Complementary observables and uncertainty relations

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Two observables A and B of an n-level system (i.e., a quantum system with n-dimensional state space) are called complementary, if knowledge of the measured value of A implies maximal uncertainty of the measured value of B, and vice versa. Such observables exist for all n, but no classification (up to equivalence) of all possible pairs of complementary observables is known except for $n \leq 4$. Complementary observables are conjectured to satisfy an "entropic" uncertainty relation of the strongest possible form. This relation has been verified for $n \leq 4$ by explicit calculations. A recent attempt of substantiating the widespread interpretation of uncertainty relations in terms of mutual disturbances between measurements is criticized.

I. INTRODUCTION

The position and momentum components $Q = \mathbf{Q}\cdot\mathbf{e}$ and $P = \mathbf{P}\cdot\mathbf{e}$ of a particle along any common direction $\mathbf{e}$ are "complementary" observables, as expressed by Heisenberg's uncertainty relation

$$\Delta Q \Delta P \geq \tfrac{1}{2}\hbar \; . \tag{1}$$

According to (1) there are no states with arbitrarily narrow statistical distributions of the measured values of both Q and P. If, in particular, there were (normalized) states with $\Delta Q = 0$, i.e., eigenstates of Q, then (1) would imply that ΔP diverges, thus rendering any prediction of the values measured for P in such states rather unreliable. However, as both Q and P have purely continuous spectra, and thus do not possess normalizable eigenstates, neither the limit $\Delta Q = 0$ nor the opposite limit $\Delta P = 0$ of (1) can be realized physically.

Relation (1) and the complementarity of Q and P are discussed in every textbook of quantum mechanics. It is not widely known, however, that pairs of observables with similar properties also exist for a quantum-mechanical n-level system, i.e., a system with an n-dimensional state space (with arbitrary finite $n \geq 2$) (Refs. 1 and 2). These observables are much simpler than Q and P, since mathematical complications such as unboundedness or continuous spectra cannot occur in finite-dimensional state spaces. We hope to demonstrate here that, nevertheless, basic structures of quantum mechanics can be nicely illustrated with such observables.

The general definition of complementary observables for n-level systems is given in Sec. II, and the particular examples discussed in Refs. 1 and 2 are reviewed. Equivalence of two pairs $\{A,B\}$ and $\{A',B'\}$ of complementary observables is defined in Sec. III. By means of examples (with $n=4$) it is shown that there exist pairs which are inequivalent to the already known ones,[1,2] and some comments on the unsolved problem of classifying—up to equivalence—all complementary pairs for arbitrarily given n are added.

An analog of the uncertainty relation (1) for arbitrary pairs of complementary observables of an n-level system, involving information-theoretic entropies as appropriate measures of uncertainty,[3] is formulated in Sec. IV. This relation has been verified by analytical calculations and numerical tests for the particular cases $n=2$, 3, and 4, and is therefore conjectured to hold true in general, although a general proof is still lacking. Being stronger than the "entropic" uncertainty relation following from known estimates,[3,4] our relation also shows that these estimates are not optimal.

The mean-square deviations ΔQ and ΔP entering (1) refer to measurements of Q and P performed on different ensembles of particles in the same state. Quite frequently, however, uncertainty relations such as (1) are interpreted heuristically as expressing mutual disturbances between measurements on the same microsystem. A recent attempt of substantiating this point of view with the help of a new type of uncertainty relation[5] is criticized, and shown to miss its goal, in Sec. V.

II. DEFINITION AND KNOWN EXAMPLES OF COMPLEMENTARY OBSERVABLES

Two observables A and B of a quantum-mechanical n-level system are called complementary (to each other) if their eigenvalues are nondegenerate, and any two normalized eigenvectors e_j of A and f_k of B satisfy

$$|(e_j,f_k)| = \frac{1}{\sqrt{n}} \; . \tag{2}$$

In an eigenstate e_j of A, then, all eigenvalues $b_1,\ldots,b_n$ of B are equally probable as measured values, and vice versa; i.e., exact knowledge of the measured value of one observable implies maximal uncertainty of the measured value of the other. Apart from distinctness (nondegeneracy), the eigenvalues a_j of A and b_k of B are irrelevant in this connection.

Such complementary observables exist for arbitrary dimensions $n=2,3,4,\ldots$, as the following "canonical" example shows.[1,2] Take a fixed orthonormal basis e_j ($j=1,\ldots,n$), let $\alpha_1,\ldots,\alpha_n$ be all nth unit roots (in any order), and set

$$f_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \alpha_k{}^j e_j \quad (k=1,\ldots,n) . \tag{3}$$

Equation (2) and the normalization of f_k are obvious. In order to prove orthonormality, note that

$$(f_k,f_l) = \frac{1}{n} \sum_j (\bar{\alpha}_k \alpha_l)^j$$

with $(\bar{\alpha}_k\alpha_l)^n = 1$ and $\bar{\alpha}_k\alpha_l \neq 1$ if $k \neq l$, and use the following simple fact, the proof of which is left to the reader.

Lemma. Every nth unit root $\alpha \neq 1$ satisfies $\sum_{j=1}^{n} \alpha^j = 0$.

The inverse of (3),

$$e_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \bar{\alpha}_j{}^k f_j , \tag{4}$$

is of the same "canonical" structure (3); e.g., with $\alpha_k = \exp(2\pi i k/n)$ we get $\bar{\alpha}_j{}^k = \bar{\alpha}_k{}^j$, and $\bar{\alpha}_1,\ldots,\bar{\alpha}_n$ are again all nth unit roots. (Note that permutations of the basis vectors $e_1,\ldots,e_n$ and $f_1,\ldots,f_n$ are irrelevant for complementarity.) The term "canonical" is used here for the following reason. With $\alpha_k = \exp(2\pi i k/n)$, Eqs. (3) and (4) become

$$f_k = \frac{1}{\sqrt{n}} \sum_j \exp(2\pi i k j/n) e_j ,$$
$$e_k = \frac{1}{\sqrt{n}} \sum_j \exp(-2\pi i k j/n) f_j . \tag{5}$$

With the particular choices

$$Ae_j = \frac{2\pi j}{n} e_j, \quad Bf_k = \frac{2\pi k}{n} f_k \tag{6}$$

for their eigenvalues, the complementary observables A and B then satisfy the relations

$$\exp(ilA)\exp(imB) = \exp(-2\pi i l m/n) \times \exp(imB)\exp(ilA) \tag{7}$$

for all integers l and m. This is a discrete analog of the Weyl relation

$$\exp(i\lambda Q/\hbar)\exp(i\mu P/\hbar) = \exp(-i\lambda\mu/\hbar) \times \exp(i\mu P/\hbar)\exp(i\lambda Q/\hbar) \tag{8}$$

for arbitrary real λ and μ, which holds true for the canonical pair of complementary observables Q and P. Equation (7) is most easily verified by first using (5) and (6) to prove

$$\exp(ilA) f_k = f_{(k+l)} \tag{9}$$

or

$$\exp(imB) e_j = e_{(j-m)} , \tag{10}$$

where (r) stands for "r modulo n," and then applying both sides of (7) to f_k or e_j.

The operators A and B defined by (5) and (6) and the corresponding unitary operators $U = \exp(iA)$ and $V = \exp(iB)$ are discussed in great detail in Refs. 1 and 2. The present investigation starts from a more general notion of complementary observables and deals with different aspects of the theory.

III. EQUIVALENCE OF COMPLEMENTARY PAIRS; CLASSIFICATION PROBLEM

Unitary equivalence is not a meaningful criterion for deciding whether or not two pairs $\{A,B\}$ and $\{A',B'\}$ of complementary observables are "essentially identical," since the eigenvalues of the operators and their ordering are irrelevant. The appropriate definition of *equivalence* of $\{A,B\}$ and $\{A',B'\}$ is, instead, the following. There exist two permutations π: $\{1,\ldots,n\} \to \{\pi(1),\ldots,\pi(n)\}$ and $\tilde{\pi}$, phase factors α_j and β_j $(j=1,\ldots,n)$, and a unitary operator U, such that

$$Ue_j = \alpha_j e'_{\pi(j)}, \quad Uf_j = \beta_j f'_{\tilde{\pi}(j)} \tag{11}$$

for all $j=1,\ldots,n$ (e'_j and f'_j being the eigenvectors of A' and B', respectively). With $E_j = |e_j\rangle\langle e_j|$, $F_j = |f_j\rangle\langle f_j|$, and similarly for the "primed" quantities, Eqs. (11) simplify to

$$UE_jU^* = E'_{\pi(j)}, \quad UF_jU^* = F'_{\tilde{\pi}(j)} . \tag{12}$$

A useful equivalence criterion involves the unitary "transition" matrix

$$T = \begin{pmatrix} (e_1,f_1) & \cdots & (e_1,f_n) \\ \vdots & & \vdots \\ (e_n,f_1) & \cdots & (e_n,f_n) \end{pmatrix} \tag{13}$$

of the pair $\{A,B\}$ and the analogously defined T' of $\{A',B'\}$. The pairs of complementary observables $\{A,B\}$ and $\{A',B'\}$ are equivalent if and only if the matrix T' may be transformed into T by (i) a permutation π of its rows, (ii) a permutation $\tilde{\pi}$ of its columns, and (iii) the multiplication of its rows and columns by phase factors $\bar{\alpha}_j$ and β_j, respectively. The necessity of this condition is obvious from (11). Sufficiency follows since $Ue_j \equiv \alpha_j e'_{\pi(j)}$ defines a unitary operator U which also satisfies

$$Uf_j = \sum_k (e_k,f_j) U e_k = \sum_k \bar{\alpha}_k \beta_j (e'_{\pi(k)}, f'_{\tilde{\pi}(j)}) \alpha_k e'_{\pi(k)} = \beta_j f'_{\tilde{\pi}(j)} .$$

The application of this criterion in practice is much facilitated if T, and similarly T', is first brought into "standard" form, with $1/\sqrt{n}$ everywhere in its first row and first column, i.e., with

$$(e_1,f_j) = (e_j,f_1) = \frac{1}{\sqrt{n}} \quad \text{for all } j . \tag{14}$$

This can be achieved by multiplying the eigenvectors e_j and f_k by suitable phase factors.

A pair of complementary observables $\{A,B\}$ is called *self-dual*, if it is equivalent to the "dual" pair $\{A',B'\} = \{B,A\}$. (The transition matrices to be compared are T and $T' = T^*$ in this case.) The canonical construction (3) yields a self-dual pair $\{A,B\}$, as already noticed [Eq. (4)].

Straightforward calculation shows that for $n=2$ and $n=3$ every pair of complementary observables is equivalent to the canonical pair defined by (3). But this is not true in general. If n is not prime, say $n=\tilde{n}\,\tilde{\tilde{n}}$, another construction is possible. Let $\tilde{e}_j,\tilde{f}_k$ and $\tilde{\tilde{e}}_l,\tilde{\tilde{f}}_m$ be the eigenvectors of complementary observables in $\tilde{n}$ and $\tilde{\tilde{n}}$ dimensions, respectively; then the product vectors

$$e_{jl}=\tilde{e}_j\otimes\tilde{\tilde{e}}_l,\quad f_{km}=\tilde{f}_k\otimes\tilde{\tilde{f}}_m \tag{15}$$

or, alternatively,

$$e'_{jl}=\tilde{e}_j\otimes\tilde{\tilde{f}}_l,\quad f'_{km}=\tilde{f}_k\otimes\tilde{\tilde{e}}_m \tag{16}$$

define complementary observables A,B or A',B' in n dimensions. (The two alternatives are equivalent if the pair $\{\tilde{\tilde{A}},\tilde{\tilde{B}}\}$ is self-dual, and $\{A,B\}$ is self-dual if both $\{\tilde{A},\tilde{B}\}$ and $\{\tilde{\tilde{A}},\tilde{\tilde{B}}\}$ are.) When applied to canonical pairs $\{\tilde{A},\tilde{B}\}$ and $\{\tilde{\tilde{A}},\tilde{\tilde{B}}\}$, this product construction (15) [or (16)] may, but need not, lead to a pair $\{A,B\}$ which is inequivalent to the canonical pair. An example for inequivalence is provided by $n=4$, $\tilde{n}=\tilde{\tilde{n}}=2$, whereas (15) for $n=6$, $\tilde{n}=2$, $\tilde{\tilde{n}}=3$ is equivalent to (3). Inspection of the corresponding matrices T reveals the crucial difference between these two examples: whereas $\exp(2\pi i/6)$, and thus every sixth unit root, is the product of a second and a third unit root, the fourth unit roots $\pm i$ cannot be represented as products of the second unit roots ± 1.

Besides the equivalence classes obtained from the canonical construction (3) and via (15) or (16), there may exist additional ones, as the example of $n=4$ shows. In this case the most general transition matrix T may be calculated explicitly. It is given in standard form (14) by

$$T(\alpha)=\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & \alpha & -1 & -\alpha\\ 1 & -1 & 1 & -1\\ 1 & -\alpha & -1 & \alpha\end{pmatrix} \tag{17}$$

with an arbitrary phase factor α. Matrices $T(\alpha)$ and $T(\alpha')$ belong to equivalent pairs of complementary observables if and only if $\alpha'=\pm\alpha$ or $\alpha'=\pm\bar{\alpha}$. For $n=4$, therefore, we obtain a one-parameter family of equivalence classes parametrized, e.g., by $\alpha=e^{i\phi}$ with $0\le\phi\le\pi/2$. [The canonical case corresponds to $\alpha=i$, the 2×2 product (15) to $\alpha=1$.] The resulting pair $\{A,B\}$ is self-dual for arbitrary α.

The situation is expected to become even more complicated with increasing n. For instance, the result (17) for $n=4$ may be used to construct explicitly a transition matrix T for $n=8$ which contains 5 arbitrary phase factors. The classification up to equivalence of all possible pairs of complementary observables for arbitrary dimensions n therefore appears to be a nontrivial problem. It is also not known whether all such pairs, like the examples encountered so far, are self-dual.

A more sophisticated method of constructing pairs of complementary observables follows from group theory. Let $\{g_1,\ldots,g_n\}$ be the elements and $\{\chi_1,\ldots,\chi_n\}$ the characters of an Abelian group G of order n. The orthogonality relations for characters imply that a pair of complementary observables is obtained by setting $(e_j,f_k)=\chi_k(g_j)/\sqrt{n}$. With C_n, the cyclic group of order n, for G one recovers the canonical construction (3). This method does not solve the classification problem, however, since there exist only two nonisomorphic Abelian groups C_4 and $C_2\times C_2$, corresponding to the particular cases $\alpha=i$ and $\alpha=1$ of (17), for $n=4$. We therefore do not go into any details here.

IV. ENTROPIC UNCERTAINTY RELATION FOR COMPLEMENTARY OBSERVABLES

Perhaps the most interesting property of complementary observables is the existence of uncertainty relations. Although very different formally, these relations are quite similar in content to the famous Heisenberg relation (1). The formal difference arises from the necessity of adopting a measure of uncertainty which, like the definition of complementary observables and unlike the more familiar mean square deviation, is independent of the eigenvalues of the observables considered. Moreover, if A and B are observables of an n-level system, then ΔA and ΔB are always finite and vanish in eigenstates of A and B, respectively; therefore no uncertainty relation of the form (1) except the trivial one, $\Delta A\,\Delta B\ge 0$, is possible at all.

In this situation, an appropriate measure of the uncertainty of an observable A in a quantum state W is the *information-theoretic entropy*.[3] It is defined as

$$S_W(A)=-\sum_j p_j \ln p_j \tag{18}$$

in terms of the probabilities $p_j=p_j(A,W)$ of measuring the particular values a_j $(j=1,2,\ldots)$ for the observable A in the given state W. This definition makes sense for any observable $A=\sum_j a_j E_j$ with purely discrete spectrum. For the quantum state described by the density matrix $W=W^*>0$ $(\mathrm{tr}\,W=1)$, the probabilities p_j are given by $p_j=\mathrm{tr}(E_jW)$ in terms of the spectral projections E_j of A.

In the case considered here the j sum in (18) is finite, $j=1,\ldots,n$. It is easily shown then that (18) implies

$$0\le S_W(A)\le \ln n\ , \tag{19}$$

in particular, $S_W(A)=0$ if and only if all p_j except one are zero (maximal knowledge), and $S_W(A)=\ln n$ if and only if $p_j=1/n$ for all j (maximal uncertainty of the measured value of A).

If W describes a pure state, $W=|g\rangle\langle g|$ with some unit vector g, we denote $S_W(A)$ by $S_g(A)$. Since $-x\ln x$ is a convex function, the additivity of probabilities

$$p_j(A,W)=\sum_\nu c_\nu p_j(A,g_\nu)$$

in a mixed state

$$W=\sum_\nu c_\nu |g_\nu\rangle\langle g_\nu| \quad \left[c_\nu>0,\ \sum_\nu c_\nu=1\right] \tag{20}$$

implies

$$S_W(A)\ge\sum_\nu c_\nu S_{g_\nu}(A)\ . \tag{21}$$

Similarly to (18) we define $S_W(B)$ for another observable $B=\sum_j b_j F_j$ in terms of the probabilities $q_j=\mathrm{tr}(F_j W)$. For any two observables A and B of this kind in an arbitrary state space $\mathscr{H}$, then, the following uncertainty relation holds true for arbitrary states W (Ref. 4):

$$S_W(A)+S_W(B)\geq 2\ln\left[2/\sup_{j,k}\|E_j+F_k\|\right], \tag{22}$$

with vertical bars denoting the operator norm. [In view of (21), it suffices to derive uncertainty relations such as (22) for arbitrary *pure* states W.]

For the observables considered here, $E_j=|e_j\rangle\langle e_j|$ and $F_k=|f_k\rangle\langle f_k|$ are one dimensional. In this case we have

$$p_j=|(e_j,g)|^2,\quad q_k=|(f_k,g)|^2 \tag{23}$$

for pure states $W=|g\rangle\langle g|$, and (22) takes the simpler form[3]

$$S_W(A)+S_W(B)\geq 2\ln\left[2/\left[1+\sup_{j,k}|(e_j,f_k)|\right]\right]. \tag{24}$$

For complementary observables, in particular, we thus obtain with (2) the uncertainty relation

$$S_W(A)+S_W(B)\geq 2\ln[2/(1+1/\sqrt{n})]. \tag{25}$$

The estimates leading to (22) (Ref. 3) or (24) (Ref. 3) are not optimal, however, and therefore (25) need not be optimal either. Explicit calculations for $n=2$, 3, and 4 indeed indicate that, most likely, the general uncertainty relation for complementary observables is

$$S_W(A)+S_W(B)\geq \ln n, \tag{26}$$

which is much stronger than (25) (in particular for large n), and obviously cannot be further improved. In view of (19), (26) is in fact the most stringent among all conceivable uncertainty relations for observables of an n-level system.

The explicit calculations supporting (26) have been performed for pure states $W=|g\rangle\langle g|$ using Eq. (23), the known representations (up to equivalence) of e_j and f_k, and a suitable parametrization of the arbitrary unit vector g. For $n=2$ they can still be done analytically, whereas for $n=3$ and 4 the minimum of the left-hand side of (26) has been determined numerically; in particular, different values of the parameter α in (17) have been used for $n=4$. It is left to the reader to judge whether or not these few examples make the conjecture (26) sufficiently plausible. In any case, she or he is invited to look for a general proof or a counterexample.

V. UNCERTAINTY RELATIONS FOR SUCCESSIVE MEASUREMENTS

The uncertainty relations discussed so far refer to independent measurements of A and B on *different* microsystems in the same state W. They therefore do not support the widespread opinion that uncertainty relations have something to do with unavoidable mutual disturbances between measurements of A and B on *the same* microsystem. For this reason another type of uncertainty relation has been introduced recently.[5] These relations connect the uncertainty $S_W(A)$ of A in an arbitrary initial state W with the uncertainty $S_{W'}(B)$ of B in a new quantum state W', which results from W if a measurement of A is performed. The A measurement is expected to disturb the subsequent measurement of B, and uncertainty relations of this kind should thus be able to substantiate the opinion mentioned above.

The basic assumption of Ref. 5 is the familiar "projection postulate," which asserts that the measurement of an observable $A=\sum_j a_j E_j$ transforms W into the new state

$$W'=\sum_j E_j W E_j. \tag{27}$$

This state is a mixture $\sum_j p_j W_j'$ of the states $W_j'=E_j W E_j/\mathrm{tr}(E_j W)$. The state W_j' describes those particular systems for which the result of the A measurement was a_j, and accordingly the weight $p_j=\mathrm{tr}(E_j W)$ of W_j' in the mixture W' coincides with the probability of obtaining the result a_j in the original state W.

The state W' given by (27) corresponds to an A measurement "of the first kind;" i.e., a repetition of the A measurements yields, for every single system, the same result as the first measurement. This follows since the states W_j' are (in general, mixed) eigenstates of A corresponding to the particular eigenvalue a_j, and implies $S_{W'}(A)=S_W(A)$. Therefore a lower bound for $S_W(A)+S_{W'}(B)=S_{W'}(A)+S_{W'}(B)$ already follows from (22). Because of the particular form (27) of W', however, this lower bound may be improved. Since W' is mixture of eigenstates of A, it suffices to derive a lower bound for *pure* eigenstates, $W'=|g\rangle\langle g|$ with $E_j g=g$ for some j, in virtue of (21). In this case we have $S_g(A)=0$ and, since

$$\begin{aligned} q_k &= (g,F_k g)\\ &= (g,E_j F_k E_j g)\leq \sup_j\|E_j F_k E_j\| \quad\text{for all } k,\\ S_g(B) &= -\sum_k q_k \ln q_k \geq -\ln\left[\sup_{j,k}\|E_j F_k E_j\|\right]. \end{aligned}$$

With (21) this implies the inequality

$$S_{W'}(B)\geq \ln\left[1/\sup_{j,k}\|E_j F_k E_j\|\right], \tag{28}$$

which expresses the randomizing effect of the A measurement (27) upon the observable B. [This is exactly the estimate (48) of Ref. 5, but it was not realized there that $\|\sum_j E_j F_k E_j\|=\sup_j\|E_j E_k E_j\|$.]

The desired "uncertainty relation for successive measurements,"[5]

$$S_W(A)+S_{W'}(B)\geq \ln\left[1/\sup_{j,k}\|E_j F_k E_j\|\right], \tag{29}$$

follows trivially from (28) and $S_W(A)\geq 0$. The right-hand side of (29) is easily shown[5] to be at least as large as the right-hand side of (22). The estimate (29) is thus interpreted in Ref. 5 as supporting the mutual disturbance interpretation of uncertainty relations mentioned above. Since

$$||EFE|| = ||EF(EF)^*|| = ||EF||^2 = ||(EF)^*||^2$$
$$= ||FE||^2 = ||FEF||$$

for any two projection operators E and F, the lower bounds in (28) and (29) remain unchanged if the roles of A and B are interchanged on the left-hand sides.

For complementary observables, Eqs. (2) imply

$$||E_j F_k E_j|| = ||(1/n)E_j|| = 1/n \ .$$

With (19), therefore, (28) becomes an equality,

$$S_{W'}(B) = \ln n \ , \tag{30}$$

while (29) becomes formally similar to (26):

$$S_W(A) + S_{W'}(B) \geq \ln n \ . \tag{31}$$

According to (30), the randomization of B due to the A measurement (27) is complete, independently of the initial state W.

The projection postulate (27) is not suitable to substantiate general statements about quantum measurements, however. In order to be universally valid, such statements have to be based on the most general description of measurements which is compatible with the principles of quantum mechanics. Contrary to what is still asserted in some textbooks, Eq. (27) does not belong to these principles. It rather describes a very idealized kind of measurement only, which need not be—and in most cases indeed is not—realized in actual measurements. The only indispensable requirement for the measurement of an observable $A = \sum_j a_j E_j$ is that the alternative readings r_j $(j = 1,2,\ldots)$ of the measuring apparatus, which correspond to the possible measured values a_j of A, occur with relative frequencies $p_j = \mathrm{tr}(E_j W)$ when the apparatus is successively applied to many microsystems in an arbitrarily given state W. If, as in the situation discussed here, one wants to admit the subsequent measurement of another observable B on the same ensemble of microsystems, one has to assume in addition that the A measurement is nondestructive—i.e., that every single microsystem is still present after its interaction with the measuring apparatus.

The state change ("operation") induced by an A measurement of this general type may be represented in the following form.[6,7] The (arbitrary) initial state W of the microsystems is transformed into

$$W' = \sum_{\nu \in I} T_\nu W T_\nu^* \ , \tag{32}$$

with a finite or countably infinite index set I consisting of disjoint subsets I_j $(j = 1,2,\ldots)$, and operators T_ν $(\nu \in I)$ satisfying

$$\sum_{\nu \in I_j} T_\nu^* T_\nu = E_j \tag{33}$$

for all j. Conversely every finite or countably infinite set of operators T_ν $(\nu \in I = \cup_j I_j)$ satisfying Eqs. (33) may be conceived as describing, via (32), the state change induced by some nondestructive measurement of the observable A. The state W' as given by (32) is a mixture $\sum_j p_j W_j'$ of the states

$$W_j' = \sum_{\nu \in I_j} T_\nu W T_\nu^* \Big/ \mathrm{tr}\left[\sum_{\nu \in I_j} T_\nu W T_\nu^*\right] . \tag{34}$$

The state W_j' describes the subensemble of those microsystems for which the apparatus reading was r_j, and accordingly its weight in the mixture W' is

$$p_j = \mathrm{tr}\left[\sum_{\nu \in I_j} T_\nu W T_\nu^*\right]$$
$$= \mathrm{tr}\left[\sum_{\nu \in I_j} T_\nu^* T_\nu W\right] = \mathrm{tr}(E_j W) \ , \tag{35}$$

by (33).

If one chooses the I_j to be one-index sets $I_j = \{j\}$ and takes $T_j = E_j$ for all j, one recovers Eq. (27), but obviously this is a very particular case of (32) only.

It is easily seen now that *no* uncertainty relations such as (29) or (31) can be derived from (32). In order to give a counterexample, consider two arbitrary observables with nondegenerate spectra,

$$A = \sum_j a_j |e_j\rangle\langle e_j| \quad \text{and} \quad B = \sum_j b_j |f_j\rangle\langle f_j| \ ,$$

in a finite- or countably infinite-dimensional state space $\mathscr{H}$. Let again the I_j be one-index sets $I_j = \{j\}$ but take now, instead of $T_j = |e_j\rangle\langle e_j|$ as in (27), $T_j = |f_j\rangle\langle e_j|$, which also satisfy Eqs. (33). In this case (34) becomes

$$W_j' = T_j W T_j^* / \mathrm{tr}(T_j W T_j^*) = |f_j\rangle\langle f_j| \ ,$$

and therefore (32) leads to

$$W' = \sum_j p_j |f_j\rangle\langle f_j| \tag{36}$$

with $p_j = \mathrm{tr}(E_j W) = (e_j, W e_j)$. The state W' after this measurement of A is thus a mixture of eigenstates of the observable B, and the probability p_j of measuring the value b_j for B in this state coincides with the probability of measuring the value a_j for A in the initial state W. With (36), therefore, $S_{W'}(B)$ and $S_W(A)$ become equal, and vanish simultaneously—thus violating any conceivable uncertainty relation—if W is an eigenstate of A.

The example (36) also illustrates that no counterparts of (28) or (30) can be derived from (32) either. [This is already obvious since (28) would imply (29).] For instance, let A and B be complementary, and take $W = |f_k\rangle\langle f_k|$, such that $S_W(B) = 0$. Since $p_j = 1/n$ in this case, (36) implies $S_{W'}(B) = \ln n$; i.e., the A measurement has indeed led to a total randomization of B. The same A measurement has exactly the opposite effect, however, for $W = |e_k\rangle\langle e_k|$, since (36) then yields $W' = |f_k\rangle\langle f_k|$, and thus $S_{W'}(B) = 0$, while $S_W(B) = \ln n$. Even for complementary observables, therefore, a measurement of A does not necessarily lead to a randomization of B.

In view of such apparently counterintuitive results, one might be tempted to regard an A measurement leading to (36) as a mere theoretical possibility which, in contrast to the "ideal" measurement (27), is hardly realizable in the laboratory. In order to refute this objection, consider the example of spin measurements on a beam of spin-$\frac{1}{2}$ particles with a Stern-Gerlach apparatus. Any two spin components, e.g., S_x and S_y, are complementary observables for $n=2$. On a particle beam moving in the z direction, S_x may be measured with a Stern-Gerlach apparatus whose inhomogeneous magnetic field is parallel to the x axis. This field splits the beam into two branches with $S_x=+\hbar/2$ and $S_x=-\hbar/2$, respectively, the relative intensities of which are given by the probabilities of finding either $S_x=+\hbar/2$ or $S_x=-\hbar/2$ in the initial spin state W, and thus the apparatus produces a new spin state W' of the form (27).

(Strictly speaking, the splitting of the beam alone is not yet a measurement of S_x, since the measured values remain undetermined unless one detects, in addition, into which branch of the beam every single particle is deflected. As this detection involves a localization of the particle, one may imagine that it leaves its spin—and thus the spin state of the split beam—unchanged. If subsequent measurements are performed on both branches of the split beam, the detection in question is not even necessary, since every subsequent measurement also involves the detection of single particles, thereby leading to a "postponed" completion of the S_x measurement as well.)

The observable S_x may also be measured, however, by first rotating the spin $\mathbf{S}$ by $\pi/2$ around the z axis, e.g., by letting the beam pass through a suitable magnetic field in z direction, and then applying a Stern-Gerlach apparatus oriented parallel to the y axis. Since now the particle beam is split into two branches with $S_y=+\hbar/2$ and $S_y=-\hbar/2$, respectively, this measurement leads to a final spin state W' of the form (36). The latter is thus almost as easily realizable as the final state (27) of an "ideal" measurement of S_x.

As other heuristic arguments, the "mutual disturbance" interpretation of uncertainty relations has had its merits in the history of quantum mechanics. As they stand, however, the presently known uncertainty relations such as (1) or (26) do not refer to mutual disturbances between *measurements*. They deal instead with limitations for the *preparation* of microsystems by expressing, e.g., the impossibility of preparing ensembles with arbitrarily narrow statistical distributions of both position and momentum.

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