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Generalized point-splitting method and Bjorken-Johnson-Low limit for the commutator anomaly

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Faddeev's method for calculating in perturbation theory the fixed-time commutator anomaly of a non-Abelian chiral gauge theory is investigated for both $1+1$ and $1+3$ dimensions. We show that this method gives results in agreement with the Bjorken-Johnson-Low (BJL) procedure in $1+1$ dimensions, while in $1+3$ dimensions it fails to give a well-defined answer. We also present a simplified version of our BJL calculation of the commutator anomaly. The result explicitly shows how the latter is related to the anomalous divergence in the same theory.

I. INTRODUCTION

The relation between the index theorem and non-Abelian anomalies has been discussed by many authors.[1] The anomaly in the divergence of the fermion source current in a $(1+n)$-dimensional non-Abelian chiral gauge theory is related to the Chern-Pontryagin density defined in $3+n$ dimensions. It is the infinitesimal form of the 1-cocycle obtained from the Chern-Pontryagin density by a well-known mathematical procedure, a dimensional descent by two units effected by the coboundary operator. Naturally, it can be suggested that the infinitesimal 2-cocycle, coming from the same Chern-Pontryagin density by a further descent by one unit, hence defined in n dimensions, gives an anomalous term $S_{ab}(\mathbf{x},\mathbf{y})$ in the commutator of gauge group generators $G_a(\mathbf{x})$ in the same anomalous gauge theory.[2,3] Thus, one is led to the following, cohomologically based conjecture for the commutator:

$$i[G_a(\mathbf{x}),G_b(\mathbf{y})]=f_{abc}G_c\delta(\mathbf{x}-\mathbf{y})+S_{ab}(\mathbf{x},\mathbf{y}) \tag{1.1a}$$

with

$$S_{ab}(x^1,y^1)=\frac{1}{4\pi}\mathrm{tr}\{T^a,T^b\}\delta'(x^1-y^1) \tag{1.1b}$$

in 1 + 1 dimensions and

$$S_{ab}(\mathbf{x},\mathbf{y})=-\frac{i}{24\pi^2}\epsilon^{ijk}\mathrm{tr}\{T^a,T^b\}\partial_i A_j\partial_k\delta(\mathbf{x}-\mathbf{y}) \tag{1.1c}$$

in 1 + 3 dimensions. Here the T^a are anti-Hermitian representation matrices of the group:

$$[T^a,T^b]=f_{abc}T^c \quad \text{and} \quad A^\mu=A^\mu_a T^a .$$

The generator $G_a(\mathbf{x})$ is composed of two parts:

$$G_a(\mathbf{x})=\delta_a(\mathbf{x})+\rho_a(\mathbf{x}) , \tag{1.2}$$

where $\delta_a(\mathbf{x})$ generates a gauge transformation on the gauge fields and $\rho_a(\mathbf{x})$ is the fermionic charge density which generates a gauge transformation on the fermionic degrees of freedom. Formally,

$$\delta_a(\mathbf{x})=-(\mathbf{D}\cdot\mathbf{E})_a(\mathbf{x}) , \tag{1.3}$$

$$\rho_a(\mathbf{x})=-i\psi^\dagger(\mathbf{x})T^a\frac{1-i\gamma^5}{2}\psi(\mathbf{x}) , \tag{1.4}$$

where

$$\gamma^5=i\gamma^0\gamma^1 \text{ in } 1+1 \text{ dimensions}$$
$$=\gamma^0\gamma^1\gamma^2\gamma^3 \text{ in } 1+3 \text{ dimensions} .$$

E^i_a is the non-Abelian electric gauge field F^{i0}_a and D_i is the gauge-covariant gradient. The vanishing of G_a is a constraint required by gauge invariance. The additional anomalous term in the commutator S_{ab} raises difficulties with implementing the constraint, since S_{ab} need not vanish with G_a. Thus, the anomalous commutator shows that imposing gauge invariance in an anomalous gauge theory is problematical.

The cohomological conjecture has been verified both in 1 + 1 dimensions and 1 + 3 dimensions by the Bjorken-Johnson-Low[4] (BJL) computation of the equal-time commutator.[5–7] In the literature,[8] there also appear derivations of the anomalous commutator from a fixed-time Hamiltonian formalism where chiral fermions, moving in a prescribed external gauge field, are canonically quantized. [I. Frenkel and I. Singer who are also cited did not complete the derivation of the commutator anomaly (private communication).] It has been claimed that this method also verifies the cohomological conjecture.

At first it appears unlikely that a fixed-time derivation of an anomalous commutator can produce the correct result. Research of two decades ago,[9] when anomalous commutators were extensively calculated in perturbation theory, indicates that the fixed-time approach *does not* yield sensible results and only the BJL procedure is reliable. The conclusion of that research is that the anomaly in the space-time theory should be calculated by a space-time method such as the BJL limit, rather than by canonical fixed-time procedures. However, the old experience is solely with fermion bilinears, whereas now we are interested in fermion bilinears supplemented by gauge field contributions δ_a above. One might therefore hope that the commutator of the total generator, which is group-theoretically significant, when computed at fixed time essentially agrees with the BJL result.

It is the purpose of this paper to examine carefully the fixed-time calculation in view of the above remarks. We conclude that the fixed-time procedure, as developed thus far, does not verify the cohomological conjecture, contrary to published assertions. A twofold difficulty appears. First, a computational oversight changes the results of Ref. 8. (This was communicated privately to the author by L. Faddeev through R. Jackiw.) Second, there is an ambiguity in the handling of products of distributions, which gives rise to an arbitrariness in the final result. (Also, we remember that in our first investigation of this problem,[5] we showed that for an external, nondynamical gauge field, even the BJL derivation fails.)

In Sec. II we review the method of Ref. 8. In Sec. III we apply the procedure to (1 + 1)-dimensional space-time, and obtain agreement with the BJL procedure. In Sec. IV the (1 + 3)-dimensional model, already discussed in Ref. 8, is reexamined. We confirm the existence of a further term, which was overlooked, and we point out the presence of the above-mentioned ambiguity. Finally, in Sec. V, we review our BJL calculation,[5,6] now presenting it in a compact fashion which quickly yields the desired result. The conclusion is that, thus far at least, the BJL method is the only way to verify the cohomological conjecture.

II. GENERALIZED POINT-SPLITTING METHOD

In order to define in a physical Hilbert space $\mathscr{H}$ an operator $\mathscr{J}(f)$ which is formally given by

$$\mathscr{J}(f)=-i\int d^nx\,\psi^\dagger(\mathbf{x})f(\mathbf{x})\frac{1-i\gamma^5}{2}\psi(\mathbf{x})\ , \tag{2.1}$$

we introduce a family of smooth kernels $F(\mathbf{x},\mathbf{y})$ which are matrices in the internal-symmetry space and have the local limit $f(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})$. For each such $F(\mathbf{x},\mathbf{y})$, define $\mathscr{J}(F)$ by

$$\mathscr{J}(F)=-i\int d^nx\,d^ny\,\psi^\dagger(\mathbf{x})F(\mathbf{x},\mathbf{y})\frac{1-i\gamma^5}{2}\psi(\mathbf{y})\ . \tag{2.2}$$

For later use we also define $\tilde{F}(\mathbf{p},\mathbf{q})$ and $\tilde{f}(\mathbf{p})$ as the Fourier transformations of $F(\mathbf{x},\mathbf{y})$ and $f(\mathbf{x})$, respectively:

$$\tilde{F}(\mathbf{p},\mathbf{q})=\int d^nx\,d^ny\,e^{i\mathbf{p}\cdot(\mathbf{x}+\mathbf{y})/2}e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}F(\mathbf{x},\mathbf{y})\ , \tag{2.3}$$

$$\tilde{f}(\mathbf{p})=\int d^nx\,e^{i\mathbf{p}\cdot\mathbf{x}}f(\mathbf{x})\ . \tag{2.4}$$

The local limit of $\tilde{F}(\mathbf{p},\mathbf{q})$ is $\tilde{f}(\mathbf{p})$. The above $\mathscr{J}(F)$ is well defined in $\mathscr{H}$ and satisfies

$$i[\mathscr{J}(F),\mathscr{J}(G)]=\mathscr{J}([F,G])\ , \tag{2.5}$$

where

$$[F,G](\mathbf{x},\mathbf{y})=\int d^nz[F(\mathbf{x},\mathbf{z})G(\mathbf{z},\mathbf{y})-G(\mathbf{x},\mathbf{z})F(\mathbf{z},\mathbf{y})]\ . \tag{2.6}$$

The vacuum expectation value (VEV) of $\mathscr{J}(F)$ can be expressed as

$$\langle\mathscr{J}(F)\rangle_A=\int d^nx\,d^ny\,\mathrm{tr}P(\mathbf{x},\mathbf{y})F(\mathbf{y},\mathbf{x})=\mathrm{tr}FP\ , \tag{2.7}$$

where

$$P(\mathbf{x},\mathbf{y})=i\frac{1-i\gamma^5}{2}\langle\psi(\mathbf{x})\psi^\dagger(\mathbf{y})\rangle \tag{2.8}$$

is the fixed-time VEV. In the local limit, we need $P(\mathbf{x},\mathbf{y})$ only for $\mathbf{x}\approx\mathbf{y}$, which, however, diverges at $\mathbf{x}=\mathbf{y}$. We extract $P^{\mathrm{inf}}(\mathbf{x},\mathbf{y})$ so that $P-P^{\mathrm{inf}}$ possesses a local limit, and define $\mathscr{J}(f)$ by

$$\mathscr{J}(f)=\text{local limit }\mathscr{J}_{\mathrm{reg}}(F)\ , \tag{2.9}$$

where $\mathscr{J}_{\mathrm{reg}}(F)=\mathscr{J}(F)-\mathrm{tr}FP^{\mathrm{inf}}$. Although the local limit of $\mathscr{J}(F)$ does not exist, after the subtraction such a limit may be taken. We also define $T(f)$ by

$$\begin{aligned}T(f)&=\int d^nx\,f_a(\mathbf{x})\delta_a(\mathbf{x})\\&=-i\int d^nx(D_if)_a\frac{\delta}{\delta A_i^a(\mathbf{x})}\ .\end{aligned} \tag{2.10}$$

With the definition given above, one obtains

$$i[T(f)+\mathscr{J}_{\mathrm{reg}}(F),T(g)+\mathscr{J}_{\mathrm{reg}}(G)]=T([f,g])+\mathscr{J}_{\mathrm{reg}}([F,G])+a(F,G)\ , \tag{2.11a}$$

$$\begin{aligned}a(F,G)&\equiv i[T(f),\mathscr{J}_{\mathrm{reg}}(G)]+i[\mathscr{J}_{\mathrm{reg}}(F),T(g)]+\mathscr{J}([F,G])-\mathscr{J}_{\mathrm{reg}}([F,G])\\&=-\int d^nx\left[(D_if)_a\frac{\delta}{\delta A_i^a(\mathbf{x})}(\mathrm{tr}GP^{\mathrm{inf}})-(D_ig)_a\frac{\delta}{\delta A_i^a(\mathbf{x})}(\mathrm{tr}FP^{\mathrm{inf}})\right]+\mathrm{tr}[F,G]P^{\mathrm{inf}}\ ,\end{aligned} \tag{2.11b}$$

and the local limit of $a(F,G)$ is claimed to be the anomaly $a(f,g)$. Note that $P^{\mathrm{inf}}(\mathbf{x},\mathbf{y})$ is ambiguous up to finite terms which leave the anomaly ambiguous. But such an ambiguity is not important, since it corresponds to a trivial 2-cocycle. A more serious ambiguity exists: we shall see that in (1 + 1)-dimensional theory $a(F,G)$ is well defined in the local limit, but in the (1 + 3)-dimensional theory such a limit does not exist, and no unique value can be given.

We first derive the general form of $P(\mathbf{x},\mathbf{y})$. From Eq. (2.8), one obtains

$$P(\mathbf{x},\mathbf{y})=\lim_{y_0-x_0\to0^+}\left[\frac{1-i\gamma^5}{2}[S(x,y)]\gamma^0\right]\ , \tag{2.12}$$

where $iS(x,y)$ is the fermion propagator satisfying

$$\left[\partial\!\!\!/+A\!\!\!/\frac{1-i\gamma^5}{2}\right]S(x,y)=\delta^{1+n}(x-y)\ . \tag{2.13}$$

The fermion propagator $iS(x,y)$ can be expanded in powers of A_μ:

$$S(x,y)=S_F(x,y)-\int d^{n+1}z\, S_F(x,z)A\!\!\!/(z)\frac{1-i\gamma^5}{2}S_F(z,y)$$
$$+\int d^{n+1}z_1 d^{n+1}z_2 S_F(x,z_1)A\!\!\!/(z_1)\frac{1-i\gamma^5}{2}S_F(z_1,z_2)A\!\!\!/(z_2)\frac{1-i\gamma^5}{2}S_F(z_2,y)+\cdots\ , \qquad (2.14)$$

where

$$S_F(x,y)=\int\frac{d^{n+1}p}{(2\pi)^{n+1}}e^{-ip\cdot(x-y)}\frac{i}{p\!\!\!/+i\epsilon}\ . \qquad (2.15)$$

We only consider the parity-odd portion as we did in the BJL limit procedure.[5,6]

III. (1 + 1)-DIMENSIONAL CASE

In the (1 + 1)-dimensional case the first term $S_F(x,y)$ is the only term which diverges as x approaches y. The second term appears to diverge logarithmically, but the parity-odd part converges to a finite value as x approaches y. After a little algebra one gets

$$S_F(x,y)\,|_{x_0=y_0}=-\frac{\gamma^1}{4\pi}\int_{-\infty}^{\infty}dp\,\epsilon(p)e^{ip(x^1-y^1)}\ , \qquad (3.1)$$

where

$$\epsilon(p)=\begin{cases}1 & \text{if } p>0\ ,\\ -1 & \text{if } p<0\ .\end{cases}$$

Substituting this into Eq. (2.14) we obtain

$$P^{\rm inf}(x^1,y^1)=\frac{i}{2}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\epsilon(p)e^{ip(x^1-y^1)}\ . \qquad (3.2)$$

Once $P^{\rm inf}(x^1,y^1)$ is found, the calculation of $a(F,G)$ in Eq. (2.12b) is straightforward:

$$a(F,G)=\frac{i}{2}{\rm tr}\int dx^1dy^1\int_{-\infty}^{\infty}\frac{dp}{2\pi}\epsilon(p)e^{ip(x^1-y^1)}\int dz^1[F(y^1,z^1)G(z^1,x^1)-G(y^1,z^1)F(z^1,x^1)]\ . \qquad (3.3)$$

In momentum space, this becomes

$$a(F,G)=\frac{i}{2}\int\frac{dq\,dp}{(2\pi)^2}\epsilon(p){\rm tr}\left[\widetilde{F}\left[q,-p-\frac{q}{2}\right]\widetilde{G}\left[-q,-p-\frac{q}{2}\right]-\widetilde{G}\left[q,-p-\frac{q}{2}\right]\widetilde{F}\left[-q,-p-\frac{q}{2}\right]\right]\ . \qquad (3.4)$$

If we take the local limit of $a(F,G)$ at this stage, we get an indefinite expression: infinity minus infinity. In fact, we can transform Eq. (3.4) so that the local limit may be safely taken. After changing the integration variable p into $-k-q/2$, and using the trace property, we obtain

$$a(F,G)=\frac{i}{2}\int\frac{dq\,dk}{(2\pi)^2}\epsilon\left[-\frac{q}{2}-k\right]{\rm tr}[\widetilde{F}(q,k)\widetilde{G}(-q,k)-\widetilde{F}(-q,k)\widetilde{G}(q,k)]\ . \qquad (3.5)$$

Finally, changing the integration variable q to $-q$ for the second term gives

$$a(F,G)=-\frac{i}{2}\int\frac{dq\,dk}{(2\pi)^2}\left[\epsilon\left[k+\frac{q}{2}\right]-\epsilon\left[k-\frac{q}{2}\right]\right]{\rm tr}\widetilde{F}(q,k)\widetilde{G}(-q,k)$$
$$=-i\int\frac{dq}{2\pi}\int_{-q/2}^{q/2}\frac{dk}{2\pi}{\rm tr}\widetilde{F}(q,k)\widetilde{G}(-q,k) \qquad (3.6)$$

which yields in the local limit

$$a(f,g)=-\frac{i}{2\pi}\int\frac{dq}{2\pi}q\,{\rm tr}\widetilde{f}(q)\widetilde{g}(-q)$$
$$=-\frac{1}{2\pi}\int dx^1{\rm tr}f(x^1)g'(x^1)\ . \qquad (3.7)$$

Also, we have

$$[T(f),\mathscr{J}(g)]=[T(g),\mathscr{J}(f)]=0\ , \qquad (3.8)$$

because $P^{\rm inf}(x^1,y^1)$ is independent of the gauge potential. In summary,

$$i[\delta_a(x^1),\delta_b(y^1)]=f_{abc}\delta_c\delta(x^1-y^1)\ , \qquad (3.9a)$$

$$i[\delta_a(x^1),\rho_b(y^1)]=0\ , \qquad (3.9b)$$

$$i[\rho_a(x^1),\rho_b(y^1)]=f_{abc}\rho_c\delta(x^1-y^1)-\frac{1}{2\pi}{\rm tr}T^aT^b\delta'(x^1-y^1)\ . \qquad (3.9c)$$

The results in Eqs. (3.9a)–(3.9c) agree with the BJL limit calculation.[6] They all combine to give Eq. (1.1b) which was constructed from a cohomological consideration. Note that we use no special property of $F(x^1,y^1)$ in order to get the anomaly except that its local limit is $f(x^1)\delta(x^1-y^1)$. However, as we show in the next section, this nice feature does not occur in the (1 + 3)-dimensional case.

IV. (1 + 3)-DIMENSIONAL CASE

In 1 + 3 dimensions, the terms which diverge at $x=y$ in Eq. (2.14) are linear in A and quadratic in A. The first term $S_F(x,y)$ vanishes when the trace is taken with γ^5. The term linear in A can be expressed in momentum space as

$$S(x,y)(\text{linear in } A)=\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}e^{-ip\cdot(x-y)}e^{-iq\cdot x}\frac{1}{\not p+\not q}\tilde{\not A}(q)\frac{1}{\not p}$$
$$=\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}e^{-ip\cdot(x-y)}e^{-iq\cdot x}\left[\frac{1}{\not p}-\frac{1}{\not p}\not q\frac{1}{\not p}+\frac{1}{\not p}\not q\frac{1}{\not p}\not q\frac{1}{\not p+\not q}\right]\tilde{\not A}(q)\frac{1}{\not p}\ . \tag{4.1}$$

The first term of Eq. (4.1), after substitution into Eq. (2.12) contributes zero to $P(\mathbf{x},\mathbf{y})$ because $\mathrm{tr}\gamma^5\not p\not q\not p\gamma^0=0$. The third term of Eq. (4.1) which seemingly diverges logarithmically around $x=y$ actually converges to a finite value as x approaches y. Therefore, it is enough to consider the second term only:

$$S(x,y)(\text{linear in } A)=-\int \frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}q_\mu\tilde{A}_\nu(q)\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{1}{\not p}\gamma^\mu\frac{1}{\not p}\gamma^\nu\frac{1}{\not p}+\text{terms finite at } x=y\ . \tag{4.2}$$

The term quadratic in A in Eq. (2.14) can be expressed in momentum space as

$$S(x,y)(\text{quadratic in } A \text{ term})=-i\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}e^{-i(q_1+q_2)\cdot x}\int\frac{d^4p}{(2\pi)^4}\frac{1}{\not q_1+\not q_2+\not p}\tilde{\not A}(q_1)\frac{1}{\not q_2+\not p}\tilde{\not A}(q_2)\frac{1}{\not p}e^{-ip\cdot(x-y)}$$
$$=-i\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}e^{-i(q_1+q_2)\cdot x}\tilde{A}_\mu(q_1)\tilde{A}_\nu(q_2)\int\frac{d^4p}{(2\pi)^4}\frac{1}{\not p}\gamma^\mu\frac{1}{\not p}\gamma^\nu\frac{1}{\not p}e^{-ip\cdot(x-y)}$$
$$+\text{terms finite at } x=y\ . \tag{4.3}$$

Substituting Eqs. (4.2) and (4.3) into Eq. (2.12) we get

$$P^{\text{inf}}(\mathbf{x},\mathbf{y})=\lim_{y_0-x_0\to 0^+}\left[-\frac{i}{2}\right]\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^6}\mathrm{tr}\gamma^5\not p\gamma^\mu\not p\gamma^\nu\not p\gamma^0(\partial_\mu A_\nu+A_\mu A_\nu)(x)\ . \tag{4.4}$$

Using $\not p\gamma^\mu\not p=2p^\mu\not p-p^2\gamma^\mu$ and

$$\lim_{y_0\to x_0+0^+}\int d^4p\, e^{-ip\cdot(x-y)}\frac{p^k}{p^4}=\frac{i\pi}{2}\int d^3p\frac{p^k}{|\mathbf{p}|^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

or

$$=-2\pi^2\frac{(x-y)^k}{|\mathbf{x}-\mathbf{y}|^2}\ , \tag{4.5}$$

$P^{\text{inf}}(\mathbf{x},\mathbf{y})$ can be written as

$$P^{\text{inf}}(\mathbf{x},\mathbf{y})=-\frac{i}{4\pi^2}\frac{(x-y)^i}{|\mathbf{x}-\mathbf{y}|^2}B^i(\mathbf{x})\ , \tag{4.6a}$$

$$B^i\equiv-\epsilon^{ijk}(\partial_jA_k+A_jA_k)\ . \tag{4.6b}$$

Furthermore, we can change the argument of B from $\mathbf{x}$ to $(\mathbf{x}+\mathbf{y})/2$. So we get

$$P^{\text{inf}}(\mathbf{x},\mathbf{y})=-\frac{i}{4\pi^2}\frac{\eta^i}{|\eta|^2}B^i(\xi)\ , \tag{4.7}$$

where $\xi=(\mathbf{x}+\mathbf{y})/2$, $\eta=\mathbf{x}-\mathbf{y}$. The contribution to B from A^2 in Eq. (4.7) was overlooked in Ref. 8. The regularized fermion charge $\mathscr{I}_{\text{reg}}(F)$ is

$$\mathscr{I}_{\text{reg}}(F)=\mathscr{I}(F)+\frac{i}{4\pi^2}\int d^3x\,d^3y\frac{\eta^i}{|\eta|^2}\mathrm{tr}B^i(\xi)F(\mathbf{y},\mathbf{x})\ . \tag{4.8}$$

Now it is straightforward to get $a(F,G)$ through Eq. (2.11b) with the above $P^{\text{inf}}(\mathbf{x},\mathbf{y})$. It can be shown that

$$i[T(f),\mathscr{I}_{\text{reg}}(G)]=\frac{i}{4\pi^2}\int d^3x\,d^3y\frac{\eta^i}{|\eta|^2}\times\mathrm{tr}[B^i,f](\xi)G(\mathbf{y},\mathbf{x}) \tag{4.9}$$

For simplicity, we choose the intermediate kernel $G(\mathbf{x},\mathbf{y})$ to be Lie-algebra valued,

$$G(\mathbf{x},\mathbf{y})=G_a(\mathbf{x},\mathbf{y})T^a\ , \tag{4.10}$$

and symmetric under the change of $\mathbf{x}$ and $\mathbf{y}$:

$$G_a(\mathbf{x},\mathbf{y})=G_a(\mathbf{y},\mathbf{x})\ . \tag{4.11a}$$

In momentum space,

$$\tilde{G}(p,q)=\tilde{G}(p,-q) \tag{4.11b}$$

and similarly for $F(\mathbf{x},\mathbf{y})$. For later use we give here one example of $G(\mathbf{x},\mathbf{y})$:

$$G_\mu(\mathbf{x},\mathbf{y})=g\left[\frac{\mathbf{x}+\mathbf{y}}{2}\right]\frac{1}{(4\pi\mu)^{3/2}}e^{-(\mathbf{x}-\mathbf{y})^2/4\mu}\ , \tag{4.12a}$$

or in momentum space

$$\tilde{G}_\mu(\mathbf{p},\mathbf{q})=\tilde{g}(\mathbf{p})e^{-\mu\mathbf{q}^2}\ . \tag{4.12b}$$

The local limit is achieved by letting μ approach zero. With this choice, Eq. (4.9) vanishes because the integrand is antisymmetric under the change of $\mathbf{x}$ and $\mathbf{y}$. Now $a(F,G)$ can be expressed in momentum space as

$$a(F,G)=-\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}B^i(q)\int\frac{d^3p'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{p^i}{p^3}\left[\tilde{F}\left[\mathbf{p}',\mathbf{p}+\frac{\mathbf{q}}{2}+\frac{\mathbf{p}'}{2}\right]\tilde{G}\left[-\mathbf{p}'-\mathbf{q},\mathbf{p}+\frac{\mathbf{p}'}{2}\right]-(F\leftrightarrow G)\right]. \tag{4.13}$$

In order to get the anomaly, we have to take the local limit of Eq. (4.13). Unfortunately, this cannot be done. In order to show this, let us define

$$\chi^i(F,G;\mathbf{p}',\mathbf{q})=\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{p^3}\tilde{F}\left[\mathbf{p}',\mathbf{p}+\frac{\mathbf{q}}{2}+\frac{\mathbf{p}'}{2}\right]\tilde{G}\left[-\mathbf{p}'-\mathbf{q},\mathbf{p}+\frac{\mathbf{p}'}{2}\right]. \tag{4.14}$$

We expand $\tilde{F}$ and $\tilde{G}$ as

$$\tilde{F}\left[\mathbf{p}',\mathbf{p}+\frac{\mathbf{q}}{2}+\frac{\mathbf{p}'}{2}\right]=\tilde{F}(\mathbf{p}',\mathbf{p})+\left[\frac{q+p'}{2}\right]^j\frac{\partial}{\partial p^j}\tilde{F}(\mathbf{p}',\mathbf{p})+\cdots, \tag{4.15a}$$

$$\tilde{G}\left[-\mathbf{p}'-\mathbf{q},\mathbf{p}+\frac{\mathbf{p}'}{2}\right]=\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p})+\left[\frac{p'}{2}\right]^j\frac{\partial}{\partial p^j}\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p})+\cdots. \tag{4.15b}$$

Substituting these into Eq. (4.14), we obtain

$$\chi^i(F,G;\mathbf{p}',\mathbf{q})=\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{p^3}\left[\tilde{F}(\mathbf{p}',\mathbf{p})\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p})+\left[\frac{p'}{2}\right]^j\frac{\partial}{\partial p^j}[\tilde{F}(\mathbf{p}',\mathbf{p})\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p})]\right.$$
$$\left.+\left[\frac{q}{2}\right]^j\left[\frac{\partial}{\partial p^j}\tilde{F}(\mathbf{p}',\mathbf{p})\right]\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p})+\text{higher-order derivative terms}\right]. \tag{4.16}$$

The first term is zero by Eq. (4.11b), and the nth-order derivative term looks like

$$\int d^3p\frac{p^i}{\mathbf{p}^3}\left[\frac{\partial}{\partial p}\right]^{n-m}\tilde{F}\left[\frac{\partial}{\partial p}\right]^m\tilde{G}\equiv I^{(n)}. \tag{4.17}$$

As we have seen in the example of Eq. (4.12b) the p dependence of $\tilde{F}(\mathbf{p}',\mathbf{p})$ appears coupled to some damping factor $\sqrt{\mu}$. By changing the integration variable from p to $s=\sqrt{\mu}p$, we obtain

$$I^n=O(\mu^{(n-1)/2}). \tag{4.18}$$

Therefore for $n\geq 2$, I^n vanishes under the local limit, which is implemented by $\mu\to 0$. Neglecting these higher-derivative terms, we get

$$\chi^i(F,G;\mathbf{p}',\mathbf{q})=\left[\frac{p'}{2}\right]^j\int\frac{d^3p}{(2\pi)^3}\left[\frac{\partial}{\partial p^j}\frac{\partial}{\partial p^i}\frac{1}{p}\right]\tilde{F}(\mathbf{p}',\mathbf{p})\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p})+\left[\frac{q}{2}\right]^j\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{p^3}\left[\frac{\partial}{\partial p^j}\tilde{F}(\mathbf{p}',\mathbf{p})\right]\tilde{G}(-\mathbf{p}'-\mathbf{q},\mathbf{p}). \tag{4.19}$$

Using the symmetry property, the first integral becomes

$$\chi_1^i(F,G;\mathbf{p}',\mathbf{q})=-\frac{1}{12\pi^2}(p')^i\tilde{F}(\mathbf{p}',0)\tilde{G}(-\mathbf{p}'-\mathbf{q},0). \tag{4.20}$$

Here one can safely take the local limit. But the second integral in Eq. (4.19) does not allow a well-defined local limit. This can be shown explicitly by using the regulator of the above example, Eq. (4.12b):

$$\tilde{F}_{\mu_f}(\mathbf{p},\mathbf{q})=\tilde{f}(\mathbf{p})e^{-\mu_f q^2},\quad \tilde{G}_{\mu_g}(\mathbf{p},\mathbf{q})=\tilde{g}(\mathbf{p})e^{-\mu_g q^2}.$$

Then the troublesome part of χ^i is

$$\chi_2^i(F,G;\mathbf{p}',\mathbf{q})=\left[\frac{q}{2}\right]^j\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{p^3}\left[\frac{\partial}{\partial p^j}\tilde{F}_{\mu_f}(\mathbf{p}',\mathbf{p})\right]\tilde{G}_{\mu_g}(-\mathbf{p}'-\mathbf{q},\mathbf{p})$$
$$=-\frac{\mu_f}{\mu_f+\mu_g}\frac{q^i}{12\pi^2}\tilde{f}(\mathbf{p}')\tilde{g}(-\mathbf{p}'-\mathbf{q}). \tag{4.21}$$

The local limit of this depends on how μ_f and μ_g approach zero. With this ambiguity, the final result for $a(f,g)$ looks like

$$a(f,g)=\frac{-i}{24\pi^2}\epsilon^{ijk}\int d^3x\, \mathrm{tr}\left[(\partial_j A_k+A_jA_k)(\partial_i fg-\partial_i gf)+\partial_i(A_jA_k)\left[\frac{\mu_f}{\mu_f+\mu_g}fg-\frac{\mu_g}{\mu_f+\mu_g}gf\right]\right] . \tag{4.22}$$

The only sensible choice of the undetermined coefficients $\mu_{f,g}/(\mu_f+\mu_g)$ is $\frac{1}{2}$ because $a(f,g)$ should be antisymmetric when f and g are interchanged while their sum must be 1.

Note that (4.22) violates the Jacobi identity, which is equivalent to the 2-cocycle condition. In fact, the Jacobi identity for three operators $J_{\mathrm{reg}}(F), J_{\mathrm{reg}}(G), J_{\mathrm{reg}}(H)$ holds before the local limit is taken. The situation here is similar to that of Ref. 10, where an ambiguity in double commutators of currents with spacelike point splitting is demonstrated. Nevertheless, there the Jacobi identity was shown to hold true. In our case the ambiguity appears in a single commutator. The existence of the ambiguity in Eq. (4.22) is required so that the Jacobi identity holds before the local limit is taken. This means that if we take the Jacobian of three operators, each commutator will contain its own ambiguous term arising from the local limit. Nevertheless in the sum making up the Jacobian they all cancel. The approach dependence in Eq. (4.22) is specific to the intermediate kernel of Eq. (4.12). A different ansatz may give rise to a different approach dependence. Therefore $a(f,g)$ cannot be well defined. This is to be contrasted with the unambiguous and path-independent situation in 1 + 1 dimensions.

V. REEXAMINING THE BJL LIMIT

In our previous paper we derived with the BJL limit method various commutators $[\rho_a(\mathbf{x}),\rho_b(\mathbf{y})]$, $[\rho_a(\mathbf{x}),\delta_b(\mathbf{y})]$, $[\delta_a(\mathbf{x}),\delta_b(\mathbf{y})]$, $[E_a^i(\mathbf{x}),E_b^j(\mathbf{y})]$, and $[E_a^i(\mathbf{x}),\rho_b(\mathbf{y})]$. From these we obtained the anomaly $S_{ab}(\mathbf{x},\mathbf{y})$ in 1 + 3 dimensions:

$$\begin{aligned}S_{ab}(\mathbf{x},\mathbf{y})=&-\frac{i}{48\pi^2}\epsilon^{ijk}\mathrm{tr}T^a\partial_i(A_jT^bA_k)\delta(\mathbf{x}-\mathbf{y})\\&-\frac{i}{48\pi^2}\epsilon^{ijk}\mathrm{tr}[T^a,T^b](\partial_iA_jA_k+A_i\partial_jA_k)\delta(\mathbf{x}-\mathbf{y})-\frac{i}{48\pi^2}\epsilon^{ijk}\mathrm{tr}[T^a,T^b]A_iA_jA_k\delta(\mathbf{x}-\mathbf{y}) .\end{aligned}\tag{5.1}$$

[This differs from Eq. (1.1c) by a trivial infinitesimal 2-cocycle.] However, if we are interested in only $S_{ab}(\mathbf{x},\mathbf{y})$, rather than in the separate commutators, the BJL limit can be simplified. The BJL limit reads

$$\begin{aligned}\int d^nx\, e^{-i\mathbf{p}\cdot\mathbf{x}}\langle\alpha\,|\,S_{ab}(\mathbf{x},0)\,|\,\beta\rangle=&\lim_{p_0\to\infty}\int d^{n+1}x\, e^{ip\cdot x}\langle\alpha\,|\,T((-\mathbf{D}\cdot\mathbf{E})_a(x)+\rho_a(x))((-\mathbf{D}\cdot\mathbf{E})_b(0)+\rho_b(0))\,|\,\beta\rangle_c\\&-f_{abc}\langle\alpha\,|\,((-\mathbf{D}\cdot\mathbf{E})_c(0)+\rho_c(0))\,|\,\beta\rangle .\end{aligned}\tag{5.2}$$

We add the subscript c because only the connected diagrams contribute. Therefore, contractions should occur so that $G_a(x)$ and $G_b(0)$ are connected to each other. One way is a direct contract of E^i and A_j:

$$\lim_{p_0\to\infty}p\int d^{n+1}x\, e^{ip\cdot x}\langle\alpha\,|\,((\mathbf{D}\cdot\overline{\mathbf{E})_a(x)(\mathbf{D}}\cdot\mathbf{E})_b(0)+(\overline{\mathbf{D}\cdot\mathbf{E})_a(x)(\mathbf{D}\cdot\mathbf{E}})_b(0))\,|\,\beta\rangle=-f_{abc}\langle\alpha\,|\,(\mathbf{D}\cdot\mathbf{E})_c(0)\,|\,\beta\rangle .\tag{5.3}$$

This is canceled by the boson part of the last term in Eq. (5.2). All the other connected diagrams involve fermion loops. Here we shall be content with fermion one-loop graphs. In this case E^i is always contracted with the A^i which is coupled to j^i in the perturbative expansion of the Gell-Mann—Low formula. That is,

$$\int d^{n+1}x\, e^{ip\cdot x}(\mathbf{D}\cdot\mathbf{E})_a(x)\underset{1\text{ loop}}{=}-i\int d^{n+1}x\, e^{ip\cdot x}(\mathbf{D}\cdot\mathbf{E})_a(x)\int d^4z\, A_\mu(z)j^\mu(z) .\tag{5.4}$$

Using the free-gauge-boson propagator

$$\begin{aligned}\Delta_{ab}^{ij}(x,y)&=\langle TA_a^i(x)A_b^j(y)\rangle_F\\&=\int\frac{d^{n+1}q}{(2\pi)^{n+1}}\frac{i\delta^{ij}}{q^2+i\epsilon}e^{-iq\cdot(x-y)}+\text{gauge-dependent terms} .\end{aligned}\tag{5.5}$$

The above becomes, under the limit $p_0\to\infty$,

$$\lim_{p_0\to\infty}\int d^{n+1}x\, e^{ip\cdot x}(\mathbf{D}\cdot\mathbf{E})_a(x)\underset{1\text{ loop}}{=}\lim_{p_0\to\infty}\left[\frac{i}{p_0}\int d^{n+1}x\, e^{ip\cdot x}(\mathbf{D}\cdot\mathbf{j})_a(x)+\text{terms which do not contribute to }S_{ab}(\mathbf{x},0)\right] .\tag{5.6}$$

Therefore, for $\mathbf{D}\cdot\mathbf{E}(x)$ we can substitute $-(i/p_0)\mathbf{D}\cdot\mathbf{j}(x)$, and likewise $\mathbf{D}\cdot\mathbf{E}(0)$ can be substituted by $(i/p_0)\mathbf{D}\cdot\mathbf{j}(0)$. [We found an anomalous, Jacobi identity violating, commutator for E_a^i and E_b^j (Ref. 6); now we see that it is related to anomalies in commutators of the j's.] This is the only occasion when the bosonic propagator appears in the fermion one-loop diagrams. Therefore, after these substitutions we can regard the gauge potential as an external field, and we only take the vacuum expectation value of two j's in the background gauge field $A_\mu(x)$. So we have

$$S_{ab}(\mathbf{x},0)=\lim_{p_0\to\infty} p_0 \int dt\, e^{ip_0 t}\left\langle T\left[\frac{i\mathbf{D}\cdot\mathbf{j}(x)}{p_0}+j^0(x)\right]_a \left[\frac{-i\mathbf{D}\cdot\mathbf{j}(0)}{p_0}+j^0(0)\right]_b\right\rangle - f_{abc}\langle j_c^0(0)\rangle\delta(\mathbf{x})$$

$$=\lim_{p_0\to\infty}\frac{1}{p_0}\hat{D}_\mu^{aa'}(x)\hat{D}_\nu^{bb'}(0)\langle Tj_{a'}^\mu(x)j_{b'}^\nu(0)\rangle - f_{abc}\langle j_c^0(0)\rangle\delta(\mathbf{x}) , \tag{5.7}$$

where

$$\hat{D}_\mu^{aa'}(x)=D_\mu^{aa'}(x)-f_{aa''a'}A_0^{a''}(x)\delta_{\mu 0} . \tag{5.8}$$

In fact, we can substitute D for $\hat{D}$, because

$$\lim_{p_0\to\infty}\langle Tj^0 j^\nu\rangle = 0 . \tag{5.9}$$

Using

$$\langle Tj_{a'}^\nu(x)j_{b'}^\mu(0)\rangle = -\frac{\delta^2 iS_{\text{eff}}}{\delta A_\nu^{a'}(x)\delta A_\mu^{b'}(0)} , \tag{5.10}$$

$$D_\nu^{aa'}\langle Tj_{a'}^\nu(x)j_{b'}^\mu(0)\rangle = i\frac{\delta}{\delta A_\mu^{b'}(0)}\left[iD_\nu^{aa'}\frac{\delta}{\delta A_\nu^{a'}(x)}iS_{\text{eff}}\right]+f_{ab'a'}\frac{\delta iS_{\text{eff}}}{\delta A_\mu^{a'}(x)}\delta(x)$$

$$= i\frac{\delta X_a(x)}{\delta A_\mu^{b'}(0)} - if_{ab'a'}\langle j_{a'}^\mu\rangle\delta(x) , \tag{5.11}$$

where

$$X\equiv D_\mu j^\mu . \tag{5.12}$$

With this

$$S_{ab}(\mathbf{x},0)=\lim_{p_0\to\infty}\frac{1}{p_0}\int dt\, e^{ip_0 t} iD_\mu^{bb'}(0)\frac{\delta}{\delta A_\mu^{b'}(0)}X_a(x) - \lim_{p_0\to\infty}\frac{1}{p_0}\int dt\, e^{ip_0 t} f_{ab'a'} iD_\mu^{bb'}(0)[\langle j_{a'}^\mu(0)\rangle\delta(x)] - f_{abc}\langle j_c^0\rangle\delta(\mathbf{x}) . \tag{5.13}$$

After the limit is taken the second term is exactly canceled by the third, and finally we have

$$S_{ab}(\mathbf{x},0)=\lim_{p_0\to\infty}\frac{i}{p_0}\int dt\, e^{ip_0 t} D_\mu^{bb'}(0)\frac{\delta}{\delta A_\mu^{b'}(0)}X_a(x) . \tag{5.14}$$

In order to get $S_{ab}(\mathbf{x},0)$ in 1 + 3 dimensions, we substitute $X_a(x)$,

$$X_a(x)=-\frac{i}{24\pi^2}\text{Tr}T^a\epsilon^{\mu\alpha\beta\gamma}\partial_\mu(A_\alpha\partial_\beta A_\gamma+\tfrac{1}{2}A_\alpha A_\beta A_\gamma) , \tag{5.15}$$

and obtain $S_{ab}(\mathbf{x},0)$ given in Eq. (5.1). Equation (5.14) gives the correct anomaly for the (1 + 1)-dimensional case as well. The result in Eq. (5.14) explicitly shows how the commutator anomaly is related to the divergence anomaly of a non-Abelian chiral gauge theory in even-dimensional space-time.

VI. SUMMARY

Even though Faddeev's method looks promising, it turns out that in the (1 + 3)-dimensional case it does not give the correct anomaly. A possible remedy might be an additional regularization of the boson part. From the BJL limit calculation we have already observed that the commutator of E^i fields is not canonical;[6] it even violates the Jacobi identity. This result has also been derived from purely mathematical considerations.[11] Thus, it seems that in 1 + 3 dimensions, the regularization of the boson part is necessary for a fixed-time calculation and may introduce those terms which cancel the ambiguous and unwanted terms.

On the other hand, the above simplified BJL procedure for the commutator anomaly actually resembles the cohomological construction of the infinitesimal 2-cocycle out of the infinitesimal 1-cocycle responsible for the divergence anomaly. Therefore, we conclude that the BJL limit is, at least so far, the only method for deriving the commutator anomaly in general.

Note added. Recently there appeared a paper by S. Hosono (Nagoya University Report No. DPNU-86-44). There he regularized the fermion current by normal ordering and obtained an expression for the anomalous term in the equal-time commutator of fermion charge densities.

$$a(\theta,\varphi)=\mathrm{Tr}P_-\langle\partial_\theta g^{-1}g\rangle P_+\langle\partial_\varphi g^{-1}g\rangle-(\theta\leftrightarrow\varphi)\ . \tag{N1}$$

This is equivalent to Faddeev's expression, which is reproduced in my Eq. (2.11b):

$$a(F,G)=\cdots+\mathrm{tr}[F,G]P^{\mathrm{inf}}\ . \tag{N2}$$

Here $\cdots$ denotes terms measuring the gauge dependence of the regularized fermion charge density, but these vanish as was shown in my calculations. The equivalence can be seen by writing $P_+=1-P_-$ in (N1). The term linear in P_- reproduced Faddeev's formula (N2) and the term quadratic in P_- vanishes by the cyclicity of the trace. We expect therefore that Hosono's result, just as Faddeev's, does not in general reproduce the cohomological expression for the commutator anomaly.

In $1+1$ dimensions Hosono evaluates his expression exactly and nonperturbatively for the case that the background is a pure gauge. His result agrees with my Eq. (3.7), which makes no reference to the nature of the background gauge. Thus in $1+1$ dimensions all is well and the cohomological result is verified.

However, when Hosono's formula is taken in $1+3$ dimensions and evaluated with a pure gauge background, the result disagrees with the cohomological conjecture.[3,11] One finds zero, whereas the cohomological formula, being gauge noninvariant, would give my Eqs. (1.1c) or (5.1). For the Hamiltonian with a pure background $H=-i\boldsymbol{\sigma}\cdot(\nabla+g\nabla g^{-1})$ the positive- and negative-energy projection operators P_+,P_- are given by $P_\pm=\frac{1}{2}g(1\pm\boldsymbol{\sigma}\cdot\nabla/|\nabla|)g^{-1}$. Therefore, Hosono's anomaly vanishes because of the tracelessness of the Pauli matrices. Note further that a vanishing answer is consistent with my corrected rederivation of the commutator by Faddeev's method. This is because, for a pure gauge field, $F_{ij}=0$ and $\epsilon^{ijk}\partial_i(A_jA_k)=-\epsilon^{ijk}\partial_i\partial_jA_k=0$. The anomaly $a(f,g)$ in my Eq. (4.22) is zero without any ambiguities in this case.

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[1]M. Atiyah and I. Singer, Proc. Natl. Acad. Sci. USA **81**, 1597 (1984); L. Alvarez-Gaumé and P. Ginsparg, Nucl. Phys. **B243**, 449 (1984); B. Zumino, *ibid.* **B253**, 447 (1985).

[2]J. Mickelsson, Commun. Math. Phys. **97**, 361 (1985); L. Faddeev, Phys. Lett. **145B**, 81 (1984). For a review see R. Jackiw, in *Quantum Field Theory and Quantum Statistics,* edited by I. Batalin, C. Isham, and G. Vilkovisky (Hilger, Bristol, England, 1985); in *Current Algebra and Anomalies,* edited by S. Treiman, R. Jackiw, B. Zumino, and E. Witten (World Scientific, Singapore, 1985).

[3]L. Faddeev and S. Shatashvili, Theor. Math. Phys. **60**, 770 (1984).

[4]J. Bjorken, Phys. Rev. **148**, 1467 (1966); K. Johnson and F. Low, Prog. Theor. Phys. **B14**, 269 (1969).

[5]S.-G. Jo, Nucl. Phys. **B259**, 616 (1985).

[6]S.-G. Jo, Phys. Lett. **163B**, 353 (1985).

[7]M. Kobayashi, K. Seo, and A. Sugamoto, Nucl. Phys. **B273**, 607 (1986).

[8]L. Faddeev and S. Shatashvili, Phys. Lett. **167B**, 225 (1986).

[9]R. Jackiw and K. Johnson, Phys. Rev. **182**, 1459 (1969); D. G. Boulware and R. Jackiw, *ibid.* **186**, 1442 (1969); M. Chanowitz, Phys. Rev. D **2**, 3016 (1970).

[10]Y. Takahashi, Phys. Rev. D **24**, 1021 (1981).

[11]A. Niemi and G. Semenoff, Phys. Rev. Lett. **55**, 927 (1985).