Distinguishing three-body from two-body nonseparability by a Bell-type inequality

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We derive an inequality, violated by quantum mechanics, that in a three-body system can detect three-body correlations that cannot be reduced to mixtures of two-body ones related locally to the third body.

Physicists generally agree that quantum mechanics gives accurate and at times remarkably accurate numerical predictions. The existing body of experimental evidence, however, is not qualitatively diverse enough to warrant the attitude that deems the present quantum-mechanical formalism universally valid. Now certain qualitative features of this formalism, such as the existence of state superposition, and the existence of a calculus of transition probabilities, are common to practically all the formalisms that can be grouped under the generic label of "quantum logic."[1] Such features are formal and do not reflect any philosophical attitudes; their existence is immune to controversies regarding interpretations, being in fact part of the ground of these controversies. We must therefore in proposing tests of quantum mechanics consider it as a very particular mathematical framework (such as the study of very particular self-adjoint differential operators in a complex Hilbert space) and not as a set of general principles prior to some mathematical expression. It is in the quantitative tests of some of its predictions that we must seek any clue as to its possible universality. Seen as a theory that describes systems of particles, we can consider the question of the applicability of the formalism under various grossly defined conditions: (1) the number of particles; (2) the spaciotemporal configuration of the system; (3) the predominant interaction type; and (4) the particle type. A comprehensive test program should try to sample in some uniform way all the possibilities that are created by independently varying each of the above features in the experimentally accessible range.

Existing convincing support of the quantum-mechanical formalism should certainly include various successes of atomic spectroscopy, various precise quantum-electrodynamical predictions, such as the values of the Lamb shifts and the anomalous magnetic moments,[2] and a series of experiments directly relevant to foundation questions, such as neutron scattering in perfect crystals (Ref. 3), K^0-$\overline{K}^0$ oscillations,[4] low-intensity photon interference,[5] and separated two-body systems used to explore Bell's inequalities.[6] Of these the neutron and kaon experiments are both significant in that several interaction types enter, and the kaon system involves strange particles. The two-photon experiments are significant in that they deal with spacelike separations. There is certainly a very large body of other favorable evidence, but which cannot be judged to be conclusive because of the approximations or the phenomenological input that must be used in the theoretical treatment. Nuclear, condensed matter, and the bulk of elementary-partice physics fall into this category.

We are thus very far from having seen a reasonable portion of the sample space presented above. What is particularly lacking is evidence on many-body systems. Such evidence would be particularly valuable since we live in a many-body world. The gross features of the macroscopic world directly accessible to the senses are apparently well described in classical terms. In spite of many ingenious attempts, there seems to be no way to reconcile this with quantum universality, maintaining any sort of philosophical conservatism. The most natural attitude would then seem to be to propose that certain quantitative features of the present quantum-mechanical formalism become modified as the number of particles increases. We do not want to speculate here on the possible mechanisms of this modification. With presently known dynamical schemes, their description could be either dynamical or extradynamical. Extradynamical features of many-body systems are already known. The necessity of symmetrizing or antisymmetrizing many-body wave functions of identical particles does not at present have any dynamical explanation. Other many-body restrictions may exist and may not become apparent even with experimental evidence for extremely accurate two-body dynamics.

It is therefore important to devise direct tests of quantum mechanics in many-body situations that are immune to any explanation in terms of mechanisms involving fewer bodies, for which a quantum-mechanical description is presumably accurate.

This is the main conceptual point of this paper. We shall now argue that experimentally feasible tests of quantum mechanics of this nature can probably be devised. We do this by deriving an inequality of a Bell type but with a completely different aim.

In separated two-body systems, the experimental situation in relation to Bell's inequalities[6] seems to rule out any local microrealistic explanation of quantum-mechanical correlations. We must admit either superluminal influences or extended physical entities that do not possess well-defined local properties.[7] Thus there seems to be no conservative alternative to the Copenhagen philosophy. Having a firm experimental ground for this thesis for certain two-particle systems, it becomes pertinent to ask how many particles can participate in such situations. Quantum mechanics allows for an arbitrary number. This is because a state vector belonging to a tensor product of N Hilbert spaces is in general not an eigenstate of any operator, other than a multiple of the identity, acting on any smaller subproduct. If such a state vector represents a quantum-mechanical state of N particles, then no subgroup of fewer particles has well-defined properties. Nature, on the other hand, may impose her own restrictions and limit this possibility. We can imagine a multiparticle system as consisting of a collection of extended entities, each one comprising a certain limited number of particles, with no subclass of these particles having well-defined properties. The extended entities themselves, however, can posess well-defined global properties that behave locally in relation to other such entities. We shall give the name of "limited entanglement" to this situation. This notion has some resemblance to the notion of "disconnectivity" of Leggett[8] who expresses concerns similar to ours.

It is this situation that we claim to give rise to a Bell-type inequality. Since our aim here is to establish this point of principle, rather than give a general and exhaustive treatment, we proceed under various simplifying assumptions. We explicitly derive our inequality in the simplest hypothetical case that the extended entity cannot consist of more than two bodies and see what restrictions this imposes on the three-body observations. Similar arguments can be brought forth for extended entities of no more than some given number N of bodies with observation of $N+1$ bodies. It is important to discuss the use we envisage for such inequalities. If limited entanglement is in fact true, we expect it to come in gradually as the number of bodies increases. Thus we should not expect a sudden breakdown of quantum mechanics at a certain number of bodies. Such an inequality is thus considered as a convenient and probably effective way to detect incipient breakdowns of quantum mechanics due to limited entanglement. In such situations, though the inequality continues to be violated, the results would first be seen to differ slightly but significantly from the quantum-mechanical prediction. If limited entanglement is true, it possibly only becomes operant for a very large number of bodies. We thus do not concern ourselves with proving a reciprocal version of the inequalities, that is, finding criteria under which observed data imply limited entanglement in its pure form, since this is not expected to be readily seen.

Imagine thus a system decaying into three subsystems which then separate in three different directions. At some later time we perform dichotomous measurements on each of the three parts, represented by observables A, B, and C, respectively, with possible results ± 1. Let us now make the following hypothesis of limited entanglement: An ensemble of such decaying systems consists of three subensembles in each one of which two given parts form an extended system which however behaves locally with respect to the third part. Let us for the time being focus our attention on one of these subensembles, the one in which the third part behaves locally with respect to the system formed by the other two. We express our locality hypothesis by assuming a factorizable expression for the probability $p(abc)$ for observing the results a, b, and c for the observables A, B, and C, respectively:

$$p(abc)=\int q(ab\lambda)r(c\lambda)d\rho(\lambda)\ , \tag{1}$$

where q and r are probabilities conditioned to the hidden variable λ with probability measure $d\rho$. Factorizability is another one of our simplifying assumptions which we adopt for its mathematical simplicity. We believe that our inequalities, just as Bell's should hold under various ways of postulating local realism among the extended entities. Formulas similar to (1) with the role of the third part taken up by the first and the second of course describe the other two ensembles. Our assumption is to be contrasted with that of absolute local realism which would be expressed by complete factorizability in which q would also be a product:

$$q(ab\lambda)=s(a\lambda)t(b\lambda)\ .$$

This would lead to Bell's inequalities in the two-particle subsystems and thus to no new insight. In view of the experimental two-photon evidence we discard this possibility.

Computing the expected value

$$\begin{aligned}E(ABC)&=\langle ABC\rangle\\&=p(+++)+p(+--)+p(-+-)\\&\quad+p(--+)-P(-++)-p(+-+)\\&\quad-p(++-)-p(---)\end{aligned}$$

of the product of the three observables, we find

$$E(ABC)=\int (AB\lambda)v(C\lambda)d\rho(\lambda)\ , \tag{2}$$

where $-1\le u(AB\lambda)\le 1$ and $-1\le v(C\lambda)\le 1$. We can now derive inequalities satisfied by the numbers $E(ABC)$ when we introduce alternative dichotomous observables $A_1,A_2,\ldots;\ B_1,B_2,\ldots,;\ C_1,C_2,\ldots$ for each of the parts. We seek inequalities of the form

$$\sum_{ijk}C_{ijk}E(A_iB_jC_k)\le M\ . \tag{3}$$

The simplest case is given by choosing two alternative observables for each subsystem and we derive the inequality that results from thus assuming that i, j, and k run from 1 to 2. The derivation follows closely the arguments used in Garuccio and Selleri[9] but is self-sufficient.

Looking at (3) we see that we must put a bound on

$$\int\sum C_{ijk}u(A_iB_j\lambda)v(C_k\lambda)d\rho(\lambda)\ .$$

Now since $\int d\rho=1$ the above integral is a weighted average of the set of numbers

$$\sum C_{ijk}u(A_iB_j\lambda)v(C_k\lambda)\ ,$$

where λ runs over all possible values of the hidden variables. Thus any λ independent bound on this expression is also a bound on the integral. Furthermore, from $-1\le u(A_iB_j\lambda)\le 1$ and $-1\le v(C_k\lambda)\le 1$ we see that the expression will be bounded by the maximum of

$$\sum C_{ijk}\xi_{ij}\eta_k\ ,$$

where the ξ_{ij} run over the four-cube $[-1,1]^4$ and the η_k

over the square $[-1,1]^2$. By convexity arguments, the expression must assume its maximum on some vertex of $[-1,1]^4\times[-1,1]^2$ and so we can take our bound to be

$$M=\max\sum C_{ijk}\xi_{ij}\eta_k ,$$

where now the maximum is taken over $\xi_{ij}=\pm1$, $\eta_k=\pm1$. Let us now for simplicity take $C_{ijk}=\sigma_{ijk}$, as a sign. Set $\eta_2=\eta\eta_1$ for some sign η. Taking into account that $\sigma_{ijk}{}^2=1$ we have

$$M=\max\sum \sigma_{ij1}\eta_1\xi_{ij}(1+\sigma_{ij1}\sigma_{ij2}\eta) .$$

The maximum being over ξ_{ij}, η, and η_1. Being now free to choose ξ_{ij} and η_1 we can arrange that for all ij, and a given η:

$$\sigma_{ij1}\eta_1\xi_{ij}=\mathrm{sgn}(1+\sigma_{ij1}\sigma_{ij2}\eta) .$$

This now immediately leads to the conclusion that M is the maximum of

$$|1+\sigma_{111}\sigma_{112}\eta|+|1+\sigma_{211}\sigma_{212}\eta| +|1+\sigma_{121}\sigma_{122}\eta|+|1+\sigma_{221}\sigma_{222}\eta| . \quad (4)$$

The smallest possible M is thus 4, in which case we must choose two of the signs among $\sigma_{111}\sigma_{112}$, $\sigma_{211}\sigma_{212}$, $\sigma_{121}\sigma_{122}$, and $\sigma_{221}\sigma_{222}$, to be equal and the other two to be opposite to that one.

At this point we must consider the other two subensembles. We do not know in any particular instance of decay to which of the subensemles the event belongs. It is precisely in this lack of knowledge of how to separate each instance of the three-part system into its two subsystems by which probabilities factorize, that our analysis cannot be reduced to the usual one involving locality between two subsystems. Our inequality must be such that the same one holds up to an overall sign of the left-hand part, if we were to treat either the first or the second part as the one having well-defined properties. This is just a question of straightforward relabeling and comparing, and under this additional condition we obtain two possible solutions, up to overall sign:

$$(\sigma_{111},\sigma_{112},\sigma_{211},\sigma_{212},\sigma_{121},\sigma_{122},\sigma_{221},\sigma_{222})=(+++-+---)$$

and

$$(\sigma_{111},\sigma_{112},\sigma_{211},\sigma_{212},\sigma_{121},\sigma_{122},\sigma_{221},\sigma_{222})=(+------+) .$$

Abbreviating $E(A_iB_jC_k)$ to $E(ijk)$ our inequalities that follow from the limited entanglement hypothesis are thus

$$|E(111)+E(112)+E(211)-E(212)+E(121)-E(122)-E(221)-E(222)|\le4 , \quad (5)$$

$$|E(111)+E(112)-E(211)-E(212)-E(121)-E(122)-E(221)+E(222)|\le4 . \quad (6)$$

To show that quantum mechanics contradicts these, consider that each subsystem is described by a two-dimensional Hilbert space with basis

$$u=\begin{bmatrix}1\\0\end{bmatrix},\ d=\begin{bmatrix}0\\1\end{bmatrix} .$$

Assume that on each subsystem we can perform a dichotomous observation labeled by an angle α and given by the transition amplitudes:

$$\langle + | u\rangle=\cos(\alpha/2)=\langle - | d\rangle ,$$

$$-\langle - | u\rangle=\sin(\alpha/2)=\langle + | d\rangle ,$$

where $A(\alpha)|\pm\rangle=\pm|\pm\rangle$, and similarly for B and C, labeled by angles β and γ. These are formally identical to rotated Stern-Gerlach apparatus for $s=\frac{1}{2}$ measurements, where u and d are eigenstates of σ_z. For the quantum-mechanical state $(udd+dud+ddu-uuu)/2$ we find after a straightforward computation that $E(A(\alpha)B(\beta)C(\gamma))=\cos(\alpha+\beta+\gamma)$. For this expression inequality (6) is obtained from (5) by summing 180° to each of α_2, β_2, and γ_2 so the two are equivalent. A numerical search reveals, for example, that for

$$\alpha_1=218°,\ \beta_1=81°,\ \gamma_1=15° ,$$

$$\alpha_2=309°,\ \beta_2=171°,\ \gamma_2=105° ,$$

the value in the expression under the modulus sign in (5) is 5.66 in evident contradiction with the inequality.

We do not have at the moment any detailed suggestion for a specific experimental arrangement. The triplet decay of positronium comes to mind as a possibility and has already been proposed for possible new tests of quantum mechanics.[10]

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[1]There is a vast amount of literature on this topic. Some representative works can be found in *Mathematical Foundations of Quantum Theory,* edited by A. R. Marlow (Academic, New York 1978); *Interpretations and Foundations of Quantum Theory,* edited by H. Neumann (Wissenschaftsliche Verlagsgesellschaft, Mannheim, 1981). See also W. Guz, Ann. Inst. Henri Poincaré A **XXXIV**, 15 (1981), for definitions of superposition and transition probabilities and S. P. Gudder,

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[2]The very readable article by V. F. Weisskopf, Phys. Today, **34** (11), 69 (1981), gives a general overview of the successes of field theory in the last fifty years. Quantum-field-theoretic results are very dependent on the dynamics and one may argue that a simultaneous change of dynamics and the overall formalism could, within experimental indistinguishability, leave the results intact. Thus these types of results are significant insofar as the validity of the formalism is established in a variety of contexts under different dynamics.

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