Monopole-antimonopole pair solution of the classical SU(3) Yang-Mills theory

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Inspired by structural similarities between the SU(2) finite-energy and the corresponding SU(3) Wu-Yang-type monopole solutions we develop a procedure that, for the first time, provides us with an explicit SU(3) Wu-Yang-type monopole-antimonopole pair solution. Exploiting these similarities the procedure could hopefully be adapted to the construction of an explicit SU(2) finite-energy monopole-antimonopole saddle-point solution whose existence is already known.

I. INTRODUCTION

The most important feature of the present theory of strong interactions is the confinement of quarks and gluons. Although QCD is widely believed to be the correct theory of strong interactions, confinement has not yet been proven a consequence of the principles of QCD. It is of common belief that a proper treatment of confinement will come about by understanding the complicated nonperturbative structure of the QCD vacuum.

One of the possibilities to bring about permanent confinement is connected to a condensation phenomenon where the vacuum closely resembles the dual analogue of the ground state of a superconductor. Since the superconducting ground state is a condensate of charged particles (Cooper pairs) the QCD ground state should be visualized as a condensate of monopoles.[1]

Except for lattice calculations, almost all investigations of the monopole condensation start with a classical gauge field configuration of finite energy or with the classical interaction between monopoles. Such configurations are well known for a single monopole or antimonopole.[2] However, it seems quite difficult to determine a proper gauge field configuration for the monopole-antimonopole pair. Several authors[3] have studied gauge field configurations of widely separated monopole-antimonopole pairs; but such configurations are rather unsuitable for the investigation of monopole-antimonopole pair condensation, where the monopole and antimonopole are expected to be close together. Taubes and Groisser[4] have proven the existence of monopole-antimonopole saddle-point solutions in SU(2)-Higgs theories, but so far the explicit structure of such solutions is not yet known.

It seems to us that more information about the realization of the SU(2) saddle-point solution should be sought by looking at other, possibly explicit, examples of monopole-antimonopole pair solutions. In this connection we think of Wu-Yang-type monopole-antimonopole pair solutions which, to our knowledge, have not been investigated so far. Although these solutions are singular at the origin the manner they are constructed could yield useful ideas for setting up a practical procedure for finding saddle-point solutions.

In this paper we will present a procedure which provides us with an SU(3) Wu-Yang-type monopole-antimonopole pair solution based on the topological background found by Marciano and Pagels.[5] This solution is valid for arbitrary separations of the monopole and antimonopole.

An important aspect of our procedure is that it takes care of a structural resemblance between the SU(2) finite-energy and the corresponding SU(3) Wu-Yang-type monopole solutions which becomes apparent in the so-called "Abelian" gauge.[6] Exploiting these similarities most steps of our procedure could also be used for constructing the SU(2) monopole-antimonopole saddle-point solution. Of course, some modifications are necessary. Apart from smoothing out the point singularities we know from the proof of Taubes that the relative global gauge between monopole and antimonopole becomes important, an ingredient which has to be incorporate in our construction scheme so as to find the nonsingular solution.

The paper is organized as follows. For a better understanding of the calculation, part of the work of Marciano and Pagels[5] is briefly reviewed in Sec. II. In Sec. III we work out the monopole-antimonopole pair ansatz in the "Abelian" gauge. Finally in Sec. IV we determine the SU(3) Wu-Yang-type monopole-antimonopole pair solution by solving a system of coupled differential equations.

II. MAGNETIC MONOPOLES IN THE PURE SU(3) YANG-MILLS THEORY

If we want to associate a magnetic charge with certain types of solutions of pure SU(N) gauge theories we have to introduce a gauge-invariant electromagnetic field tensor $F_{\mu\nu}$ in terms of the parent fields. This is possible because we can construct from the SU(N) field tensor an octet of scalar fields

$$\Phi^a = d_{abc} G_{\mu\nu}{}^b G^{\mu\nu c} , \tag{2.1}$$

where d_{abc} is the completely symmetric Gell-Mann SU(N) tensor. The definition (2.1) is only valid for SU(N) gauge groups with $N \geq 3$. Then, in analogy to 't Hooft's definition of the electromagnetic field tensor in the SU(2) Higgs-boson theory, $F_{\mu\nu}$ can be written as

$$F_{\mu\nu}=\hat{\Phi}^a G_{\mu\nu}{}^a+\frac{4}{3g}f_{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c$$
$$=\partial_\mu A_\nu^{\rm em}-\partial_\nu A_\mu^{\rm em}+\frac{4}{3g}f_{abc}\hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c\ ,\tag{2.2}$$
$$A_\mu^{\rm em}=\hat{\Phi}^a A_\mu{}^a\ .$$

Here we have introduced the normalized field such that $\hat{\Phi}^a\hat{\Phi}^a=1$ and $\sqrt{3}\,d^{abc}\hat{\Phi}^b\hat{\Phi}^c=\hat{\Phi}^a$. The monopole strength M is defined by

$$M=-\frac{1}{4\pi}\oint_s\tilde{F}_{\mu\nu}dx^\mu\wedge dx^\nu\ ,\tag{2.3}$$

where $dx^\mu\wedge dx^\nu$ denotes the exterior product and $\tilde{F}_{\mu\nu}$ the dual of $F_{\mu\nu}$. In a nonsingular gauge (where $A_\mu^{\rm em}$ has no singular Dirac string) the contribution of the first two terms in $F_{\mu\nu}$ of (2.2) to the magnetic flux vanishes and M is only given in terms of $\hat{\Phi}^a$ alone. Instead, if we choose the singular "Abelian" gauge[7] where all but the three and eight components of $\hat{\Phi}^a$ vanish, $F_{\mu\nu}$ is given by

$$F_{\mu\nu}=\partial_\mu A_\nu^{\rm em}-\partial_\nu A_\mu^{\rm em}\ ,$$
$$A_\mu^{\rm em}=-\tfrac{1}{2}\sqrt{3}\,A_\mu^3+\tfrac{1}{2}A_\mu^8\ ,\tag{2.4}$$

and the monopole strength M is only determined by the Dirac-type string structure.

A static monopole solution of the pure SU(3) gauge field equations that belongs to the SO(3) subembedding in SU(3) is[5]

$$\mathbf{A}_m=\left[\begin{pmatrix}xz\\-yz\\y^2-x^2\end{pmatrix}\lambda_1+\begin{pmatrix}-yz\\-xz\\2xy\end{pmatrix}\lambda_3+\begin{pmatrix}-xy\\x^2-z^2\\yz\end{pmatrix}\lambda_4+\begin{pmatrix}z^2-y^2\\xy\\-xz\end{pmatrix}\lambda_6+\begin{pmatrix}\sqrt{3}yz\\-\sqrt{3}xz\\0\end{pmatrix}\lambda_8\right]\frac{1}{g\sqrt{8}r^3}$$
$$+\left[\begin{pmatrix}-y\\x\\0\end{pmatrix}\lambda_2+\begin{pmatrix}-z\\0\\x\end{pmatrix}\lambda_5+\begin{pmatrix}0\\-z\\y\end{pmatrix}\lambda_7\right]\frac{1-\sqrt{8}}{g\sqrt{8}r^2}\ ,\tag{2.5}$$

where $\partial_\mu A^{\mu a}=0$ and $A_0{}^a=0$; g is the universal coupling constant of the theory. For the normalized scalar field one obtains, with (2.1),

$$\hat{\Phi}_m=\frac{\sqrt{3}}{4}\frac{1}{r}(z\lambda_2-y\lambda_5+x\lambda_7)-\frac{\sqrt{3}}{8}\frac{1}{r^2}\left[2xy\lambda_1+(x^2-y^2)\lambda_3+2xz\lambda_4+2yz\lambda_6+\frac{1}{\sqrt{3}}(x^2+y^2-z^2)\lambda_8\right]\ .\tag{2.6}$$

The corresponding antimonopole solution is simply given by charge conjugation: $\mathbf{A}_{\bar{m}}=-(\mathbf{A}_m)^*$ and $\hat{\Phi}_{\bar{m}}=(\hat{\Phi}_m)^*$.

Of course this solution has a point singularity at the origin which leads to infinite energy; this is in agreement with the Coleman-Derrick theorem according to which there exist no finite-energy stable solution to the pure classical gauge field equations. But since the monopole strength is nonzero the solution will be topologically stable. This means, in the sense in which Coleman defined topological stability,[8] that the solution does not dissipate.

III. THE MONOPOLE-ANTIMONOPOLE PAIR ANSATZ IN THE "ABELIAN" GAUGE

In order to study the $m\bar{m}$ interaction we need an ansatz for the $m\bar{m}$ gauge field configuration which is consistent with the required asymptotic properties of magnetic flux. In the gauge valid for the monopole solution (2.5) these constrains refer only to the normalized scalar field which renders the search for a proper gauge field ansatz more difficult. There fore we will switch over to the singular "Abelian" gauge[7] which is defined by the requirement that the normalized scalar field is constant in coordinate space and diagonal in group space. The advantages of this gauge are that (a) all topological considerations can directly be carried out in terms of the gauge fields and (b) the formulas will be simplified, facilitating the search for the $m\bar{m}$ solution.

In our case the "Abelian" gauge can be realized by the by the gauge transformation

$$U=\exp\left[i\frac{\pi}{4}\lambda_1\right]\exp(-i\theta\lambda_5)\exp(i\phi\lambda_2)\ ,$$

where θ and ϕ are the spherical angles. Applying this gauge transformation to (2.5) and (2.6) we obtain

$$\Phi'=U\hat{\Phi}_m U^{-1}=-\frac{\sqrt{3}}{4}\lambda_3+\tfrac{1}{4}\lambda_8\ ,\tag{3.1a}$$

$$\mathbf{A}'_m=U\mathbf{A}_m U^{-1}-\frac{1}{g}(i\nabla U)U^{-1}$$
$$=\frac{-z}{g\rho r}\lambda_3\hat{e}_\phi-\frac{1}{2gr}(\lambda_4\hat{e}_\phi+\lambda_5\hat{e}_\theta)\ ,\tag{3.1b}$$

where $x=\rho\cos\phi$, $y=\rho\sin\phi$, $z=r\cos\theta$, and $\rho=r\sin\theta$. The corresponding antimonopole formulas are simply given by charge conjugation which is equivalent to gauge transforming the antimonopole solution by U^* instead of U:

$$\hat{\Phi}'=(\hat{\phi}'_m)^*=-\frac{\sqrt{3}}{4}\lambda_3+\tfrac{1}{4}\lambda_8\ , \tag{3.2a}$$

$$\mathbf{A}'_{\bar{m}}=-(\mathbf{A}'_m)^* = \frac{z}{g\rho r}\lambda_3\hat{e}_\phi-\frac{1}{2gr}(-\lambda_4\hat{e}_\phi+\lambda_5\hat{e}_\theta)\ . \tag{3.2b}$$

In both cases the electromagnetic field tensor is given by (2.4) with

$$\mathbf{A}_m^{\text{em}}=\frac{\sqrt{3}}{g}\frac{z}{\rho r}\hat{e}_\phi \quad\text{and}\quad \mathbf{A}_{\bar{m}}^{\text{em}}=-\frac{\sqrt{3}}{g}\frac{z}{\rho r}\hat{e}_\phi \tag{3.3}$$

for the monopole and antimonopole, respectively.

Note that if we transform the antimonopole solution with U and the monopole solution with U^* the resultant gauge fields belong to another SO(3) subalgebra characterized by λ_6 and λ_7 instead of λ_4 and λ_5. As a matter of fact $\hat{\Phi}$ is the same for all the gauge fields belonging to the same SO(3) subalgebra because the gauge transformation "transfers" all topological informations from the scalar field to the gauge fields. The λ_3 part of the gauge field carries the topological string which comes solely from the term $(i\nabla U)U^{-1}$; the absence of other contributions due to $U\mathbf{A}U^{-1}$ and proportional to λ_3 and λ_8 is related to the condition $\nabla\cdot\mathbf{A}=0$ of the original gauge.[9]

Just as the monopole and antimonopole solutions (3.1b) and (3.2b) so the $m\bar{m}$ solution must belong to the same SO(3) subalgebra; thus we make the following ansatz for the $m\bar{m}$ solution:

$$\mathbf{A}'_{m\bar{m}}=\mathbf{s}\lambda_3+\mathbf{A}_4\lambda_4+\mathbf{A}_5\lambda_5\ , \tag{3.4}$$

where $\mathbf{s}$ stands for the topological string function resulting from the $(i\nabla U)U^{-1}$ term of the corresponding singular gauge transformation for the $m\bar{m}$ solution. The electromagnetic gauge field is then simply given by

$$\mathbf{A}_{m\bar{m}}^{\text{em}}=\hat{\Phi}'^{a}\mathbf{A}'^{a}_{m\bar{m}}=-\sqrt{3}\,\mathbf{s}\ . \tag{3.5}$$

On the other hand the electromagnetic gauge group is U(1) and the electromagnetic gauge field of the $m\bar{m}$ pair must therefore be the sum of $\mathbf{A}_m^{\text{em}}$ and $\mathbf{A}_{\bar{m}}^{\text{em}}$; thus for a monopole sitting at $z=a$ and an antimonopole at $z=-b$ we obtain, from (3.3),

$$\mathbf{A}_{mm}^{\text{em}}=-\frac{\sqrt{3}}{g}\left[-\frac{z-a}{\rho r_-}+\frac{z+b}{\rho r_-}\right]\hat{e}_\phi\ , \tag{3.6}$$

where $r_-=[\rho^2+(z-a)^2]^{1/2}$ and $r_+=[\rho^2+(z+b)^2]^{1/2}$. Comparing (3.5) and (3.6) it follows immediately that

$$\mathbf{s}=\left[-\frac{z-a}{g\rho r_-}+\frac{z+b}{g\rho r_+}\right]\hat{e}_\phi\ . \tag{3.7}$$

Consequently we are left to determine $\mathbf{A}_4$ and $\mathbf{A}_5$. To that end we discuss the following two limiting cases: (i) $b\to\infty$ for which $\mathbf{A}'_{m\bar{m}}$ should describe a monopole and (ii) $a\to\infty$ for which we should be left with an antimonopole solution. Given $\mathbf{s}$ by (3.7) we obtain, from (3.4),

$$\text{(i)}\quad \lim_{b\to\infty}\mathbf{A}'_{m\bar{m}}=\frac{r_--(z-a)}{g\rho r_-}\lambda_3\hat{e}_\phi+\lim_{b\to\infty}(\mathbf{A}_4\lambda_4+\mathbf{A}_5\lambda_5)\ ,$$

$$\text{(ii)}\quad \lim_{a\to\infty}\mathbf{A}'_{m\bar{m}}=\frac{r_++(z+b)}{g\rho r_+}\lambda_3\hat{e}_\phi+\lim_{a\to\infty}(\mathbf{A}_4\lambda_4+\mathbf{A}_5\lambda_5)\ .$$

Apart from the different locations of the monopole and antimonopole on the z axis we notice that the string functions of (i) and (ii) are not those of the monopole and antimonopole solutions given by (3.1b) and (3.2b). But the difference between the corresponding string functions is only a relative Abelian gauge transformation $\chi=\exp(i\phi\lambda_3)$ which leaves $\hat{\Phi}'$ invariant. Thus the monopole and antimonopole gauge fields we have to identify with (i) and (ii) are

$$\mathbf{A}_m=\frac{r_--(z-a)}{g\rho r_-}\lambda_3\hat{e}_\phi-\frac{1}{2gr_-}(\cos\phi\lambda_4-\sin\phi\lambda_5)\hat{e}_\phi-\frac{z-a}{2g{r_-}^2}(\sin\phi\lambda_4+\cos\phi\lambda_5)\hat{e}_\rho+\frac{\rho}{2g{r_-}^2}(\sin\phi\lambda_4+\cos\phi\lambda_5)\hat{e}_z \tag{3.8a}$$

and

$$\mathbf{A}_{\bar{m}}=\frac{r_++(z+b)}{g\rho r_+}\lambda_3\hat{e}_\phi+\frac{1}{2gr_+}(\cos\phi\lambda_4-\sin\phi\lambda_5)\hat{e}_\phi-\frac{z+b}{2g{r_+}^2}(\sin\phi\lambda_4+\cos\phi\lambda_5)\hat{e}_\rho+\frac{\rho}{2g{r_+}^2}(\sin\phi\lambda_4+\cos\phi\lambda_5)\hat{e}_z\ , \tag{3.8b}$$

where, for further purposes, we switched over to the cylindrical coordinates ρ, ϕ, and z.

Note, that upon the replacement $a\to -b$ and $b\to -a$ the monopole and antimonopole gauge fields given by (3.8a) and (3.8b) are not related via charge conjugation. Apparently, the string function $\mathbf{s}$ fixes the so-far unrestricted relative local gauge between the monopole and the antimonopole of the $m\bar{m}$ configuration in a definite way. What is still left free is a relative global gauge which would become important when we consider finite-energy configurations.

Finally, if we assume cylindrical symmetry around the string axis and note that for $a\to\infty$ or $b\to\infty$ the resulting gauge fields (3.8a) and (3.8b) have exactly the same ϕ dependence it seems reasonable to use this ϕ dependence also for the $m\bar{m}$ ansatz; thus (3.4) goes over into

$$\mathbf{A}_{m\bar{m}}=s(\rho,z)\frac{\lambda_3}{2}\hat{e}_\phi+\left[\cos\phi a_\phi^{(4)}(\rho,z)\frac{\lambda_4}{2}-\sin\phi a_\phi^{(5)}(\rho,z)\frac{\lambda_5}{2}\right]\hat{e}_\phi + \left[\sin\phi a_\rho^{(4)}(\rho,z)\frac{\lambda_4}{2}+\cos\phi a_\rho^{(5)}(\rho,z)\frac{\lambda_5}{2}\right]\hat{e}_\rho+\left[\sin\phi a_z^{(4)}(\rho,z)\frac{\lambda_4}{2}+\cos\phi a_z^{(5)}(\rho,z)\frac{\lambda_5}{2}\right]\hat{e}_z \tag{3.9}$$

with

$$s(\rho,z)=2\left[-\frac{z-a}{g\rho r_-}+\frac{z+b}{g\rho r_+}\right].$$

Here $a_\phi^{(i)}$, $a_\rho^{(i)}$, $a_z^{(i)}$, $i=4,5$, are still unknown functions which we have to determine from the gauge field equations and the fact that $\hat{\Phi}'$ given by (2.1) must have the "Abelian" gauge form, namely, $\hat{\Phi}'=-(\sqrt{3}/4)\lambda_3+\frac{1}{4}\lambda_8$.

IV. THE MONOPOLE-ANTIMONOPOLE PAIR SOLUTION

We consider the static case and take $A_0=0$; therefore the gauge field equations are given by

$$0=\partial_i\partial_i A_j^a(x)-\partial_j\partial_i A_i^a(x)+gf_{abc}\{A_i^b(x)[2\partial_i A_j^c(x)-\partial_j A_i^c(x)]+[\partial_i A_i^b(x)]A_j^c(x)\}$$
$$+g^2 f_{abc}f_{ceh}A_i^b(x)A_i^e(x)A_j^h(x),\quad a=1,2,\ldots,8,\quad i=1,2,3\ . \tag{4.1}$$

Furthermore, we obtain, from (2.1),

$$G_{ij}{}^3(x)G_{ij}{}^a(x)-\frac{1}{\sqrt{3}}G_{ij}{}^8(x)G_{ij}{}^a(x)=0,\quad a=4,5 \tag{4.2}$$

which guarantees that $\hat{\Phi}'$ has the "Abelian" gauge form. Here, we have already used the fact that our ansatz has only λ_3, λ_4, and λ_5 components.

Inserting the ansatz (3.9) into (4.1) and (4.2) a lengthy but more or less straightforward calculation leads us to a system of differential equations that must be solved. First of all one derives from (4.1) a useful relation between the coefficient of λ_4 and λ_5: namely,

$$\frac{a_\phi^{(5)}(\rho,z)}{a_\phi^{(4)}(\rho,z)}=\frac{a_\rho^{(5)}(\rho,z)}{a_\rho^{(4)}(\rho,z)}=\frac{a_z^{(5)}(\rho,z)}{a_z^{(4)}(\rho,z)}\ . \tag{4.3}$$

This encourages us to use the notation $a_i^{(5)}(\rho,z)=f(\rho,z)a_i^{(4)}(\rho,z)$ for $i=\rho,\phi,z$ and $a_\phi^{(4)}(\rho,z)=a(\rho,z)/\rho, a_\rho^{(4)}(\rho,z)=h(\rho,z)a(\rho,z)/\rho, a_z^{(4)}(\rho,z)=k(\rho,z)a(\rho,z)/\rho$, $s(\rho,z)=\tilde{s}(\rho,z)/\rho$. We thus obtain the following set of equations:

$$(\partial_z\tilde{s})\partial_\rho f-(\partial_\rho\tilde{s})\partial_z f=0\ ,\qquad (\partial_z\tilde{s})\partial_\rho a-(\partial_\rho\tilde{s})\partial_z a=0\ ; \tag{4.4a}$$

$$\partial_\rho a=(1-\tfrac{1}{2}f\tilde{s})ha/\rho\ ,\qquad \partial_z a=(1-\tfrac{1}{2}f\tilde{s})ka/\rho\ ; \tag{4.4b}$$

$$\partial_z h-\partial_\rho k=-k/\rho\ ; \tag{4.4c}$$

$$\partial_\rho\tilde{s}=2fha^2/\rho,\quad \partial_z\tilde{s}=2fka^2/\rho\ ; \tag{4.4d}$$

$$\partial_\rho f=\tfrac{1}{2}\tilde{s}h(f^2-1)/\rho\ ,\qquad \partial_z f=\tfrac{1}{2}\tilde{s}k(f^2-1)/\rho\ ; \tag{4.4e}$$

$$\partial_\rho\left[\frac{1}{\rho}\partial_\rho\tilde{s}\right]+\frac{1}{\rho}\partial_z{}^2\tilde{s}=0\ . \tag{4.4f}$$

For simplicity we temporarily suppress the arguments ρ and z.

Although this system of differential equations looks quite complicated it can be solved step by step. The general solutions of (4.4a) for any $\tilde{s}$ can be written as $f=F(\tilde{s})$ and $a=A(\tilde{s})$ where F and A are arbitrary functions of $\tilde{s}$. Injecting F and A into (4.4b) and solving these equations for h and k yields

$$h=\frac{\rho A'\partial_\rho\tilde{s}}{(1-\frac{1}{2}F\tilde{s})A}\quad\text{and}\quad k=\frac{\rho A'\partial_z\tilde{s}}{(1-\frac{1}{2}F\tilde{s})A}\ . \tag{4.5}$$

Here the prime denotes differentiation with respect to $\tilde{s}$. By a simple calculation we can convince ourselves that (4.5) is consistent with (4.4c). Next, we insert (4.5) in (4.4d) and obtain

$$(A')^2=\frac{1}{F}(1-\tfrac{1}{2}F\tilde{s})\ . \tag{4.6}$$

If we now impose $f=F=\pm1$ which, as can readily be seen, satisfies (4.4e) as well as (4.4a), Eq. (4.6) can easily be integrated leading to

$$A^2=F\tilde{s}-\tfrac{1}{4}(F\tilde{s})^2+C\ , \tag{4.7}$$

where C is an arbitrary integration constant. Note that (4.7) is valid only if

$$-\sqrt{1+C}\le\tfrac{1}{2}F\tilde{s}-1\le\sqrt{1+C}\ . \tag{4.8}$$

To summarize the results so far, we have expressed all unknown functions in terms of $\tilde{s}$ [see (4.5) and (4.7)] and we have verified that this is consistent with Eqs. (4.4a)–(4.4e). What remains to be done is to verify the validity of (4.4f) and (4.8) for the string function $s(\rho,z)$ appearing in (3.9). Since $s(\rho,z)$ is nothing else but the Dirac $m\overline{m}$ string function it, except on the string, obviously solves the Laplace equation (4.4f). Equation (4.8) can easily be analyzed for all possible z values; the result is that the inequality holds for $s(\rho,z)$ in the case $F=+1$ and $C\ge0$. But since for physical reasons the gauge field should vanish for $|\mathbf{r}|\to\infty$ one finds that only $C=0$ is acceptable. Finally, putting all things together we obtain

$$s(\rho,z)=\frac{2}{g}\left[-\frac{z-a}{\rho r_-}+\frac{z+b}{\rho r_+}\right],\quad a(\rho,z)=\pm\left[\left[-\frac{z-a}{r_-}+\frac{z+b}{r_+}\right]\left[2+\frac{z-a}{r_-}+\frac{z+b}{r_+}\right]\right]^{1/2},$$
$$h(\rho,z)=\frac{\rho^2[(z-a)r_+{}^3-(z+b)r_-{}^3]}{r_+r_-[-(z-a)r_++(z+b)r_-][2r_+r_-+(z-a)r_--(z+b)r_+]},\tag{4.9}$$
$$k(\rho,z)=\frac{\rho^3(r_-{}^3-r_+{}^3)}{r_+r_-[-(z-a)r_++(z+b)r_-][2r_+r_-+(z-a)r_--(z+b)r_+]},$$

and $f=1$. Thus in the "Abelian" gauge the SU(3) Wu-Yang-type $m\bar{m}$ solution is given by

$$\mathbf{A}'_{m\bar{m}}=s(\rho,z)\frac{\lambda_3}{2}\hat{e}_\phi+\frac{a(\rho,z)}{\rho}\left[\cos\phi\frac{\lambda_4}{2}-\sin\phi\frac{\lambda_5}{2}\right]\hat{e}_\phi+\frac{a(\rho,z)}{\rho}\left[\sin\phi\frac{\lambda_4}{2}+\cos\phi\frac{\lambda_5}{2}\right][h(\rho,z)\hat{e}_\rho+k(\rho,z)\hat{e}_z],\tag{4.10}$$

where $s(\rho,z)$, $a(\rho,z)$, $h(\rho,z)$, and $k(\rho,z)$ are as in (4.9). Both signs are possible for $a(\rho,z)$; the two corresponding gauge fields are simply related via a global Abelian gauge transformation $\exp(in\pi\lambda_3)$ with $n\in N$.

Of course the energy of our solution (4.10) is infinite but by subtracting the infinite self-energy of each monopole we can calculate the interaction energy

$$H_{\rm int}=\int d^3r[\mathbf{B}^a_{m\bar{m}}\mathbf{B}^a_{m\bar{m}}-(\mathbf{B}^a_m\mathbf{B}^a_m+\mathbf{B}^a_{\bar{m}}\mathbf{B}^a_{\bar{m}})]\ .\tag{4.11}$$

In the singular "Abelian" gauge the magnetic field strength $\mathbf{B}^a$ is given by[10]

$$\mathbf{B}^a=-\nabla\times\mathbf{A}^a-\tfrac{1}{2}gf_{abc}(\mathbf{A}^b\times\mathbf{A}^c)-\mathbf{B}^a_s\ ,\tag{4.12}$$

where $\mathbf{B}^a_s$ cancels the fictitious magnetic field strength resulting from the gauge string. It turns out that $\mathbf{B}^a_{m\bar{m}}$ is purely Abelian and essentially nothing else but the magnetic field strength of a Dirac monopole-antimonopole pair:

$$\mathbf{B}^a_{m\bar{m}}=\frac{\sqrt{3}}{4\rho}\{\hat{e}_\rho\partial_z[\rho s(\rho,z)]-\hat{e}_z\partial_\rho[\rho s(\rho,z)]\}(\sqrt{3}\,\delta^{a3}-\delta^{a8})\ .\tag{4.13}$$

(Note that the magnetic field strength is not gauge invariant; thus the Abelian character of $\mathbf{B}^a_{m\bar{m}}$ is a property of the singular "Abelian" gauge.)

Therefore the resulting interaction energy is, as expected in the case of massless particles, the Coulomb potential

$$H_{\rm int}=4\pi\frac{(-\sqrt{3/g}\,)(\sqrt{3/g}\,)}{a+b}\ .\tag{4.14}$$

The same result has been obtained for widely separated monopole-antimonopole pairs in SU(2) Higgs-boson theories.[3]

V. REMARKS

We have found a monopole-antimonopole pair solution for the pure SU(3) Yang-Mills equations. The calculation, complicated to start with, can be simplified using the "Abelian" gauge. This enabled us to extract the topological and group-theoretical structure of the monopole-antimonopole gauge field. We were then left with the problem of solving a system of coupled differential equations. This was done only for the monopole-antimonopole string function. But it is very likely that other interesting multimonopole solutions can be constructed by a careful choice of topological string functions. Such solutions will be discussed in a forthcoming paper.

The "Abelian" gauge expression for the monopole-antimonopole pair is, except on the string, a solution of the gauge field equations. Since the field equations are gauge invariant this proves that the corresponding expression in the nonsingular "non-Abelian" gauge will also be a solution of these equations over all space (except for the locations of the monopole and antimonopole). A "non-Abelian" gauge field configuration can be realized, for example, via the gauge transformation

$$U=\exp\left[i\frac{\pi}{4}\lambda_1\right]\exp(i\theta\lambda_5)\exp(i\phi\lambda_2)\ ,$$

where θ is the angle between $\mathbf{r}_-$ and $\mathbf{r}_+$.

Our solution is not yet suitable for the investigation of an eventual monopole-antimonopole pair condensation in the nonperturbative vacuum. To study such a condensation requires a consistent procedure for smeering out the point singularities and thus producing finite-energy configurations. Unfortunately such a procedure is not presently available for pure Yang-Mills theory.

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