CP violation and Yukawa couplings in superstring models: A four-generation example

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We present a detailed discussion of the pattern of Yukawa couplings and *CP* violation in a superstring E_6 model built on a CP^4-based Calabi-Yau manifold, with $Z_5 \times Z'_5$ discrete symmetry. Embedding the two Z_5's into the E_6 group leads to a drastically different pattern of Yukawa couplings. The model is presented more as an illustration of the steps involved in the calculation of Yukawa couplings in a generic Calabi-Yau manifold rather than as a realistic model of quarks and leptons.

I. INTRODUCTION

Superstring theories[1-3] have, during the past year, raised the exciting possibility[4] that an ultimate theory of everything (TOE) including gravitation and quark-lepton interactions may be at hand. These theories arise from a fundamental heterotic string theory[2] and lead, in the zero-slope limit, to an $N=1$ locally supersymmetric Yang-Mills theory based on the $E_8 \times E_8$ gauge group in ten-space dimensions. On compactifying the extra six dimensions[3] on a Kahler manifold with SU(3) holonomy (known as the Calabi-Yau manifold), an $N=1$ supergravity model with an $E_6 \times E'_8$ grand-unified-theory (GUT) group emerges with N_g number of chiral multiplets transforming as $\{27\}$-dimensional representations of E_6 and a certain number of $\{27\} = \{\overline{27}\}$ representations. The main euphoria stems from the fact that in addition to giving rise to a massless $\{27\}$-dimensional representation of E_6 which has the right quantum numbers to incorporate all the known quarks and leptons (plus more) per generation, there exists a natural mechanism to implement breaking of E_6 down to the standard model. A great deal of attention has therefore been rightly focused on the phenomenological analysis[5] of these models to see whether or not they can be candidates for a realistic description of quarks, leptons, and their interactions.

An added advantage of the superstring model is the possibility that one may be able to calculate the Yukawa couplings among the fermions and the Higgs bosons (the latter being picked out of the scalar components of the supersymmetric multiplets). The reason for this optimism is as follows. Since the four-dimensional theory arises from ten-dimensional super Yang-Mills theory, the Yukawa couplings[6] in the former arise from the gaugino—gauge-boson coupling in ten dimensions which has only one parameter, the gauge coupling. So, if the nature of the manifold on which the latter is compactified is known, then in principle the various Yukawa couplings can be exactly calculated. In practice the situation closely resembles this scenario. Under favorable circumstances one can evaluate all Yukawa couplings in terms of a few unknown constants. The essential steps in the calculations have been given by Witten,[7] Strominger,[6] and Candelas.[8] In particular we follow the methods of Candelas[8] for computing Yukawa couplings of algebraic varieties. It should also be mentioned that some couplings have been computed by Strominger[6] and nonperturbative corrections to couplings have been computed by Dine *et al.*[9]

In this paper we will use their techniques for a four-generation model originally proposed by Candelas *et al.*[3] to obtain Yukawa couplings and *CP* violation. We consider different symmetry-breaking patterns needed in these models and various embeddings of the discrete Z_5 groups and present Yukawa couplings for various cases. We find only one case, which has a slim chance of being a realistic model, while the rest of the cases face phenomenological difficulties of one kind or another. On the whole we present our calculations more as an illustration of the techniques involved rather than as a quest for a realistic model of quark-lepton interactions. We therefore have no predictions that we could exhort our experimental colleagues to look for.

The paper is planned as follows. In Sec. II we present the manifold and list its properties. Section III contains a discussion of general rules for calculating Yukawa couplings. In Sec. IV we consider different symmetry-breaking patterns and associated transformation properties of quarks and leptons. Sections V, VI, and VII, respectively, contain our calculations of Yukawa couplings for the cases in which Z_5, Z'_5, and $Z_5 \times Z'_5$ are embedded in the E_6 group. In Sec. VIII we discuss some phenomenological aspects whereas Sec. IX contains our conclusions.

II. THE CP^4-BASED CALABI-YAU MANIFOLD

Knowledge of the Calabi-Yau manifold on which the string theory is compactified is essential for calculation of Yukawa couplings as it is for studying many other aspects of these models such as the number of generations, number of Higgs bosons, the pattern of symmetry breaking, etc. In this paper we will consider the manifold suggested in Ref. 3, which is defined as the hypersurface in the complex four-dimensional CP^4 space ($Z_i \simeq \lambda Z_i$, $i=1,\ldots,5$) defined by the following equation (called the defining polynomial):

$$\frac{1}{5}\sum_i Z_i{}^5 - CZ_1Z_2Z_3Z_4Z_5=0 \; . \tag{1}$$

We choose C to be complex, so that the model defined by $\{Z_i\}$ and $\{Z_i^*\}$ is not the same, leading to CP violation. (As in the work of Stominger and Witten,[6] we define the CP transformation as the one that takes $Z_i \to Z_i^*$.) This manifold K has a Bette-Hodge number $b_{1,1}=1$, Euler characteristic $\chi=-200$, and is simply connected. One can define $G=Z_5\times Z'_5$ on this manifold by transformations:

$$\begin{aligned} &Z_5,\; Z_i \to \alpha^i Z_i, \quad \alpha=e^{2\pi i/5}\; ;\\ &Z'_5,\; Z_i \to Z_{i+1}\; . \end{aligned} \tag{2}$$

It has been shown that these two symmetries act freely on the manifold; i.e., they do no have any fixed points. Dividing the manifold K by G reduces the Euler characteristics of the manifold to $-200/(5\times 5)=-8$, leading to a model with four generations. They also make the manifold multiply connected, providing a mechanism to break the E_6 symmetry at the compactification scale.

The properties of this manifold have been investigated in detail by Witten[7] and others.[5] The analysis of global symmetries reveals that the manifold is endowed with the following exact symmetries:

$$\begin{aligned} &B,\; Z_i \to \alpha^{2i^2} Z_i\; ;\\ &Y,\; Z_i \to Z_{2i}\; . \end{aligned} \tag{3}$$

In addition the manifold is also invariant under the pseudosymmetries (P) (i.e., the symmetries of K but not of K/G) which are

$$\begin{aligned} &Z_i \to \alpha^{n_i} Z_i, \quad \sum n_i = 0\; ,\\ &Z_i \to Z_j\; . \end{aligned} \tag{4}$$

We will make extensive use of these symmetries in what follows.

III. YUKAWA COUPLINGS: PRELIMINARIES

The Yukawa couplings in the effective four-dimensional theory arise from the gauge interaction term in the ten-dimensional action

$$L_g = \int d^{10}w\sqrt{-g}\;\overline{\Psi}_A \gamma^m A_{mB} \Psi_C f^{ABC}\; , \tag{5}$$

where A,B,C are $E_8\times E_8$ indices and f^{ABC} are structure constants. After compactification on the manifold $K\times M_4$, the 10-dimensional fields can be expanded in harmonics on the internal manifold K. We choose K to be the Calabi-Yau manifold defined in the last section. On this manifold (and other Calabi-Yau manifolds with a negative Euler number) the chiral massless fields transforming as 27-plet of E_6 are elements of $H_1(T)$, i.e., closed but nonexact one-forms with values in tangent space, denoted by $A^\mu_\rho dx^\rho$ (Ref. 3). The effective four-dimensional trilinear couplings are given by the triple overlap integral

$$\int_K A\wedge B\wedge C\wedge\Omega\; , \tag{6}$$

where Ω is the ($\bar\partial$ closed) holomorphic three-form. The calculation of Yukawa couplings consists of two parts: the evaluation of the triple overlap integral [Eq. (6)] [we will refer to these trilinear couplings as raw Yukawa couplings (RYC)] and the normalization given by

$$\int_K A\wedge A^*\; . \tag{7}$$

To compute the above expressions we need to know the explicit representations of the elements of $H_1(T)$. Fortunately for the manifold under consideration (and other Calabi-Yau manifolds defined as the transverse intersection of the hypersurfaces in CP^N) there is a one-to-one correspondence between elements of $H_1(T)$ and the linearly independent polynomials one can add to the defining polynomial.[10] This remarkable connection can be understood by noting that the different choices of defining polynomials give rise to physically distinct but topologically equivalent vacua. This freedom in choice of vacuua leads to flat directions in the theory which in turn give rise to massless scalars. Since the theory is supersymmetric, the massless scalars are accompanied by massless fermions.

For example, in our case (before dividing K by $Z_5\times Z'_5$) we may add any polynomial of the type

$$C_{ABCDE}Z_AZ_BZ_CZ_DZ_E \tag{8}$$

in the defining polynomial [Eq. (1)]. Equation (8), the most general quintic polynomial, has 126 parameters. However, due to the freedom in change in basis $Z_A \to C_A^B Z_B$, 25 of the parameters in Eq. (8) are redundant. Hence the description of the vacuum requires (126-25) parameters. Thus on this manifold there are 101 flat directions leading to 101 massless scalars which in turn lead to 101 massless fermions.

A more rigorous connection between polynomials and the elements of $H_1(T)$ has been obtained by Candelas.[8] Using the deformation theory he finds

$$A^\mu_{\bar\rho i}dx^{\bar\rho} = q_i \chi^\mu_{\bar\rho} dx^{\bar\rho}\; , \tag{9}$$

where q_i are the polynomials and χ^μ_ρ is the extrinsic curvature of the hypersurface embedded in CP.[4] Moreover, he also finds that A^μ is exact when

$$A^\mu_{\bar\rho}dx^{\bar\rho} = C^A_B Z_A(\partial P/\partial Z_B)\chi^\mu_{\bar\rho}dx^{\bar\rho}\; , \tag{10}$$

where P is the defining polynomial [Eq. (1)]. For calculational purposes we will only need to exploit the one-to-one correspondence between the massless zero modes and the polynomials. The results can be summarized as

$$\phi_i \sim q_i, \quad q_i \simeq q_i + f^A(\partial P/\partial Z_A)\; , \tag{11}$$

where ϕ_i is the zero mode and f^A is any function linear in Z. Furthermore one can always add a multiple of the defining polynomial P in q_i without altering the values of the integrals in Eqs. (6) and (7). Note that Eqs. (6) and (7) depend only on the cohomological classes of A^μ, i.e., changing $A \to A + \partial F$ has no effect on the integral. In terms of the polynomials, this means

$$q \simeq \tilde{q} = q + f^A(\partial P/\partial Z_A) + gP \ . \quad (12)$$

In other words A^μ defined with different q's are in the same cohomological class if they differ only by the multiple or the derivative of the defining polynomial. Using either q or $\tilde{q}$ will give the same results.

Now the Yukawa couplings between three fields are essentially given by the multiplication of three polynomials of degree 5 and integration over the whole manifold, K. At this point one may use Eq. (12) to connect various couplings.

In addition to the relations implied by the use of Eq. (12), there are strong constraints from the symmetries of the theory. For the integral to be nonzero, the integrand must be invariant under all the symmetries of the manifold. The set of 15th-order polynomials which are invariant under the symmetries Y, B, and P, discussed in the last section is $(Z_1Z_2Z_3Z_4Z_5)^3$, $(Z_1Z_2Z_3Z_4Z_5)^2Z_i^5$, $Z_1Z_2Z_3Z_4Z_5Z_i^5Z_j^5$, $Z_1Z_2Z_3Z_4Z_5Z_i^{10}$, Z_i^{15}, $Z_i^{10}Z_j^5$, $Z_i^5Z_j^5Z_k^5$.

Using Eq. (12) all of them can be written as $k(Z_1Z_2Z_3Z_4Z_5)^3$ and hence the relative value of RYC is given by k:

Polynomial	RYC
$(Z_1Z_2Z_3Z_4Z_5)^3$	μ
$(Z_1Z_2Z_3Z_4Z_5)^2Z_i^5$	C_μ
$(Z_1Z_2Z_3Z_4Z_5)Z_i^5Z_j^5$	$C^2\mu$
$(Z_1Z_2Z_3Z_4Z_5)Z_i^{10}$	$C^2\mu$
Z_i^{15}	$C^3\mu$
$Z_i^{10}Z_j^5$	$C^3\mu$
$Z_i^5Z_j^5Z_k^5$	$C^3\mu$

(13)

In what follows we will omit the overall constant μ from RYC. Now the task of computing the Yukawa coupling is straightforward. The procedure one may follow is (a) identify the fifth-order polynomial associated with the zero modes, (b) multiply these polynomials to obtain the 15th-order resulting polynomial, and (c) if the 15th-order polynomial is not invariant under the aforementioned symmetries of the manifold, the Yukawa coupling is zero; otherwise the Yukawa coupling can be picked out from the above table.

Now we turn our attention to the normalization matrix N_{ab} [Eq. (7)]. N_{ab} can also be evaluated using previously discussed methods. In terms of the polynomials it is essentially given by the integral of $q_a q_b^*$ over the whole manifold. Before dividing the manifold K by $G = Z_5 \times Z_5'$ the normalization matrix N_{ab} is a 101×101 matrix. On K/G it will generically reduce to a block diagonal matrix of 5×5 and 24 copies of 4×4 matrices. However in our case N_{ab} does not have any off-diagonal elements. This can be easily verified using the pseudosymmetric $Z_i \to \alpha^{n_i} Z_i$; $\sum n_i = 0$, whereas using the symmetry $Z_i \to Z_j$ various diagonal elements can be related. We find

Polynomial of the type	Normalization
$Z_1Z_2Z_3Z_4Z_5$	N_0
$Z_i^3Z_j^2 (i \neq j)$	N_1
$Z_i^3Z_jZ_k (i \neq j \neq k)$	N_2
$Z_i^4Z_j (i \neq j)$	N_3
$Z_i^2Z_j^2Z_k (i \neq j \neq k)$	N_4
$Z_j^2Z_iZ_kZ_l (i \neq k \neq l)$	N_3/c

(14)

Combining these results with RYC one can obtain the normalized Yukawa couplings (NYC).

We would also like to point out at this stage that the fourth-order terms of the type $(27\times\overline{27}\times27\times\overline{27})$ can also be evaluated in similar fashion. Since $\overline{27}$ corresponds to the Kahler form, it is invariant under all symmetries of the manifold and since $b_{1,1}=1$ for this manifold it is also unique. Thus we only need to concentrate on the $\mathbf{27}\times\mathbf{27}$ part of the above expression, which is essentially a tenth-order polynomial. The set of tenth-order polynomials invariant under all symmetries is $(Z_1Z_2Z_3Z_4Z_5)^2$, $(Z_1Z_2Z_3Z_4Z_5)Z_i^5$, Z_i^{10}, $Z_i^5Z_j^5$. The relative values of the couplings are given by 1, C, $C_,^2$ C^2, respectively.

IV. SYMMETRY BREAKING

As is by now quite well known, a multiply connected Calabi-Yau manifold can lead to the breakdown of the E_6 gauge symmetry at the scale of compactification. Study of this breakdown is facilitated by embedding the discrete group of the manifold into the E_6 gauge group, i.e., the generating elements of the discrete group G are expressed as

$$U_g = \exp\left[i \sum_k \lambda_k H_k\right] , \quad (15)$$

where H_k are the elements of the Cartan subalgebra (or diagonal generators) of E_6 and λ_k are a set of six real parameters. U_g is actually the Wilson loop integral $\exp(i\int A_m dy_m)$ where $m - 6, \ldots, 10$ in the Calabi-Yau manifold. If the manifold is simply connected, application of Stokes's theorem and that $F_{mn}=0$ for the ground state described by the manifold would tell us that $U_g = 1$; for general multiply connected manifolds with Z_n symmetry, $[U_g]^n = 1$. Clearly, U_g breaks the gauge symmetry group without reducing its rank. There are however many rank-6 subgroups of E_6 and superstring physics cannot decide between them and phenomenological constraints must be invoked.

The fact that a low-energy gauge group has to include the standard model [i.e., $SU(3)_c \times SU(2)_L \times U(1)_Y$] implies that

$$\{\lambda_i\} = (-c, c, a, b, c, 0) \ , \quad (16)$$

where we have used the notation of Slansky.[11] Alternatively one can consider a $SU(3)\times SU(3)\times SU(3)$ subgroup of E_6 and choose

$$U_g = [1] \times \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix} \times \begin{bmatrix} \delta & & \\ & \eta & \\ & & \gamma \end{bmatrix} : \delta\eta\gamma = 1 \ .$$

A further set of constraints emerges from requirements that some component of the $(27+\overline{27})$ pair remains light in order to act as Higgs bosons serving various purposes such as symmetry breaking, generating quark, lepton masses, etc. The first question that may be asked is why cannot these multiplets come from extra fields in $\{27\}$-dimensional multiplets that also contain quark and lepton fields, since these fields always remain massless by the virtue of the index theorem. The answer is dictated by the requirement that we preserve supersymmetry down to the low-energy scale of about 1 TeV, which in turn requires that the F and D terms in the potential be at most of the order of 1 TeV^2. Because of this, any Higgs-boson fields ϕ that acquires a vacuum expectation value (VEV) at a scale above 1 TeV must be accompanied by an anti-Higgs-boson field $\bar{\phi}$, so that the D terms vanish. Since the ϕ fields from the "matter" $\{27\}$'s do not have a $\bar{\phi}$, only Higgs-boson fields that can be chosen out of the $\{27\}$'s are the light-Higgs-boson doublets; any $SU(2)_L$-singlet fields that have VEV's above 1 TeV must come from a $(27+\overline{27})$ pair. All these considerations lead to the following three interesting possibilities.

(a) Two light-Higgs-boson doublets from $(27+\overline{27})$. The electroweak gauge group in this case is $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \times U(1)_N$.

(b) Two light SU(5)-singlet-fields from $(27+\overline{27})$. (To be denoted below by ν^c and n^0.) The electroweak symmetry below plank scale is given by $SU(3)_c \times SU(2)_L \times SU(2)_{\tilde{R}} \times U(1) \times U(1)$.

(c) Only one light SU(5) singlet (n^0) from $(27+\overline{27})$. The electroweak group here is given by $SU(3)_c \times SU(2)_L \times U(1)^3$ in general.

We will consider these possibilities in Secs. IV—VII.

After E_6 is broken due to Wilson loops, quarks and leptons transform nontrivially under $G=Z_5 \times Z'_5$. Their exact transformation properties depend upon the way we choose to embed G in E_6. The polynomials which represent these massless fields reflect their appropriate transformation properties. Here we list the structure of the polynomials and their transformations under $Z_5 \times Z'_5$. For brevity we have devised a shorthand notation of the form (a,b,c,d,e) to denote the polynomial $Z_1^a Z_2^b Z_3^c Z_4^d Z_5^e + \cdots$, and $+\cdots$ denotes cyclic permutation. We find

Transformation property: $\alpha^0\beta^0$,

$Z_i{}^5$	(50000)
$Z_{i+1}Z_i{}^3Z_{i-1}+\cdots$	(31001)
$Z_{i+2}Z_i{}^3Z_{i-2}+\cdots$	(30110)
$Z_{i+1}{}^2Z_iZ_{i-1}{}^2+\cdots$	(12002)
$Z_{i+2}{}^2Z_iZ_{i-2}{}^2+\cdots$	(10220)
$Z_1Z_2Z_3Z_4Z_5$	(11111) , (17)

$\alpha^1\beta^0$,

(21110), (30020), (30101), (12200), (41000) ; (18)

$\alpha^2\beta^0$,

(21101), (32000), (30011), (20210), (40100) ; (19)

$\alpha^3\beta^0$,

(21011), (23000), (03110), (02012), (40010); (20)

$\alpha^4\beta^0$,

(20111), (30200), (03101), (00221), (40001) . (21)

To construct polynomials which are invariant under Z_5 but transform nontrivially under Z'_5, let us consider $(a,b,c,d,e)_m$, defined as

$$(a,b,c,d,e)_m \equiv \sum Z_i^a Z_{i+1}^b Z_{i+2}^c Z_{i+3}^d Z_{i+4}^e \beta^{-m(i-1)} , \quad (22)$$

where β is a typical element of Z'_5, $\beta^5=1$. Under $Z_5 \times Z'_5$ symmetry the above polynomials transform as

$$\begin{aligned} Z_5&: (a,b,c,d,e)_m \to \alpha^{b+2c+3d+4e}(a,b,c,d,e)_m , \\ Z'_5&: (a,b,c,d,e)_m \to \beta^m (a,b,c,d,e)_m . \end{aligned} \quad (23)$$

Since we want the polynomials to be invariant under Z_5, we choose $b+2c+3d+4e$ to zero (mod 5). Thus the polynomials transforming as $\alpha^0\beta^m$ are given by $(50000)_m$, $(31000)_m$, $(30110)_m$, $(12002)_m$, $(10220)_m$. The polynomials transforming as $\alpha^m\beta^n$ can be easily constructed by appropriately choosing a, b, c, d, and e.

Note that not all of the above polynomials represent massless fields. Using Eq. (12) it is easy to show that one of the polynomials from each set corresponds to A^μ which is exact. Alternatively one can show that the polynomial of the types $Z_i{}^5$ and $Z_i{}^4Z_j$ are equivalent to $Z_1Z_2Z_3Z_4Z_5$ and $Z_j{}^2Z_kZ_lZ_m$ ($i\neq j\neq k\neq l\neq m$), respectively. To reproduce the results of Witten in the limit of $C=0$ (Ref. 7), we choose the latter set to correspond to massless fields. These four polynomials will represent the four generations, respectively. For instance, in the case of Z_5 embedding, the first four polynomials in Eqs. (18)—(21) represent the first, second, third, and the fourth generation, respectively.

TABLE I. Transformation of various fields in the case of Z_5 embedding, where

$$Q=\begin{vmatrix} u \\ d \end{vmatrix},\ L=\begin{vmatrix} \nu \\ e \end{vmatrix},\ \Psi \equiv \begin{vmatrix} \nu & E^0 \\ e & E_d \end{vmatrix},\ E_u \equiv \begin{vmatrix} E_u^+ \\ E_u^0 \end{vmatrix},\ N \equiv \begin{vmatrix} \nu^c \\ n_0 \end{vmatrix},$$

and

$$\phi \equiv \begin{vmatrix} E_d^0 & E_u^+ \\ E_d^- & E_u^0 \end{vmatrix} .$$

For Z'_5 embedding transformation properties of the fields can be obtained by replacing α with β.

Case (a)	Case (b)	Transformations
Q	Q,u^c,e^+	α
L,g^c	Ψ,D^c	α^2
L^c,g	E_u,g	α^3
Q^c,	None	α^{-1}
ϕ,n_0	N	α^0

TABLE II. Define $\lambda_1^{abc}=h_1^{abc}/N_aN_bN_c$, where N_a are the normalizations coming from D terms. We also omit an overall scale factor.

h^{111}	$5C$	h^{112}	5	h^{113}	5	h^{114}	5
h^{131}	$5C$	h^{132}	$5C$	h^{133}	5	h^{134}	$5C$
h^{121}	$5C$	h^{122}	$5C^2$	h^{123}	0	h^{124}	5
h^{141}	5	h^{142}	$5C$	h^{143}	5	h^{144}	5
h^{211}	$5C$	h^{212}	$5C^2$	h^{213}	0	h^{214}	5
h^{221}	5	h^{222}	5	h^{223}	$5C$	h^{224}	$5C^3$
h^{231}	$5C^2$	h^{232}	$5C^2$	h^{233}	5	h^{234}	0
h^{241}	$5C^3$	h^{242}	5	h^{243}	$5C$	h^{244}	$5C$
h^{311}	$5C$	h^{312}	$5C$	h^{313}	5	h^{314}	$5C$
h^{331}	5	h^{332}	0	h^{333}	$5C$	h^{334}	0
h^{321}	$5C^2$	h^{322}	$5C^2$	h^{323}	5	h^{324}	0
h^{341}	$5C^2$	h^{342}	5	h^{343}	$5C$	h^{344}	5
h^{411}	5	h^{412}	$5C$	h^{413}	5	h^{414}	5
h^{431}	$5C^2$	h^{432}	5	h^{433}	$5C$	h^{434}	5
h^{421}	$5C^3$	h^{422}	5	h^{423}	$5C$	h^{424}	$5C$
h^{441}	5	h^{442}	$5C$	h^{443}	$5C^3$	h^{444}	$5C^2$

V. YUKAWA COUPLINGS FOR THE Z_5-EMBEDDING CASE

In this section we will consider the case where the first Z_5 group is embedded in the E_6 group (i.e., the Z_5 defined by the transformations $Z_i\rightarrow\alpha^iZ_i$, $\alpha=e^{2\pi/5}$). In Table I we list the chiral fermion multiplets and their transformation properties. For the sake of notation we give below the decomposition of $\{27\}$-dimensional representation of E_6 under [SO(10), SU(5)] subgroups and identify the various particles they represent:

$$\{27\}\rightarrow[16,10]+[16,\bar{5}]+[16,1]+[10,5] \times(u,d;u_c,e^c)+(d^c,\nu,e)+\nu^c +(g^c,E_d^0,E_d^-)[10,\bar{5}]+[1,1](g,E_u^+,E_u^0)+n_0 \ . \quad (24)$$

Case (a). The superpotential consistent with the gauge symmetry can be written as

$$W_a=\lambda_1^{adb}Q_a\phi_bQ_d^c+\lambda_2^{adb}L_a\phi_bL_d^c+\lambda_3^{adb}g_ag_d^cn_{0,b}+\lambda_4^{adb}(Q_aQ_bg_d+Q_a^cQ_b^cg_d^c)\lambda_5^{abd}(Q_aL_bg_d^c+Q_a^cL_b^cg_d)+\lambda_6^{abd}\phi_a\phi_bn_{0,d} +\tilde{\lambda}_6^{abd}\bar{\phi}_a\bar{\phi}_b\bar{n}_{0,d} \ . \quad (25)$$

The coupling $\lambda_i(i=1,\ldots,6)$ in Eq. (25) is further restricted by the Y symmetry present in the manifold under which

$$L^cg\overset{Y}{\rightarrow}Q\overset{Y}{\rightarrow}L,g^c,\quad L,g^c\overset{Y}{\rightarrow}Q^c\overset{Y}{\rightarrow}L^c,g, \qquad Q\overset{Y^2}{\rightarrow}Q^c,\quad L,g^c\overset{Y^2}{\rightarrow}L^c,g \ . \quad (26)$$

This implies that

$$\lambda_1=\lambda_2=\lambda_3 \ \text{and} \ \lambda_4=\lambda_5 \ . \quad (27)$$

This leaves us with four couplings to evaluate, i.e., λ_1, λ_4, λ_6, and $\tilde{\lambda}_6$, of which λ_6 has already been evaluated by Witten[7] for the CP-conserving case, $C=0$ and the result in the presence of CP-violating parameter $C\neq0$ is given by

TABLE III. Yukawa couplings for Z_5 embedding case. $\lambda_4=g^{abc}/N_aN_bN_c$ where N_a's are normalization constants.

g^{111}	5	g^{112}	0	g^{113}	$5C$	g^{114}	5
g^{121}	0	g^{122}	$5C$	g^{123}	0	g^{124}	5
g^{131}	5	g^{132}	0	g^{133}	$5C$	g^{134}	0
g^{141}	5	g^{142}	$5C$	g^{143}	$5C^2$	g^{144}	0
g^{221}	$5C^2$	g^{222}	5	g^{223}	0	g^{224}	0
g^{231}	$5C$	g^{232}	0	g^{233}	0	g^{234}	$5C^3$
g^{241}	0	g^{242}	0	g^{243}	$5C$	g^{244}	$5C$
g^{331}	0	g^{332}	0	g^{333}	0	g^{334}	$5C$
g^{341}	0	g^{342}	0	g^{343}	0	g^{344}	$5C$
g^{441}	$5C$	g^{442}	0	g^{443}	0	g^{444}	0

TABLE IV. Nonzero RYC's for Z_5' embedding case.

h_{221}^{0-22}	$5\beta^4+5C\beta^2+5C^3$
h_{221}^{02-2}	$5\beta^{-4}+5C\beta^{-2}+5C^3$
h_{221}^{2-20}	$5+5C\beta^4+5C^3$
h_{-2-2-1}^{-220}	$5\beta^4+5C\beta^3+5C^3$
h_{-2-2-1}^{-220}	$5\beta^{-4}+5C\beta^{-3}+5C^3$
h_{-2-2-1}^{02-2}	$5\beta^{-2}+5C\beta+5C^3$
h_{-2-2-1}^{0-22}	$5\beta^2+5C\beta^4+5C^3$
h_{2-2-1}^{0-22}	$5\beta^{-2}+5C+5C^2\beta^4$
h_{2-2-1}^{2-20}	$5\beta^4+5C+5C^2\beta^2$
h_{2-2-1}^{-220}	$5\beta^{-4}+5C+5C^2\beta^{-2}$

TABLE V. Transformation properties of various fields in the case of $Z_5\times Z_5'$ embedding.

Fields	Transformation
Q	α
g	α^{-2}
u^c	$\alpha^{-1}\beta$
d^c	$\alpha^{-1}\beta^{-1}$
g^c	α^2
L	α^2
E_u	β^{-1}
E_d	β
e^c	$\beta^{-1}\alpha^{-2}$
ν^c	$\alpha^{-2}\beta$
n_0	$\alpha^0\beta^0$

$$\lambda_6{}^{000}=I,\quad \lambda_6{}^{0,2,-2}=5I\ ,$$
$$\lambda_6{}^{2,-1-1}=(5+5C+5C^2)I\ ,$$
$$\lambda_6{}^{2,2,1}=5I(1+C+C^3)\ , \tag{28}$$
$$\lambda_6{}^{0,1,-1}=5IC,\quad \tilde{\lambda}_6^{000}=\delta\ .$$

We note that the above couplings are complex and the phases cannot all be removed by redefinition of fields in Eq. (19), implying that there is generic *CP* violation in the model. As already noted by Witten, there is an interesting symmetry of the manifold (B symmetry) which implies that we get nonvanishing coupling only when the subscripts of the fields add to zero (mod 5). We list the results for the other couplings in Tables II and III. A point worth emphasizing is that the operation Y^2 is equivalent to invariance under left-right symmetry of weak interaction.

Case (b). The superpotential in this case can be written as

$$W_a=\gamma_1^{adb}Q_a\psi_b D_d^c+\gamma_2^{adb}(Q_aE_{u,b}u_d^c+D_a^cN_bg_d)+\gamma_3^{abd}\mathrm{Tr}(\psi_a\psi_b)e_d^+ +\gamma_4^{adb}(E_{u,a}\psi_bN_d)\ . \tag{29}$$

Under the Y symmetry we get in this case

$$Q,u^c,e^+6\rightarrow\psi,D_-^c\ ,E_ug\rightarrow Q,u^c,e^+\ , \tag{30}$$

implying $\gamma_1=\gamma_2$. Again, by looking at the polynomial structure we can conclude that $\gamma_2=\lambda_2=\gamma_4$ and $\gamma_3=\lambda_5$. Therefore, these couplings can be read from Tables II and III.

Case (c). In this case the transformation of quarks and leptons after symmetry breaking depends on two independent phase factors α and β. Therefore to realize this case for the manifold at hand one has to embed both the Z_5 groups into E_6, a case we treat in a subsequent section. If we embed only one Z_5, it reduces to case (b), which corresponds to $\beta=\alpha^{-3}$.

VI. Z_5' EMBEDDING AND YUKAWA COUPLINGS

In this section we consider the implications of embedding the second Z_5 (i.e., $Z_i\rightarrow Z_{i+1}$) into the symmetry group. Transformation properties of various fields are

TABLE VI. Yukawa couplings for $Z_5\times Z_5'$ embedding case. Defining $h_1^{abc}=h^{abc}/N_aN_bN_c$, where N_a are the normalizations coming from D terms. We also omit an overall scale factor. $\tilde{h}_1^{abc}$ can be obtained by replacing β with β^{-1}.

h^{111}	$5C\beta^{-1}$	h^{112}	5	h^{113}	5β	h^{114}	$5\beta^{-2}$
h^{131}	$5C$	h^{132}	$5C$	h^{133}	$5\beta^{-2}$	h^{134}	$5C\beta^{+2}$
h^{121}	$5C$	h^{122}	$5C^2\beta^2$	h^{123}	0	h^{124}	5β
h^{141}	$5\beta^{-2}$	h^{142}	$5C\beta^{-1}$	h^{143}	5β	h^{144}	$5\beta^{-1}$
h^{211}	$5C\beta^2$	h^{212}	$5C^2\beta^2$	h^{213}	0	h^{214}	$5\beta^{-1}$
h^{221}	$5\beta^{-2}$	h^{222}	$5\beta^{-1}$	h^{223}	$5C\beta^{-1}$	h^{224}	$5C^3\beta^2$
h^{231}	$5C^2\beta^{-2}$	h^{232}	$5C^2$	h^{233}	$5\beta^2$	h^{234}	0
h^{241}	$5C^3\beta^{-1}$	h^{242}	0	h^{243}	$5C$	h^{244}	$5C\beta^{-2}$
h^{311}	$5C\beta$	h^{312}	$5C\beta^{-2}$	h^{313}	$5\beta^2$	h^{314}	$5C\beta^{-2}$
h^{331}	$5\beta^2$	h^{332}	0^{333}	h^{334}	$5C^2\beta^{-1}$	h	0
h^{321}	$5C^2\beta^2$	h^{322}	$5C^2$	h^{323}	$5\beta^2$	h^{324}	0
h^{341}	$5C^2\beta^{-2}$	h^{342}	$5\beta^{+2}$	h^{343}	$5C\beta^{+2}$	h^{344}	$5\beta^{-2}$
h^{411}	5	h^{412}	$5C\beta^{-1}$	h^{413}	$5\beta^{-1}$	h^{414}	5
h^{431}	$5C^2$	h^{432}	$5\beta^2$	h^{433}	$5C\beta$	h^{434}	$5\beta^{-1}$
h^{421}	$5C^3\beta^{-1}$	h^{422}	5β	h^{423}	$5C$	h^{424}	$5C\beta^{-2}$
h^{441}	$5\beta^2$	h^{442}	$5C\beta^{-2}$	h^{443}	$5C^3\beta^{-1}$	h^{444}	$5C^2\beta$

TABLE VII. Yukawa couplings for $Z_5\times Z_5'$ case. $\tilde{\kappa}_4^{abc}=k^{abc}/N_aN_bN_c$. $\tilde{\tilde{\kappa}}_4$ can be obtained by replacing β with β^{-1}.

k^{111}	$5\beta^2$	k^{112}	0	k^{113}	$5C\beta^{-2}$	k^{114}	$5\beta^{-1}$
k^{121}	0	k^{122}	$5C$	k^{123}	0	k^{124}	$5\beta^2$
k^{131}	5	k^{132}	0	k^{133}	$5C$	k^{134}	0
k^{141}	$5\beta^2$	k^{142}	$5C\beta^{-1}$	k^{143}	$5C^2\beta^{-1}$	k^{144}	0
k^{221}	$5C^2$	k^{222}	$5\beta^{-2}$	k^{223}	0	k^{224}	0
k^{231}	$5C\beta$	k^{232}	0	k^{233}	0	k^{234}	$5C^2\beta^{-2}$
k^{241}	0	k^{242}	0	k^{243}	$5C$	k^{244}	$5C$
k^{331}	0	k^{332}	0	k^{333}	0	k^{334}	$5C$
k^{341}	0	k^{342}	0	k^{343}	0	k^{344}	$5C\beta$
k^{441}	$5C\beta^{-1}$	k^{442}	0	k^{443}	0	k^{444}	0

given in Table I and the corresponding polynomials were obtained in Sec. III. If we denote $\chi_{n,m}$ as

$$\chi_{+1,m}\equiv(31001)_m:\chi_{-1,m}\equiv(30110)_m\ :\ ,$$
$$\chi_{+2,m}\equiv(12002)_m:\chi_{-2,m}\equiv(10220)_m\ :\ , \tag{31}$$
$$\chi_{0,m}\equiv\chi_{0,0}=(50000)\ ,$$

then the Yukawa couplings are of the form

$$W=\sum\chi_{n_1,m_1}\chi_{n_2,m_2}\chi_{n_3,m_3}h_{n_1n_2n_3}^{m_1m_2m_3} \tag{32}$$

with the nonzero h's being those that satisfy $n_1+n_2+n_3=0$ (mod5) and $m_1+m_2+m_3=0$ (mod5). Nonzero elements of h are given in Table IV.

VII. $Z_5\times Z_5'$ EMBEDDING

In this section we will consider the embedding of both the groups in order to realize case (c), where the E_6 group breaks to the $SU(3)_c\times SU(2)_L\times U(1)^3$ group. The transformation properties of the fields under the discrete group are given in Table V. The polynomials corresponding to the different fields can be picked out from Sec. III. The most general gauge-invariant superpotential in this case is

$$W=h_1^{abc}Q_aE_{u,b}u_c^c+\tilde{h}_1^{abc}Q_aE_{d,b}d_c^c+h_2^{abc}L_aE_{d,b}e_c^c+\tilde{h}_2^{abc}L_aE_b\nu_c^c h_3^{abc}g_ag_b^c n_{0,c}+h_4^{abc}Q_aQ_bg_c+\tilde{h}_4^{abc}u_a^cd_b^cg_c^c$$
$$+h_5^{abc}Q_aL_ag_c^c\tilde{\kappa}_4^{abc}u_a^ce_b^cg_c^c+\tilde{\tilde{\kappa}}_4^{abc}d_a^cg_b\nu_c^c+h_6^{abc}E_{u,a}E_{d,b}n_{0,c}\ .$$

Of these h_5, h_3, h_4, and h_6 have been evaluated previously in Tables II, III, and V. Furthermore B symmetry implies $h_2=\tilde{h}_1$; $h_1=\tilde{h}_2$. The values of the couplings h_1, $\tilde{h}_2$, $\tilde{\kappa}_5$, and $\tilde{\tilde{\kappa}}_5$ are listed in Tables VI and VII.

VIII. PHENOMENOLOGICAL OUTLOOK

We now wish to discuss whether any of the models presented in this paper have a chance of being realistic. Let us, for instance, consider models of type (a), where the Higgs multiplet which gives masses to quarks and leptons comes from a $(27+\overline{27})$ pair. We see that in this case we automatically have the SO(10)-singlet field of the above pair light. This enables us to introduce an intermediate-mass scale using the dimension-4 terms in the superpotential. This will have the desirable effect of suppressing proton decay[12] which can arise from the couplings of λ_4 and λ_5 in Eq. (25). Coming to the quark-lepton masses, we first note that in this model, even though there is no quark-lepton symmetry in the gauge sector, the Y symmetry implies that quark and lepton masses are equal at the GUT scale, as is evident from Table II. The most general up- and down-quark mass matrices (at the GUT scale) that follow in this case are

$$\begin{pmatrix} Ck_1+k_2+k_3+k_4 & (Ck_1+C^2k_3+k_4) & C(k_1+k_2+k_4)+k_3 & (k_1+Ck_2+k_3k_4) \\ & k_1+k_2+Ck_3+C^3k_4 & C^2(k_1+k_2)+k_3 & C^3k_1+k_2{}^3+Ck_3+Ck_4 \\ & & k_1+C_1{}^2k_3 & C^2k_1+k_2+Ck_3+k_4 \\ & & & (k_1+Ck_2+C^3k_3+C^2k_4) \end{pmatrix},$$

where $k_i=\langle\phi_{u,i}^0\rangle$. The down-quark mass matrix is obtained by replacing k_i with k_i' where $k_i'=\langle\phi_{d,i}^0\rangle$. (We have not exhibited the normalization factors for different generations.) It is not clear whether these mass matrices can be realistic for some choice of parameters k_i and k_i'. In any case it is not in obvious conflict with observations. Cases (b) and (c) will have similar features. More detailed phenomenological analysis of these models is in progress.

IX. CONCLUSIONS

In this paper we have presented a detailed calculation of Yukawa couplings for the four-generation superstring model for the CP^4 based Calabi-Yau manifold. We find that none of these models lead to a transparently realistic pattern of quark and lepton masses. Our main purpose has been to illustrate the techniques of the calculations in detail.

One of the noteworthy features of these explicit calculations of Yukawa couplings has been the emergence of an interesting structure within these couplings. Typically not all the couplings are of the same order but differ from each other by various powers of C. This may be one way to understand the mass hierarchy of different families. The search for a model in which such a possibility is realized is in progress.

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