral density (1.1) and the fermion vector and axial-vector currents which can be used to characterized solutions to the Dirac equation. It also gives expressions for the components of the SU(2) "color" current which serves as the source for the non-Abelian gauge field. Section IV presents the two illustrative examples of extended solutions to the Yang-Mills Dirace equation mentioned above; the first involves fermions in a trivial vacuum and the second involves massive fermions in the field of a Wu-Yang monopole. In both cases the feedback of the fermionic currents on the gauge field configuration provides important constraints. Section V gives some brief conclusions.

II. GAUGE AND SPINOR FIELDS WITH SPHERICAL SYMMETRY

The dynamical equations for a non-Abelian gauge theory such as SU(2) contain intrinsic nonlinearities which preclude a general construction of solutions. In order to make progress, it is often convenient to make the simplifying assumptions necessary to produce a specialized ansatz. The type of simplification which offers the most clear-cut interpretation involves underlying space-time symmetries. For example, in an SU(2) gauge theory, it is common to assume time independence,[6] spherical symmetry,[7] or both.[8]

In this paper, we will be considering SU(2) gauge fields and fermion fields which are consistent with a spherical symmetry. For objects which transform nontrivially under both rotations and gauge transformations, consistency of the theory requires that the effect of a gauge transformation can be compensated by an appropriate rotation or that the effect of a rotation can be compensated by a gauge transformation.[9] Symmetries, therefore, restrict the form of the fields.

For the vector gauge potential A^a_μ the assumption of spherical symmetry leads to the form[3,7]

$$gA^a_0 = A_0(r,t)\hat{\mathbf{r}}_a ,$$

$$gA^a_i = A_1(r,t)\rho_{ia} + \frac{a(r,t)}{r}\sin\omega(r,t)\delta^T_{ia} + \frac{a(r,t)\cos\omega(r,t)-1}{r}\epsilon^T_{ia} . \tag{2.1}$$

The tensors ρ_{ia}, δ^T_{ia}, and ϵ^T_{ia} which relate spatial three-vector indices and internal SU(2) indices in the adjoint representation are defined to be

$$\rho_{ia} = \hat{\mathbf{r}}_i\hat{\mathbf{r}}_a, \quad \delta^T_{ia} = \delta_{ia} - \rho_{ia}, \quad \epsilon^T_{ia} = \epsilon_{ial}\hat{\mathbf{r}}_l . \tag{2.2}$$

A more complete discussion of the interpretation of this ansatz and the structure of these tensors can be found in Refs. 3, 4, and 7. In the framework of this ansatz, it is often useful to introduce the "gauge-dependent" tensors

$$e^S_{ia}(\omega) = \delta^T_{ia}\cos\omega(r,t) - \epsilon^T_{ia}\sin\omega(r,t) ,$$

$$e^A_{ia}(\omega) = \delta^T_{ia}\sin\omega(r,t) + \epsilon^T_{ia}\cos\omega(r,t) , \tag{2.3}$$

where $\omega(r,t)$ is the gauge angle in the expression for the vector potential (2.1), while δ^T_{ia} and ϵ^T_{ia} are defined in (2.2). Using these tensors, a spherically symmetric gauge-covariant object which transforms as a vector under rotations can be written using gauge-invariant coefficients. For example, the gauge-covariant current j^a_μ can be written

$$4\pi r^2 j^a_0 = J_0(r,t)\hat{\mathbf{r}}_a ,$$

$$4\pi r^2 j^a_i = J_1(r,t)\rho_{ia} + J_S(r,t)e^S_{ia}(\omega) + J_A(r,t)e^A_{ia}(\omega) , \tag{2.4}$$

where the coefficients J_0, J_1, J_S, and J_A are gauge-invariant functions of r and t. A radially directed rotation or gauge transformation merely serves to rotate the tensors (2.3) and leave the coefficients alone.

One of the most useful properties of the spherically symmetric ansätz defined above is that it forms an analog two-dimensional Abelian gauge theory.[1,2] Using the ansatz (2.1) we can see

$$\int d^4x\, \mathscr{L}_g = 4\pi \int dr\, dt\, (r^2\mathscr{L}_g) , \tag{2.5}$$

where

$$r^2\mathscr{L}_g = r^2(F_{lm}F^{lm}) + 2D^l\Phi D_l\Phi^* + \frac{1}{r^2}(|\Phi|^2-1)^2 \tag{2.6}$$

and $l,m=0,1$. In this equation the elements of A^a_μ in Eq. (2.1) enter in the form

$$F_{lm} = \partial_l A_m - \partial_m A_l, \quad D_l = \partial_l - ieA_l \tag{2.7}$$

with $e=1$ an imbedded U(1) charge and

$$\Phi = a(r,t)e^{i\omega(r,t)} . \tag{2.8}$$

The time and radial components A_0 and A_1 of the SU(2) vector potential in (2.1) are seen to play the role of a two-dimensional vector potential while $\Phi(r,t)$, which specifies the transverse degrees of freedom in (2.1), becomes a charged scalar. The Lagrange density (2.6) differs from that of the usual two-dimensional Abelian Higgs model only in the factors of r^2 which are associated with the different terms. These factors can be absorbed in the metric

$$g^{(2)}_{lm} = r^2\eta^{(2)}_{lm} , \tag{2.9}$$

which specifies a curved two-dimensional manifold.

Finally, it is worth noting that the imposition of spherical symmetry preserves the important topological properties of the original (3+1)-dimensional SU(2) gauge theory. To see that we have not lost any crucial features of the original theory, we observe that imposing vacuum boundary conditions at infinity in the Euclidean two-dimensional theory involves the maps

$$M:(S^1 \to \mathrm{U}(1)) , \tag{2.10}$$

while, in the four-dimensional theory, it involves

$$M:(S^3 \to \mathrm{SU}(2)) . \tag{2.11}$$

The reduction specified by (2.1) and (2.6) is safe topologically since the two homotopy groups coincide

$$\Pi_1(\mathrm{U}(1)) = \Pi_3(\mathrm{SU}(2)) = Z , \tag{2.12}$$

where Z is the group of integers under addition.

The relationship between (3+1)-dimensional SU(2) gauge theory with spherical symmetry and the (1+1)-dimensional Abelian Higgs model is very valuable in establishing general properties of the non-Abelian theory. The correspondence can be strengthened further by the introduction of fermions in the fundamental representation. It is instructive to see how this occurs. Following Ref. 4 we can parametrize $\hat{\mathbf{r}}$-directed spinors $\xi^\pm_I$ in three-space and similar spinors $\eta^\pm_A$ in group space such that

$$\sigma^{\hat{\mathbf{r}}J}_I \xi^\pm_J = \pm\xi^\pm_I, \quad I,J=1,2 , \tag{2.13}$$

and

$$\tau^{\hat{\mathbf{r}}B}_A \eta^\pm_B = \pm\eta^\pm_A, \quad A,B=1,2 , \tag{2.14}$$

where $\sigma^{\hat{\mathbf{r}}}$ and $\tau^{\hat{\mathbf{r}}}$ are, respectively, the 2×2 Pauli matrices in three-space and group space.

Bispinors of the form

$$X^{\pm\pm}_{IA}(\theta,\phi) = \xi^\pm_I \eta^\pm_A , \tag{2.15}$$

with one spatial index and one SU(2) index form appropriate objects for describing a set of spherically symmetric SU(2) fermion fields in the fundamental representation. They play a role similar to ρ_{ia}, δ_{ia}^T, and ϵ_{ia}^T in the description of gauge-covariant vectors. Following the pattern of the pure gauge theory, we therefore seek to construct spinors in which an $\hat{\mathbf{r}}$-directed rotation can be compensated by gauge-transformation and vice versa. This suggests that we parametrize a Dirac spinor in terms of two Weyl spinors:

$$\psi_R{}^j{}_B=\frac{1}{r}[R^+(r,t)X^{++j}{}_B+R^-(r,t)X^{--j}{}_B] , \quad (2.16)$$

$$\psi_{LIA}=\frac{1}{r}[L^+(r,t)X_{IA}^{-+}+L^-(r,t)X_{IA}^{+-}] . \quad (2.17)$$

In terms of these functions, the SU(2) Dirac equation can then be decomposed

$$\sigma^{\mu}_{Ij}\left[\partial_\mu-igA_\mu^a\left[\frac{\tau^a}{2}\right]_A{}^B\right]\psi_R{}^j{}_B=M\psi_{LIA} ,$$

$$\sigma^{\mu Ij}\left[\partial\mu-igA_\mu^a\left[\frac{\tau^a}{2}\right]_A{}^B\right]\psi_{LJB}=-m\psi_R{}^i{}_A . \quad (2.18)$$

Combining the parametrizations (2.16) and (2.17) of the spinor with the ansatz (2.1) for the gauge potential leads to the set of equations[4]

$$\frac{1}{r}(-D_0-D_r)R^++\frac{\Phi}{r^2}R^-=\frac{m}{r}L^+ , \quad (2.19a)$$

$$\frac{1}{r}(D_0-D_r)R^-+\frac{\Phi^*}{r^2}R^+=\frac{m}{r}L^- , \quad (2.19b)$$

$$\frac{1}{r}(-D_0+D_r)L^++\frac{\Phi}{r^2}L^-=\frac{-m}{r}R^+ , \quad (2.19c)$$

$$\frac{1}{r}(D_0+D_r)L^-+\frac{\Phi^*}{r^2}L^+=\frac{-m}{r}R^- . \quad (2.19d)$$

In writing (2.19), we have taken a clue from (2.7) and used a two-dimensional Abelian covariant derivative

$$D_l=\partial_l-ieA_l \quad (2.20)$$

and have made the corresponding identification of an imbedded Abelian charge such that

$$e\eta_A^{\pm}=\pm\tfrac{1}{2}\eta_A^{\pm} . \quad (2.21)$$

It is clear that the fermions defined by (2.16) and (2.17) fit snugly into the two-dimensional analog mentioned earlier. We can display this feature by introducing a convenient basis for the two-dimensional γ matrices

$$\gamma_0^{(2)}=r\begin{bmatrix}0&1\\1&0\end{bmatrix},\ \gamma_1^{(2)}=r\begin{bmatrix}0&-1\\1&0\end{bmatrix} \quad (2.22)$$

such that

$$\{\gamma_l^{(2)},\gamma_m^{(2)}\}=2g_{lm}^{(2)} , \quad (2.23)$$

and introduce the new "chirality" operator

$$\gamma_5^{(2)}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\frac{1}{r^2}\gamma_0^{(2)}\gamma_1^{(2)} . \quad (2.24)$$

Using this representation of the Dirac matrices we can then introduce the two-spinors

$$R=\begin{bmatrix}R^-\\-R^+\end{bmatrix},\ L=\begin{bmatrix}L^+\\L^-\end{bmatrix} \quad (2.25)$$

and the pseudoscalar field

$$\Phi^{(2)}=\tfrac{1}{2}\Phi(1+\gamma_5^{(2)})+\tfrac{1}{2}\Phi^*(1-\gamma_5^{(2)})$$
$$=a(r,t)e^{i\omega(r,t)\gamma_5^{(2)}} . \quad (2.26)$$

We can, using (2.20)—(2.26), transcribe Eq. (2.19) into the suggestive two-dimensional form

$$(\gamma_l^{(2)}D^l+\gamma_5^{(2)}\Phi^{(2)})R=mL , \quad (2.27a)$$

$$(\gamma_l^{(2)}D^l+\gamma_5^{(2)}\Phi^{(2)*})L=-mR . \quad (2.27b)$$

Note that the new two-dimensional chirality operator $\gamma_5^{(2)}$ acts on the internal charge space in the original 3+1 formulation and that the original four-dimensional spinors do not appear. There are other convenient alternatives to the identification of the pseudospinors (2.25) and γ matrices (2.22) and (2.24) in two dimensions which lead to a different appearance for (2.27). However, the substance of this "dimensional reduction" involves the observation that the transverse degrees of freedom of the SU(2) vector potential, characterized by the potential $\Phi(r,t)$ couple to the spinors (2.16) and (2.17) as a charged scalar while the radial and time components of the four-dimensional SU(2) vector potential couple as a two-dimensional vector *Abelian* potential. The usefulness of the two-dimensional analog theory represented by (2.5)—(2.8) is therefore reinforced by the introduction of fermions.

It is important to keep in mind that the trivial SU(2) "vacuum" state is characterized by

$$|\Phi|=1 \quad (2.28)$$

while the state

$$|\Phi|=0 \quad (2.29)$$

corresponds to presence of a Wu-Yang monopole.[5] There are thus two cases where further simplification to the equations in (2.19) or (2.27) is obviously possible, the chiral limit when $m=0$ and the monopole limit when $|\Phi|=0$. We will be dealing with some simple illustrative examples of solutions of the equations in these limits in Sec. IV.

We should further point out that the "physical" 't Hooft-Polyakov monopole is to be understood as a solution to an (imbedded) SU(2)-Higgs theory in 3+1 dimensions.[10,11] The fermion masses in (2.19) are assumed to arise in a more complete theory from the Yukawa couplings to the Higgs field. At values of r large compared to the assumed grand unified scale, the Eq. (2.19) with $|\Phi|=0$ and $m=$const are sufficient to describe the semiclassical behavior of fermions. However, solutions can depend sensitively on the boundary conditions at the origin where to be compatible with the hidden dynamics we must consider

$$\Phi\rightarrow\Phi(r),\quad m\rightarrow m(r) . \quad (2.30)$$

We will not be dealing with the full complications inherent in (2.30) as these involve considerations beyond the semiclassical limit.

III. FERMION CURRENTS AND THE SU(2) MAXWELL'S EQUATIONS

The parametrization of the SU(2) Dirac spinor given by (2.16) and (2.17) in terms of the four functions $R^{\pm}(r,t)$ and $L^{\pm}(r,t)$ allows the explicit characterization of fermionic currents. These currents characterize the observables which determine the interactions of the fermions with hypothetical external forces. The SU(2) current also serves to complete the classical dynamic system by acting as a source for the gauge fields. It is instructive to have concrete expressions for the various currents. Because of the introductory nature of this paper, we will for simplicity be considering only one "flavor" of fermion. We will be interested in the chiral density

$$\chi=\bar{\psi}\psi \tag{3.1}$$

as well as the "quark" vector current

$$q_\mu=\bar{\psi}\gamma_\mu\psi \tag{3.2}$$

and the "quark" axial-vector current

$$q_\mu^5=\bar{\psi}\gamma_\mu\gamma_5\psi\ . \tag{3.3}$$

Within our classical approximation χ is a scalar order parameter and using the forms (2.16) and (2.17) we have the simple expression

$$\chi=\frac{1}{r^2}(R^{-*}L^{-}-R^{+*}L^{+}+L^{-*}R^{-}-L^{+*}R^{+})\ . \tag{3.4}$$

We see that χ vanishes if both charge states of either L or R are zero. The presence of a nonzero value of χ is associated with the Nambu[12] realization of a spontaneously broken chiral symmetry.

The currents defined by (3.2) and (3.3) provide an important description of physical states. In terms of the two-dimensional analogy of Sec. II, the currents q_μ and q_μ^5 are two-vectors with time and radial components:

$$4\pi r^2 q_\mu=(Q_0,Q_1\hat{\mathbf{r}}_i),\quad 4\pi r^2 q_\mu^5=(Q_0^5,Q_1^5\hat{\mathbf{r}}i)\ . \tag{3.5}$$

The components of the quark-vector current are found to be

$$\begin{aligned}Q_0&=(R^{+*}R^{+}+R^{-*}R^{-}+L^{+*}L^{+}+L^{-*}L^{-})\ ,\\ Q_1&=(-R^{+*}R^{+}+R^{-*}R^{-}+L^{+*}L^{+}-L^{-*}L^{-})\ ,\end{aligned} \tag{3.6}$$

while the components of the quark axial-vector current are

$$\begin{aligned}Q_0^5&=(-R^{+*}R^{+}-R^{-*}R^{-}+L^{+*}L^{+}+L^{-*}L^{-})\ ,\\ Q_1^5&=(R^{+*}R^{+}-R^{-*}R^{-}+L^{+*}L^{+}-L^{-*}L^{-})\ .\end{aligned} \tag{3.7}$$

Consistency of the formalism obviously demands the conservation of the vector current

$$\partial^l Q_l=0\ , \tag{3.8}$$

while the divergence of the axial-vector current is ultimately related to the presence of topologically nontrivial gauge fields through the Adler-Bell-Jackiw[13] anomaly. With our normalization, the anomaly of the two-dimensional current gives

$$\partial^l Q_l^5=\frac{g^2r^2}{4\pi}{}^*G_{\mu\nu}^aG_{\mu\nu}^a\ . \tag{3.9}$$

It is not within the scope of the classical approximation to deal with the field-theoretic loop corrections which lead to (3.9). However, there are dynamical consequences associated with structure in the axial-vector current which appear in the classical approach. To see this we observe that there is another topological current involving the gauge potential (2.1) with a divergence which can be written in terms of (3.9) (Ref. 14). If we define

$$\begin{aligned}K_0&=(a^2-1)A_1-a^2\frac{\partial\omega}{\partial r}\ ,\\ K_1&=-(a^2-1)A_0+a^2\frac{\partial\omega}{\partial t}\ ,\end{aligned} \tag{3.10}$$

it is easy to verify explicitly that

$$\partial^l K_l=\tfrac{1}{2}g^2r^2G_{\mu\nu}^{*a}G^{a\mu\nu}\ . \tag{3.11}$$

This current is not gauge invariant and transform as the dual of a vector potential. In dealing with quantum effects, it is important that the combination

$$\hat{K}_l=K_l-\frac{Q_l^5}{2\pi} \tag{3.12}$$

be divergence-free. From the dynamical point of view, it is also significant that the current K_l defined in (3.10), appears explicitly in the definition of transverse components of the field-strength tensor as we shall demonstrate below.

We will also be considering the SU(2) "color" current generated by the spinor field. Unlike the currents Q_μ and Q_μ^5, the SU(2) current can have transverse components as specified in (2.4). Using the form of the spinors in (2.16) and (2.17) we can write $J_\mu^a=4\pi r^2 j_\mu^a$ in terms of its invariant components

$$\begin{aligned}J_0(r,t)&=\tfrac{1}{2}(-R^{+*}R^{+}+R^{-*}R^{-}-L^{+*}L^{+}+L^{-*}L^{-})\ ,\\ J_1(r,t)&=\tfrac{1}{2}(R^{+*}R^{+}+R^{-*}R^{-}-L^{+*}L^{+}-L^{-*}L^{-})\ ,\\ J_S(r,t)&=\tfrac{1}{2}(R^{+*}R^{-}e^{i\omega}+R^{-*}R^{+}e^{-i\omega}\\ &\qquad+L^{-*}L^{+}e^{-i\omega}+L^{+*}L^{-}e^{i\omega})\ ,\\ J_A(r,t)&=\frac{i}{2}(R^{+*}R^{-}e^{i\omega}-R^{-*}R^{+}e^{-i\omega}\\ &\qquad+L^{+*}L^{-}e^{i\omega}-L^{-*}L^{+}e^{-i\omega})\ .\end{aligned} \tag{3.13}$$

As indicated in Sec. II, the transverse components J_A and J_S are invariant under radial gauge transformations since under a gauge transform $U=\exp(i\theta\tau_{\hat{r}}/2)$

$$R^- \to \tilde{R}^- = R^- e^{-i\theta/2} ,$$
$$R^+ \to \tilde{R}^+ = R^+ e^{+i\theta/2} , \qquad (3.14)$$
$$\omega \to \tilde{\omega} = \omega + \theta .$$

Notice that the current j_μ^a is not conserved. It is, however, covariantly conserved and obeys

$$\mathscr{D}_\mu^{ab} j_b^\mu = 0 . \qquad (3.15)$$

When we apply this constraint to (3.13) we observe

$$\partial^l J_l(r,t) = \frac{2a(r,t)}{r} J_S(r,t) . \qquad (3.16)$$

This constraint follows from the form of the Dirac equation (2.19). Because of the connection between spin and color built into the ansatz (2.16) and (2.17), we observe a relation

$$J_0 = -\tfrac{1}{2} Q_1^5 , \quad J_1 = -\tfrac{1}{2} Q_0^5 . \qquad (3.17)$$

Equation (3.17) can be understood by observing that the condition on the compensation of gauge transformation and rotations inherent in (2.16) and (2.17) enforces a relation between the "color" and the "chirality." The components of the axial-vector current then enter the dynamics since the SU(2) current given by (3.13) serve as sources for the non-Abelian field-strength tensor. The electrical and magnetic components of the color field-strength tensor can be written

$$gE_i^a = E_L(r,t)\rho_{ia} + E_S(r,t) e_{ia}^S(\omega) + E_A(r,t) e_{ia}^A(\omega) ,$$
$$(3.18)$$
$$gB_i^a = B_L(r,t)\rho_{ia} + B_S(r,t) e_{ia}^S(\omega) + B_A(r,t) e_{ia}^A(\omega) .$$

As indicated in Sec. II, the components are gauge invariant. In terms of the ansatz (2.1) these components are written

$$E_L = \partial A_0/\partial r - \partial A_1/\partial t, \quad B_L = \frac{a^2-1}{r^2} ,$$
$$E_S = -\frac{a}{r}(\partial\omega/\partial t - A_0), \quad B_S = -\frac{1}{r}\partial a/\partial r , \qquad (3.19)$$
$$E_A = -\frac{1}{r}\partial a/\partial t, \quad B_A = -\frac{a}{r}(\partial\omega/\partial r - A_1) .$$

Because of the existence of the "topological" current K_l, it is sometimes convenient to explicitly display the topological content of $E_S(r,t)$ and $B_A(r,t)$ by writing

$$E_S(r,t) = \frac{1}{ar}(K_1 - A_0), \quad B_A(r,t) = \frac{1}{ar}(K_0 + A_1) , \qquad (3.20)$$

where K_l is given by (3.10).

In the generalization of Maxwell's equations to the spherically symmetric SU(2) system, the equations

$$(D_\mu G_{\mu\nu})^a = j_\nu^a \qquad (3.21)$$

become

$$-\frac{\partial}{\partial r}(r^2 E_L) + 2arE_S = QJ_0(r,t) , \qquad (3.22a)$$

$$-\frac{\partial}{\partial t}(r^2 E_L) + 2arB_A = QJ_1(r,t) , \qquad (3.22b)$$

$$-\frac{\partial}{\partial t}(arE_S) + \frac{\partial}{\partial r}(arB_A) = Q\frac{a}{r}J_S(r,t) , \qquad (3.22c)$$

$$a\Box_{(2)}a - r^2(E_S{}^2 - B_A{}^2) - \frac{a^2(a^2-1)}{r^2} = Q\frac{a}{r}J_A(r,t) , \qquad (3.22d)$$

where the charge Q absorbs the normalization of the currents. The generalization of the Bianchi constraints gives

$$\frac{\partial}{\partial r}(arE_A) - \frac{\partial}{\partial t}(arB_S) = 0 , \qquad (3.23a)$$

$$-E_L + \frac{\partial}{\partial r}(arE_S) + \frac{\partial}{\partial t}(arB_A) = \partial_l K^l$$
$$= \tfrac{1}{2} g^2 r^2 G_{\mu\nu}^{*a} G^{a\mu\nu} . \qquad (3.23b)$$

In writing (3.22) and (3.23) it is convenient to continue to use $a(r,t)$ as defined in the vector potential instead of the equivalent quantity $[r^2 B_L(r,t)+1]^{1/2}$. More details about these equations can be found in Ref. 3.

The solution of the classical dynamical system represented by Eqs. (2.19), (3.22), and (3.23) can be approached in a manner similar to the analogous systems in electrodynamics. The chief difference involves the nonlinearities inherent in (3.22) and (3.23). An iterative variational approach was suggested in Ref. 3 and we will be using some of these ideas below. Because of the nonlinearities, we have no superposition principle and it is not possible to build up solutions involving complicated current configurations from a set of more elementary solutions.

Still, there does exist a promising variational algorithm for dealing with the system. Given a starting configuration for the vector potential (2.1) which is a solution of the source-free field equations (3.22), we can solve the linear equations (2.19) to find a set of fermion fields. The next step in the construction is to calculate the SU(2) currents due to these fermions. We then assume that the current thus found is proportional to a small parameter $\lambda \ll 1$:

$$J_l(r,t) = \lambda j_l^\lambda(r,t) \quad (l=0,1) ,$$
$$(3.24)$$
$$J_S(r,t) = \lambda j_S^\lambda(r,t), \quad J_A(r,t) = \lambda j_A^\lambda(r,t) .$$

We can then formulate a linearized set of equations for the variation of the field-strength tensor associated with these currents in terms of $a(r,t)$ and $\epsilon(r,t) = r^2 E_L$ by expanding

$$\epsilon(r,t) = \epsilon_0(r,t) + \lambda\epsilon_\lambda(r,t) + \cdots ,$$
$$(3.25)$$
$$a(r,t) = a_0(r,t) + \lambda a_\lambda(r,t) + \cdots ,$$

given the fact that ϵ_0 and a_0 were solutions to the source-free equations. The form of the linearized equations depends explicitly on the starting point ϵ_0, a_0. For example, with $\epsilon_0 = 0$ and $a_0 = 1$, we assume that there exists a solution to (2.19) with a nonzero value of J_l but with $J_A = 0$. We then arrive at the linearized equations

$$\Box_2\epsilon_\lambda(r,t)-\frac{2\epsilon_\lambda(r,t)}{r^2}=Q\epsilon^{lm}\partial_l j_m^\lambda\ ,$$

$$\Box_2 a_\lambda-\frac{2a_\lambda(r,t)}{r^2}=0 \tag{3.26}$$

(where $\Box_2$ is the two-dimensional Laplacian) which replace the nonlinear equations (3.22) in this limit. The partial differential equations can now be solved with standardized techniques such as Green's functions. The remaining components of the field-strength tensor are implicitly given by

$$E_S=\frac{1}{2ar}(J_0+\partial\epsilon/\partial r),\quad B_S=\frac{1}{r}\partial a/\partial r\ ,$$

$$E_A=-\frac{1}{r}\partial a/\partial t,\quad B_A=\frac{1}{2ar}(J_1+\partial\epsilon/\partial t)\ . \tag{3.27}$$

We can then iterate the process by solving the Dirac equation (2.19) in the presence of the new vector potential. There is no guarantee that the approach will converge to a consistent set of gauge fields and fermions but the initial steps can prove very restrictive.

IV. SOME SIMPLE SOLUTIONS

It is interesting to investigate some of the very simple solutions which emerge from the spherically symmetric Dirac equation (2.19) in order to become familiar with their basic properties. These can provide an introduction to the possible range of solutions with more physical significance.

We will start with the most trivial vacuum configuration of gauge fields, given by setting $A_0=A_1=0$, $a=1$, and $\omega=0$ in (2.1). The fermion mass will also be assumed to vanish $m=0$. The set of equations (2.19) then reduces to

$$(-\partial_0-\partial_r)R^++\frac{1}{r}R^-=0\ ,$$

$$(\partial_0-\partial_r)R^-+\frac{1}{r}R^+=0\ ,$$

$$(-\partial_0+\partial_r)L^++\frac{1}{r}L^-=0\ ,$$

$$(\partial_0+\partial_r)L^-+\frac{1}{r}L^+=0\ . \tag{4.1}$$

An obvious static solution to this set of equations exists in the form

$$R^+=C_Rr+\frac{D_R}{r},\quad R^-=C_Rr-\frac{D_R}{r}\ ,$$

$$L^+=C_Lr+\frac{D_L}{r,}\quad L^-=-C_Lr+\frac{D_L}{r}\ . \tag{4.2}$$

We observe in the set (4.2) the simple pattern which sets the stage for possible chiral-symmetry breaking. At large r, (4.2) gives

$$R^+=R^-+O\left[\frac{1}{r}\right],\quad L^+=-L^-+O\left[\frac{1}{r}\right]\ . \tag{4.3}$$

Of course these relations are not gauge invariant. We can perform a simple gauge transformation by taking $\omega\to\pi$ in (2.1). This changes $\Phi=1$ to $\Phi=-1$ in (2.19), interchanges the roles of the R and L spinors in (4.1), and leads to the opposite set of boundary conditions from those in (4.3). Taking the $D_R=D_L=0$ case of a solution in (4.2) provides a simple example of a solution which represents a "medium" with uniform fermion density. It is instructive to look at some of the simple properties of this solution. The chiral order parameter X of Eq. (3.5) is given as

$$X=2(-C_R^*C_L-C_L^*C_R)\ . \tag{4.3'}$$

It is nonvanishing whenever both C_L and C_R are nonzero as that according to the usual arguments it is possible to have "spontaneous" chiral-symmetry breaking starting from a classical solution to (4.1). It is also interesting to look at the currents (3.6), (3.7), and (3.13) for this case. The components of the quark-vector current are found to be

$$Q_0=2r^2(C_R^*C_R+C_L^*C_L),\quad Q_1=0\ , \tag{4.4}$$

while the axial-vector quark current is given by

$$Q_0^5=2r^2(-C_R^*C_R+C_L^*C_L),\quad Q_1^5=0\ . \tag{4.5}$$

In this simple configuration ${}^*G^a_{\mu\nu}G^a_{\mu\nu}=0$ so that both the axial-vector and vector currents are conserved. Finally, the various components of the color current are seen to be

$$J_0=0,\quad J_1=r^2(C_R^*C_R-C_L^*C_L)\ ,$$

$$J_S=r^2(C_R^*C_R-C_L^*C_L),\quad J_A=0\ . \tag{4.6}$$

The constraint of covariant current conservation is automatically satisfied since, with $a=1$,

$$\partial_rJ_1=2r(C_R^*C_R-C_L^*C_L)=\frac{2a}{r}J_S\ . \tag{4.7}$$

When $|C_R|\neq|C_L|$, since J_1 and J_S become nonzero, the color currents will introduce new gauge fields as discussed in Sec. III. The gauge fields thus introduced have a very simple form in this example. From (3.22) we see that there appears a contribution to the field-strength tensor form

$$B_A(r)=\frac{Q}{2r}J_1(r)\ , \tag{4.8}$$

when $|C_R|\neq|C_L|$, with all other components of B_i^a and E_i^a remaining at zero. From (3.17), this component of the magnetic field is related to Q_0^5. Moreover, if we enforce left-right symmetry so that all the color currents vanish we can still have a nonvanishing chiral order parameter

$$\chi=\frac{Q_0}{r^2}\neq 0 \ . \tag{4.9}$$

The rich content of the "vacuum" configuration is perhaps surprising. The set of extended solutions represented by (4.2) already has the possibility of interesting structure which we can amplify by adding more "flavors" of fermions.

Another simple gauge field configuration which deserves attention involves a Wu-Yang monopole. As can be seen from (2.1), a monopole field involves $a(r,t)=0$. The gauge angle $\omega(r,t)$ does not enter into the specification of the transverse components of A_μ^a in the presence of Wu-Yang monopole. We will assume $m\neq 0$ and observe that the SU(2) Dirac equation then reduces to the set

$$(-D_0-D_r)R^+=mL^+, \quad (+D_0-D_r)R^-=mL^- ,$$
$$(-D_0+D_r)L^+=-mR^+, \quad (D_0+D_r)L^-=-mR^- . \tag{4.10}$$

We now choose a gauge such that the radial and time components of the vector potential vanish and we take the extreme assumption that the functions $R^\pm$ and $L^\pm$ have no spatial dependence. We then obtain the equations

$$-\partial_0 R^+=mL^+, \quad +\partial_0 L^+=mR^+ ,$$
$$+\partial_0 R^-=mL^-, \quad -\partial_0 L^-=mR^- . \tag{4.11}$$

These equations contain the possibility of interesting phenomena due to the underlying L-R asymmetry. For example, we can choose a set of solutions to be

$$R^+(m,t)=c_+e^{imt}+d_+e^{-imt} ,$$
$$L^+(m,t)=-ic_+e^{imt}+id_+e^{-imt} ,$$
$$L^-(m,t)=c_-e^{imt}+d_-e^{-imt} ,$$
$$R^-(m,t)=-ic_-e^{imt}+id_-e^{-imt} , \tag{4.12}$$

where the constants $c_\pm$ normalize the positive-frequency modes and $d_\pm$ fix the negative-frequency modes.

We can now proceed in the usual manner to calculate some of the currents which appears in this limit. The scalar order parameter (3.1) is given by

$$\chi=\frac{2i}{r^2}[(d_+^*c_+-d_-^*c_-)e^{2imt}+(c_-^*d_--c_+^*d_+)e^{-2imt}] . \tag{4.13}$$

The two components of the quark vector current are seen to be

$$Q_0=2(c_+^*c_++d_+^*d_++c_-^*c_-+d_-^*d_-) ,$$
$$Q_1=-2[(d_+^*c_++d_-^*c_-)e^{2imt}+(c_+^*d_++c^*d_-)e^{-2imt}] . \tag{4.14}$$

The corresponding components of the axial-vector current are

$$Q_0^5=2[(d_+^*c_+-d_-^*c_-)e^{2imt}+(c_+^*d_+-c_-^*d_-)e^{-2imt}]$$
$$Q_1^5=2(c_+^*c_++d_+^*d_+-c_-^*c_--d_-^*d_-) . \tag{4.15}$$

The solutions in (4.12) therefore give an explicit value for the divergence of the axial-vector current

$$\partial^l Q_l^5=\partial_t Q_0^5=4im[(d_+^*c_+-d_-^*c_-)e^{2imt} -(c_+^*d_+-c_-^*d_-)e^{-2imt}] . \tag{4.16}$$

A nonvanishing value of this divergence must be associated with nontrivial topological structure in the gauge fields from (3.9). With $m\neq 0$, the divergence vanishes only when

$$d_+^*c_+=d_-^*c_- . \tag{4.17}$$

We do not need (3.9) to see this structure however. The dynamics of the system are determined by the color currents. The transverse components of the color current represented by J_A and J_S are decoupled in the background field of a Wu-Yang monopole. This can be seen in (3.22) where these components are shown to enter the dynamics proportional to $a(r,t)$. The condition for a Wu-Yang monopole is $a(r,t)=0$ and so the right-hand sides of (3.22c) and (3.22d) vanish. The nontrivial components of the color current which remain are

$$J_0=-(c_+^*c_++d_+^*d_+-c_-^*c_--d_-^*d_-) ,$$
$$J_1=-[(d_+^*c_+-d_-^*c_-)e^{2imt}+(c_+^*d_+-c^*d_-)e^{-2imt}] . \tag{4.18}$$

In the situation described above in (4.19) where the axial-vector current has nonvanishing derivative, we see that we also have a source term in the equation for $\epsilon=r^2E_L$ in (3.26). That is, we have an equation[3]

$$\Box_2\epsilon=-\epsilon^{lm}\partial_l J_m \tag{4.19}$$

with

$$\epsilon^{lm}\partial_l J_m=\partial_0 J_1=-\tfrac{1}{2}\partial^l Q_l^5 . \tag{4.20}$$

In the presence of a Wu-Yang monopole a nonzero component for ϵ automatically leads to a nonvanishing value of $E_i^aB_i^a$. Therefore, if we insist that the solution to (3.22) does not generate a dyon, we must have (4.17) and the quantum anomaly condition is satisfied. The dynamical stability conditions which lead to this constraint can be understood within the classical framework of our simple approach.

Many of the interesting physics questions involving the interaction of SU(2) fermions with a Wu-Yang monopole have been addressed elsewhere but these analyses usually start with the neglect of the fermion mass.[15] We see here that with nonvanishing fermion mass, the condition for the absence of the dynamically dangerous currents lead to (4.17). We can obviously choose to implement this constraint in a number of ways. For example, if we choose c_- and d_- to be real in (4.12) we can implement (4.17) by

$$c_+=e^{i\eta}c_- , \quad d_+=e^{i\eta}d_- , \tag{4.21}$$

which leads to

$$R^+(m,t)=e^{i\eta}L^-(m,t)\ ,\quad L^+(m,t)=e^{i\eta}R^-(m,t)\ , \tag{4.22}$$

or we can choose to relate opposite frequency components

$$c_+=e^{i\eta}d_-\ ,\quad d_+=e^{i\eta}c_-\ , \tag{4.23}$$

which gives

$$\begin{aligned} R^+(m,t)&=e^{i\eta}L^-(-m,t)=e^{i\eta}L^-(m,-t)\ ,\\ L^+(m,t)&=e^{i\eta}R^-(-m,t)=e^{i\eta}R^-(m,-t)\ , \end{aligned} \tag{4.24}$$

and exhibits some nontrivial time-reversal properties. The exact nature of the boundary condition to be chosen obviously depends on further input. However, this simple analysis in terms of the massive SU(2) Dirac equation supports the idea that there can exist large-scale structure in the fermion fields around a grand unified monopole.

V. CONCLUSIONS

The structure of non-Abelian dynamics is nourished by the nonlinearities which appear in its formulation. One of the simplest known classical non-Abelian systems is that which represents an SU(2) gauge theory with spherical symmetry and we have examined that system in two isolated simple limits. The fact that the content of the theory was nontrivial even in these extreme limits is remarkable. Perhaps still more remarkable is the fact that the structure has emerged from considerations involving only the classical limit. The full quantum field theory is bound to be much richer in content.

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