PHYSICAL REVIEW D VOLUME 35, NUMBER 10 15 MAY 1987

Constraints on nonlinear extensions of the Lorentz group

Jean Gauthier and Louis Marchildon
Département de Physique, Université du Québec, Trois-Rivières, Québec, Canada G9A 5H7
(Received 31 October 1986)

We investigate *a priori* possible extensions of the Lorentz group to nonlinear coordinate transformations between equivalent frames. We consider nonlinear transformations preserving uniform rectilinear motion, or mapping the world lines of points at rest to uniform rectilinear motion with a fixed velocity. In each case, we implement the requirement that coordinate transformations between equivalent frames must form a group. We obtain strong constraints on nonlinear extensions of the Lorentz group which generalize earlier results on linear extensions.

I. INTRODUCTION

In the past 15 years, many attempts have been made to generalize the Lorentz transformations to include inertial frames moving faster than the speed of light in vacuum.[1] From a theoretical point of view, this is the most appealing way to apprehend the behavior of hypothetical faster-than-light particles. If superluminal frames exist and are equivalent to our frames, then tachyons behave in their frames exactly as bradyons behave in ours. The postulate of the equivalence of superluminal inertial frames to the ones of special relativity is known as the extended principle of relativity.

Appealing as the postulate may be, its consequences have not, however, always been fully appreciated. To understand the problem, it is best to formulate it in group-theoretical language. It is well known that coordinate transformations between equivalent reference frames must form a group.[2] Let $L_+^\uparrow$ denote the proper orthochronous Lorentz group. This is the set of all linear coordinate transformations which leave the Minkowski quadratic form invariant, have determinant $+1$, and preserve the direction of time. The fundamental laws of physics are widely believed to be invariant under $L_+^\uparrow$. Suppose we want to enlarge $L_+^\uparrow$ by adding to it at least one superluminal transformation s (or, for that matter, any coordinate transformation), with s relating two equivalent reference frames. The extended principle of relativity then implies the existence of many more equivalent frames, namely, all those related by repeated products of s, s^{-1}, and elements of $L_+^\uparrow$. These repeated products make up a group G, which is the smallest group that contains s and $L_+^\uparrow$. Every element of G must be a coordinate transformation between equivalent frames. That is, under the extended principle of relativity, G is a group of symmetry of the fundamental laws of nature.

This approach was investigated in Ref. 3, where linear extensions of the Lorentz group were considered. Let $SL(4;R)$ $[\overline{SL}(4;R)]$ denote the group of all linear real coordinate transformations in four dimensions with unit determinant [with determinant ± 1]. Then the following theorem holds.[3]

Theorem 0. Let s be any element of $\overline{SL}(4;R)$ outside the full Lorentz group. The smallest Lie group that includes s and $L_+^\uparrow$ is either $SL(4;R)$ or $\overline{SL}(4;R)$, depending on whether $\det s = +1$ or $\det s = -1$.

This theorem can be interpreted as follows. From an experimental point of view, $SL(4;R)$ [*a fortiori* $\overline{SL}(4;R)$] does not appear to be a group of coordinate transformations between equivalent frames. That is, the fundamental laws of physics as we presently know them (e.g., Maxwell's equations) are not invariant under $SL(4;R)$. By Theorem 0, if s belongs to $\overline{SL}(4;R)$ and does not belong to the full Lorentz group, it is impossible to find a Lie group that includes $L_+^\uparrow$ and s and does not include $SL(4;R)$. Therefore, any extension of $L_+^\uparrow$ by linear real elements (of determinant ± 1) outside the full Lorentz group, in particular by linear real superluminal coordinate transformations, is, as of now, empirically ruled out.[4]

The hypothesis of linearity, which was made in Ref. 3, was certainly appropriate from several points of view. Most superluminal transformations proposed to this day in the literature are linear. Furthermore, linear coordinate transformations will always relate frames moving with uniform relative speeds, as the Lorentz transformations do. Nevertheless, there are several instances where nonlinear coordinate transformations have important uses as symmetry transformations. Here one thinks of the quasitranslations in Robertson-Walker or de Sitter cosmologies.[5] And perhaps even more of the conformal group, relevant in the context of massless fields, which involves nonlinear coordinate transformations, the so-called uniform accelerations.[6]

In this paper we intend to investigate possible nonlinear extensions of the Lorentz group. We shall not, however, deal with completely arbitrary nonlinear transformations. We know that Lorentz transformations relate the coordinates of frames moving with uniform relative speed. We want the nonlinear extensions to retain, at least partly, this character of uniform (and possibly superluminal) relative motion.

Accordingly we will explore two different avenues. In Sec. II, we will consider inertia-preserving transformations, that is, coordinate transformations which map uniform rectilinear motion in four-dimensional space to motion of the same nature. These transformations are of fractional linear form. We will examine the effect of adding any such transformation s to the proper orthochronous Lorentz group $L_+^\uparrow$. For any choice of s, we will explicitly obtain the smallest group G that contains s and $L_+^\uparrow$. We will show that G is either a 10-parameter or a 19-parameter group, neither of which is a symmetry group of the laws of physics as we presently know them. In Sec. IV we will show that this raises serious doubts on the relevance of these groups as viable extensions of $L_+^\uparrow$. We will also address the question of what happens if, as is likely the case, our present knowledge of the laws of physics is only approximate.

In Sec. III we will investigate coordinate transformations which map the world lines of points at rest to uniform rectilinear motion with a fixed speed. This concept seems not to have been much explored in the literature. Nevertheless, it appears to correspond rather well to a minimal concept of uniform relative motion between frames. We will obtain the most general coordinate transformations which map points at rest into uniform rectilinear motion with a fixed speed. We will then show that if the group property is to be maintained, there is no viable extension of the Lorentz group by means of these transformations.

II. INERTIA-PRESERVING TRANSFORMATIONS

Consider the set of all space-time points. A reference frame is a mapping of this set onto R^4, the four-dimensional real continuum. Three dimensions of the continuum are associated with space coordinates, the fourth dimension being associated with the time coordinate. A physical meaning is given to space and time coordinates once one prescribes a way to measure them, for instance, by means of clocks and meter sticks.

A straight line in R^4 represents the uniform rectilinear motion of a particle. Let x represent the coordinates of a point in R^4. In matrix notation, x is a column of four real numbers. It is well known that the most general mapping of R^4 to itself which transforms any straight line into a straight line has the form[7]

$$x' = \frac{Ax}{1+w^t x} + x^{(0)} . \tag{2.1}$$

Here $x^{(0)}$ and w are constant four-columns and the superscript t denotes matrix transposition. A is an arbitrary real 4×4 matrix with nonvanishing determinant, that is, an element of $\mathrm{GL}(4;R)$. For $w\neq0$, the mapping (2.1) is singular on the hyperplane $w^t x = -1$.

Transformations such as (2.1) make up a 24-parameter group. Five of these, however, are rather trivial. They are the space-time translations $x^{(0)}$ and the overall scale factor in the matrix A, which corresponds to uniform space-time dilatations. They represent transformations between frames relatively at rest. We shall henceforth restrict ourselves to inertia-preserving transformations of the form

$$x' = \frac{Ax}{1+w^t x} , \tag{2.2}$$

where w is arbitrary and A is restricted to $\overline{\mathrm{SL}}(4;R)$, the set of all real 4×4 matrices with determinant ±1. Transformations such as (2.2) make up a 19-parameter subgroup of (2.1). For reasons that will be apparent we denote this group by $\overline{\mathrm{SL}}(4;R)\cdot T_4$. An arbitrary element of the group can be denoted by (A,w).

The group product law is very easy to obtain, by making two successive transformations like (2.2). One checks that

$$(A_2,w_2)(A_1,w_1) = (A_2A_1, w_1 + A_1^t w_2) . \tag{2.3}$$

The inverse of (A,w) is $(A^{-1}, -(A^t)^{-1}w)$.

Several subgroups of $\overline{\mathrm{SL}}(4;R)\cdot T_4$ are of interest. Let 0 denote the column with four zeros. The set of all elements of the form $(A,0)$ is a subgroup isomorphic to $\overline{\mathrm{SL}}(4;R)$. Let I denote the unit matrix. The set of all elements of the form (I,w) is an invariant subgroup isomorphic to T_4, the Abelian group of all four-dimensional translations. The action of (I,w) on x through (2.2), however, should not be confused with a space-time translation, even though their group properties are the same.

The group structure of $\overline{\mathrm{SL}}(4;R)\cdot T_4$ is essentially the one of a semidirect product, although the semidirect product law is usually defined slightly differently. Elements of the form (M,w), with M in $L_+^\uparrow$, also form a subgroup of $\overline{\mathrm{SL}}(4;R)\cdot T_4$, which we will denote by $L_+^\uparrow\cdot T_4$.

An important remark should be made on what is meant by the statement that transformations (2.2) map straight lines to straight lines. Let t denote the time coordinate and x_i $(i=1,2,3)$ the three spatial coordinates of a given event. Let x_{0i} and v_i be six arbitrary constants. The set of three equations

$$x_i = x_{0i} + v_i t \tag{2.4}$$

represents a rectilinear motion with velocity components v_i, going through the point x_{0i} at $t=0$. It is easy to check that these three equations, when substituted in (2.2), imply three equations such as

$$x_i' = x_{0i}' + v_i' t' , \tag{2.5}$$

where x_{0i}' and v_i' are constants which depend on x_{0i} and v_i. The fact that v_i' depends on x_{0i} means that two straight lines with the same velocity are mapped in general to two straight lines with different velocities. It is not difficult to show that v_i' is independent of x_{0i} if and only if the denominator in (2.2) is identical to 1 (that is, if w vanishes). In other words, $\overline{\mathrm{SL}}(4;R)$ is the subgroup of $\overline{\mathrm{SL}}(4;R)\cdot T_4$ which maps straight lines with the same velocity $\mathbf{v}$ to straight lines with the same velocity $\mathbf{v}'$.

Let us now investigate the effect of enlarging the proper orthochronous Lorentz group by one transformation of the form (2.2). Let s denote an element of $\overline{\mathrm{SL}}(4;R)\cdot T_4$ which we add to $L_+^\uparrow$. We look for the smallest group that includes s and $L_+^\uparrow$. From Ref. 3, we already know the answer if s is restricted to the subgroup $\overline{\mathrm{SL}}(4;R)$, that is, if s is a linear transformation. This forms the content of Theorem 0 recalled in the Introduction. The case where s is outside $\overline{\mathrm{SL}}(4;R)$ will now be dealt with in two separate lemmas. We first consider a nonlinear element s of the form (M,w), with M in $L_+^\uparrow$.

Lemma 1. Let s belong to $L_+^\uparrow \cdot T_4$, with s outside $L_+^\uparrow$. The smallest group G which includes s and $L_+^\uparrow$ is $L_+^\uparrow \cdot T_4$.

Proof. s can be written as (M,w), with M in $L_+^\uparrow$ and $w \neq 0$. For any M_1 in $L_+^\uparrow, {M_1}^t$ is also in $L_+^\uparrow$ and the following equations hold:

$$\begin{aligned} &(M^{-1},0)(M,w)=(I,w) \ , \\ &(({M_1}^t)^{-1},0)(I,w)({M_1}^t,0)=(I,M_1 w) \ , \\ &(I,M_1 w)^{-1}=(I,-M_1 w) \ . \end{aligned} \tag{2.6}$$

Each element on the left-hand side of (2.6) clearly is in G. Elements on the right-hand side therefore belong to G.

It is easy to check that if w' is any four-column and w is any four-column different from zero, one can always find matrices M_1, M_2, M_3, and M_4 in $L_+^\uparrow$ so that

$$w' = \pm M_1 w \pm M_2 w \pm M_3 w \pm M_4 w \ , \tag{2.7}$$

with the signs chosen properly. But the general element (I,w') can thus be obtained as the product

$$(I,\pm M_1 w)(I,\pm M_2 w)(I,\pm M_3 w)(I,\pm M_4 w) \ , \tag{2.8}$$

which certainly belongs to G. Finally, an arbitrary element (M',w') of $L_+^\uparrow \cdot T_4$ is obtained as

$$(M',w')=(M',0)(I,w') \tag{2.9}$$

and the lemma is proved.

Thus, enlarging the proper orthochronous Lorentz group $L_+^\uparrow$ by a single element (M,w) generates the full group $L_+^\uparrow \cdot T_4$. Note that Lemma 1 can be straightforwardly adapted to the case where the extra element $s=(M,w)$ is such that M belongs to a discrete extension of $L_+^\uparrow$. In this case, the group generated is a discrete extension of $L_+^\uparrow \cdot T_4$.

With the foregoing remark we know the result of extending $L_+^\uparrow$ by elements of the form $(A,0)$, with A in $\overline{\mathrm{SL}}(4;R)$, or of the form (M,w), with M in the full Lorentz group L. There remains to deal with elements of the form (A,w), with A outside L and $w \neq 0$.

Lemma 2. Let s belong to $\overline{\mathrm{SL}}(4;R)\cdot T_4$, with s outside $\overline{\mathrm{SL}}(4;R)$ and $L\cdot T_4$. The smallest Lie group G which includes s and $L_+^\uparrow$ is either $\mathrm{SL}(4;R)\cdot T_4$ or $\overline{\mathrm{SL}}(4;R)\cdot T_4$, depending on whether $\det A$ is $+1$ or -1.

Proof. s can be written as (A,w) with A outside L and $w \neq 0$. Let us look at the group product law, Eq. (2.3). The matrices A_1, and A_2 are multiplied in a way entirely independent of w_1 and w_2. But Theorem 0 essentially means that repeated products of an element such as $(A,0)$ with itself and elements of $L_+^\uparrow$ generate the full $\mathrm{SL}(4;R)$ [or $\overline{\mathrm{SL}}(4;R)$ if $\det A=-1$]. Clearly, the same holds for an element such as (A,w), in the following sense. Repeated products of (A,w) with itself and elements of $L_+^\uparrow$ generate at least one element of the form (A',w^*) for any A' in $\mathrm{SL}(4;R)$ [or $\overline{\mathrm{SL}}(4;R)$]. Here w^* depends on A'.

With some thought, one can convince oneself that there cannot be only one w^* corresponding to each A'. That is, there is at least one A_0 in $\mathrm{SL}(4;R)$ [or $\overline{\mathrm{SL}}(4;R)$] for which (A_0,w_1^*) and (A_0,w_2^*), with $w_1^* \neq w_2^*$, both belong to G. But, since

$$(A_0,w_1^*)^{-1}(A_0,w_2^*)=(I,w_2^*-w_1^*) \ , \tag{2.10}$$

we find that $(I,w_2^*-w_1^*)$ is also in G. By Lemma 1, all elements of the form (I,w'), for arbitrary w', therefore belong to G. The general element (A',w') can then be decomposed as

$$(A',w')=(A',w^*)(I,w'-w^*) \ , \tag{2.11}$$

which shows that (A',w') is in G. The lemma is thus proved.

We can summarize the two lemmas proved in this section together with Theorem 0 in the form of a single theorem.

Theorem 1. Let s belong to $\overline{\mathrm{SL}}(4;R)\cdot T_4$, with s outside L, the full Lorentz group. Let G be the smallest Lie group that includes s and $L_+^\uparrow$.

(i) If $s=(A,0)$ with A in $\overline{\mathrm{SL}}(4;R)$, then G is either $\mathrm{SL}(4;R)$ or $\overline{\mathrm{SL}}(4;R)$, depending on whether $\det A=+1$ or -1.

(ii) If $s=(M,w)$ with M in (a possibly discrete extension of) $L_+^\uparrow$ and $w \neq 0$, then G is (a possibly discrete extension of) $L_+^\uparrow \cdot T_4$.

(iii) If $s=(A,w)$ with A outside the full Lorentz group and $w \neq 0$, then G is either $\mathrm{SL}(4;R)\cdot T_4$ or $\overline{\mathrm{SL}}(4;R)\cdot T_4$, depending on whether $\det A=+1$ or -1.

We shall discuss the physical meaning and consequences of this theorem in Sec. IV.

III. TRANSFORMATIONS MAPPING REST TO UNIFORM RECTILINEAR MOTION

Consider two arbitrary reference frames S and S'. What can one mean by the statement that S and S' move with respect to each other with uniform relative velocity? A possible answer is that any rectilinear motion with uniform velocity $\mathbf{v}$ in S is transformed into a rectilinear motion with uniform velocity $\mathbf{v}'$ in S', with $\mathbf{v}'$ depending only on $\mathbf{v}$. Recalling the discussion which follows Eq. (2.5), we see immediately that this definition of uniform relative motion between frames implies linear coordinate transformations.

It can be argued, however, that the definition just given is somewhat too strong. That is, it translates the concept of reference frames moving with uniform relative velocity in a way which may be too restrictive. We propose, instead, the following weaker definition. Two reference frames S and S' move with respect to each other with uniform relative velocity if the world line of any point at rest in S is transformed into a rectilinear motion with velocity $\mathbf{u}$ in S', with $\mathbf{u}$ independent of the spatial coordinates of the point at rest in S. Likewise, any point at rest in S' is transformed into a rectilinear motion with velocity $\mathbf{v}$ in S. There is no *a priori* relation between $\mathbf{u}$ and $\mathbf{v}$.

Let x_i $(i=1,2,3)$ and t denote, respectively, the space and time coordinates in S. Let x_i' and t' denote the coordinates in S'. The most general coordinate transformations from (x_i,t) to (x_i',t') can be written as

$$x_i'=f_i(x_j,t), \quad t'=g(x_j,t) \ . \tag{3.1}$$

Equations (3.1) are assumed to be invertible and the functions f_i and g differentiable at least once. They are otherwise arbitrary.

We now want to investigate the constraints imposed on

f_i and g by the requirement that rest is mapped into uniform rectilinear motion. More specifically, we want to find the most general f_i and g such that (i) $\mathbf{x}'=\mathbf{c}'$ implies that $\mathbf{x}=\mathbf{x}_0+\mathbf{v}t$, with $\mathbf{c}'$, $\mathbf{x}_0$, and $\mathbf{v}$ constant, and (ii) $\mathbf{x}=\mathbf{c}$ implies that $\mathbf{x}'=\mathbf{x}_0'+\mathbf{u}t'$, with $\mathbf{c}$, $\mathbf{x}_0'$, and $\mathbf{u}$ constant.

In (i) the components of $\mathbf{v}$ are three constants which do not depend on $\mathbf{c}'$, and similarly for $\mathbf{u}$ and $\mathbf{c}$ in (ii).

First consider requirement (i). It means that there exists a $\mathbf{v}$ for which $\mathbf{x}'=\mathbf{c}'$ implies that $\mathbf{x}=\mathbf{x}_0+\mathbf{v}t$. Call $\mathbf{r}$ the variable $\mathbf{x}-\mathbf{v}t$. We define f_i^* as

$$f_i(x_j,t)=f_i^*(r_j,t) . \tag{3.2}$$

Keeping x_i' constant means that $dx_i'=0$ or, from (3.1) and (3.2),

$$\sum_j \frac{\partial f_i^*}{\partial r_j}dr_j+\frac{\partial f_i^*}{\partial t}dt=0 . \tag{3.3}$$

This will imply that $dr_j=dx_j-v_jdt=0$ for all j if and only if

$$\frac{\partial f_i^*}{\partial t}=0 \quad\text{and}\quad \det\left[\frac{\partial f_i^*}{\partial r_j}\right]\neq 0 . \tag{3.4}$$

Therefore, we have

$$x_i'=f_i^*(r_j)=f_i(x_j-v_jt) . \tag{3.5}$$

The coordinate transformations (3.1) are thus reduced to

$$x_i'=f_i(x_j-v_jt),\quad t'=g(x_j,t) . \tag{3.6}$$

Next, we assume that $\mathbf{x}=\mathbf{c}$ and investigate the conditions under which this implies that $d\mathbf{x}'/dt'=\mathbf{u}$. We have

$$\left[\frac{dx_i'}{dt'}\right]_{\mathbf{x}=\mathbf{c}}=\left[\frac{\partial f_i}{\partial t}\right]_{\mathbf{x}=\mathbf{c}}\left[\frac{\partial g}{\partial t}\right]_{\mathbf{x}=\mathbf{c}}^{-1} . \tag{3.7}$$

If the left-hand side of (3.7) is to equal a constant u_i, we must have

$$\left[\frac{\partial g}{\partial t}\right]_{\mathbf{x}=\mathbf{c}}=u_i^{-1}\left[\frac{\partial f_i}{\partial t}\right]_{\mathbf{x}=\mathbf{c}} . \tag{3.8}$$

Here no summation is implied on the index i, and the equation holds for each value of i. We can integrate (3.8) as

$$g(x_j,t)=u_i^{-1}f_i(x_j-v_jt)+F_i(x_j) . \tag{3.9}$$

Again, no summation is implied and Eq. (3.9) holds for $i=1$, 2, and 3. The functions $F_i(x_j)$ come from the indefinite integration over dt. Note, however, that (3.9) puts constraints upon them. Take, for instance, Eq. (3.9) for $i=1$ and $i=2$. We obtain

$$u_1^{-1}f_1(\mathbf{x}-\mathbf{v}t)-u_2^{-1}f_2(\mathbf{x}-\mathbf{v}t)=F_2(\mathbf{x})-F_1(\mathbf{x}) . \tag{3.10}$$

The only way a function of $\mathbf{x}-\mathbf{v}t$ can be equal to a function of $\mathbf{x}$ is for both of them to depend only on $\mathbf{x}_\perp$, the component of $\mathbf{x}$ perpendicular to $\mathbf{v}$. Thus we have

$$u_2^{-1}f_2(\mathbf{x}-\mathbf{v}t)=u_1^{-1}f_1(\mathbf{x}-\mathbf{v}t)+F(\mathbf{x}_\perp) \tag{3.11}$$

and similarly

$$u_3^{-1}f_3(\mathbf{x}-\mathbf{v}t)=u_1^{-1}f_1(\mathbf{x}-\mathbf{v}t)+F^*(\mathbf{x}_\perp) . \tag{3.12}$$

Here F and F^* are two arbitrary functions of the two variables $\mathbf{x}_\perp$.

The functions f_1, f_2, and f_3 can be taken as the components of a vector $\mathbf{f}$ so that

$$\mathbf{f}(\mathbf{x}-\mathbf{v}t)=f_1\hat{\mathbf{i}}+f_2\hat{\mathbf{j}}+f_3\hat{\mathbf{k}} . \tag{3.13}$$

Substituting Eqs. (3.11) and (3.12) in (3.13), we get

$$\begin{aligned}\mathbf{f}(\mathbf{x}-\mathbf{v}t)=&\,u_1^{-1}\mathbf{u}f_1(\mathbf{x}-\mathbf{v}t)+u_2\hat{\mathbf{j}}F(\mathbf{x}_\perp)\\&+u_3\hat{\mathbf{k}}F^*(\mathbf{x}_\perp) .\end{aligned} \tag{3.14}$$

The component of the vector $u_2\hat{\mathbf{j}}F+u_3\hat{\mathbf{k}}F^*$ which is parallel to $\mathbf{u}$ can be added to the first term on the right-hand side of (3.14). Thus we can write

$$\mathbf{f}(\mathbf{x}-\mathbf{v}t)=\hat{\mathbf{u}}f(\mathbf{x}-\mathbf{v}t)+\mathbf{F}(\mathbf{x}_\perp) . \tag{3.15}$$

In (3.15), $\hat{\mathbf{u}}$ is a unit vector parallel to $\mathbf{u}$, $\mathbf{x}_\perp$ is perpendicular to $\mathbf{v}$, and $\mathbf{F}$ is perpendicular to $\mathbf{u}$. If $x_{\parallel}$ denotes the component of $\mathbf{x}$ parallel to $\mathbf{v}$, the function $f(\mathbf{x}-\mathbf{v}t)$ can be taken as a function of $x_{\parallel}-vt$ and $\mathbf{x}_\perp$. The Jacobian $(\partial f_i/\partial x_j)$ is different from zero if and only if both $\partial f/\partial x_{\parallel}\neq 0$ and the Jacobian of $\mathbf{F}$ with respect to $\mathbf{x}_\perp$ is different from zero.

The function $g(\mathbf{x},t)$, given in Eq. (3.9) (with no summation), can now be expressed more simply. Multiplying both sides of (3.9) by $\hat{u}_i\hat{u}_i$ and summing over i, we get

$$\begin{aligned}g(\mathbf{x},t)&=\sum_i u_i^{-1}\hat{u}_i\hat{u}_if_i(\mathbf{x}-\mathbf{v}t)+\sum_i \hat{u}_i\hat{u}_iF_i(\mathbf{x})\\&=u^{-1}\hat{\mathbf{u}}\cdot\mathbf{f}(\mathbf{x}-\mathbf{v}t)+G(\mathbf{x}) .\end{aligned} \tag{3.16}$$

Here u is the magnitude of $\mathbf{u}$ and G is a function of $\mathbf{x}$. We finally put together Eqs. (3.6), (3.15), and (3.16) to obtain

$$\begin{aligned}\mathbf{x}'&=\hat{\mathbf{u}}f(\mathbf{x}-\mathbf{v}t)+\mathbf{F}(\mathbf{x}_\perp) ,\\ t'&=u^{-1}f(\mathbf{x}-\mathbf{v}t)+G(\mathbf{x}) .\end{aligned} \tag{3.17}$$

In Eqs. (3.17), $\mathbf{u}$ and $\mathbf{v}$ are two arbitrary constant vectors. For (3.17) to be invertible, $\partial f/\partial x_{\parallel}$, $\partial G/\partial x_{\parallel}$, and the Jacobian of $\mathbf{F}$ with respect to $\mathbf{x}_\perp$ must all be different from zero. Apart from this, f and G are two arbitrary functions of three variables and $\mathbf{F}$ is a two-dimensional arbitrary function of the two variables $\mathbf{x}_\perp$. Note that $\mathbf{x}_\perp$ is perpendicular to $\mathbf{v}$ and $\mathbf{F}$ is perpendicular to $\mathbf{u}$.

Equations (3.17) represent the most general coordinate transformation equations which map the world lines of points at rest to rectilinear motion with a fixed uniform velocity, that is, which meet requirements (i) and (ii) stated after Eqs. (3.1). We have not seen these equations discussed elsewhere in the literature.

Several properties of the coordinate transformations (3.17) are worth mentioning. It is easy to check that the inverse of (3.17), that is, the equations for $\mathbf{x}$ and t in terms of $\mathbf{x}'$ and t', have the same general form as (3.17), with $\mathbf{u}$ and $\mathbf{v}$ interchanged. In fact, $\mathbf{v}$ is the velocity of S' in S or, in other words, the velocity which a point at rest in S' has in S. Likewise, $\mathbf{u}$ is the velocity of S in S'. In general, there is no connection between $\mathbf{u}$ and $\mathbf{v}$. This is interesting in view of several derivations of the Lorentz transformations which simply postulate that the velocity of S in S' is equal in magnitude and opposite in direction to the velocity of S' in S.

Equations (3.17) map rest to uniform rectilinear motion. In general they do not, however, map uniform rectilinear motion to uniform rectilinear motion. They will do so if they are linear. One should note also that the set of all transformations such as (3.17) does not form a group. The product of two transformations does not result in a transformation of the same type.

We took time to derive Eqs. (3.17) in detail because they represent a very large class of possible coordinate transformations. In fact, they involve two arbitrary functions of three variables and two arbitrary functions of two variables. There are no *a priori* restrictions on the parameters $\mathbf{u}$ and $\mathbf{v}$. They can be superluminal as well as subluminal.

Our purpose is to investigate possible extensions of the Lorentz group. We will formulate the problem as follows. Let Σ be the set of all coordinate transformations of the form of Eqs. (3.17). Every element of Σ maps rest to uniform rectilinear motion. The proper orthochronous Lorentz group $L_+^\uparrow$ is a subset of Σ. Can there be a larger subset G of Σ which (i) forms a group and (ii) contains $L_+^\uparrow$? The answer to this question is contained in the following theorem.

Theorem 2. Let Σ be the set of all coordinate transformations which map the world lines of points at rest to uniform rectilinear motion with a fixed velocity. Let s be an element of Σ. If the product of s with arbitrary elements of $L_+^\uparrow$ belongs to Σ, then s is the product of a linear transformation and a space-time translation.

Proof. Let $\mathbf{x}_0$ and $\mathbf{V}$ be two constant vectors. We shall denote by $s\{\mathbf{x}=\mathbf{x}_0+\mathbf{V}t\}$ the set of all space-time points each of which is the image, under s, of a point belonging to the straight line $\mathbf{x}=\mathbf{x}_0+\mathbf{V}t$.

Assume $\mathbf{V}$ is subluminal. (Note: $\mathbf{V}$ is not the velocity parameter of the transformation s, which can be superluminal. $\mathbf{V}$ parametrizes a straight line in space-time.) There exists a Lorentz transformation M^{-1} in $L_+^\uparrow$ which maps $\mathbf{x}=\mathbf{x}_0+\mathbf{V}t$ to a point at rest, $\mathbf{x}''=\mathbf{x}_0''$, so that

$$M^{-1}\{\mathbf{x}=\mathbf{x}_0+\mathbf{V}t\}=\{\mathbf{x}''=\mathbf{x}_0''\} \ . \tag{3.18}$$

By hypothesis, sM belongs to Σ. Thus there exists a constant vector $\mathbf{W}$ such that sM transforms rest into uniform rectilinear motion with velocity $\mathbf{W}$. In other words,

$$(sM)\{\mathbf{x}''=\mathbf{x}_0''\}=\{\mathbf{x}'=\mathbf{x}_0'+\mathbf{W}t'\} \ . \tag{3.19}$$

Putting Eqs. (3.18) and (3.19) together yields

$$\begin{aligned} s\{\mathbf{x}=\mathbf{x}_0+\mathbf{V}t\} &= sMM^{-1}\{\mathbf{x}=\mathbf{x}_0+\mathbf{V}t\} \\ &= sM\{\mathbf{x}''=\mathbf{x}_0''\} \\ &= \{\mathbf{x}'=\mathbf{x}_0'+\mathbf{W}t'\} \ . \end{aligned} \tag{3.20}$$

The upshot is that s maps any straight line with $|\mathbf{V}|<c$ to another straight line with uniform velocity $\mathbf{W}$. This, as the proof in Ref. 7 shows, is enough for s to have the form of Eq. (2.1).

It is easy to see that changing $\mathbf{x}_0$ while leaving $\mathbf{V}$ unaltered in Eq. (3.18) will result in Eq. (3.20) with a different $\mathbf{x}_0'$ and the same $\mathbf{W}$. That is, s transforms straight lines with the same velocity $\mathbf{V}$ to straight lines with the same velocity $\mathbf{W}$. Therefore, the argument following Eq. (2.5) immediately shows that s has the form of Eq. (2.1) with $w=0$. QED.

IV. DISCUSSION

The two theorems we have proved in Secs. II and III entail strong limitations on possible nonlinear extensions of the Lorentz group. We will now investigate these consequences.

It was pointed out in Sec. II that $\overline{\mathrm{SL}}(4;R)$, together with space-time translations and dilatations, is the largest group which maps straight lines with the same velocity $\mathbf{v}$ to straight lines with the same velocity $\mathbf{v}'$. A nonlinear extension of the Lorentz group (other than space-time translations) therefore cannot have this mapping property.

In Sec. II we considered nonlinear coordinate transformations which map straight lines with the same velocity to straight lines with different velocities. Such transformations have the form of Eq. (2.2) and represent the group $\overline{\mathrm{SL}}(4;R)\cdot T_4$. The proper orthochronous Lorentz group $L_+^\uparrow$ is a group of coordinate transformations between equivalent reference frames. Suppose s is a nonlinear element of $\overline{\mathrm{SL}}(4;R)\cdot T_4$ which also transforms the coordinates of these frames to another equivalent reference frame. Then, as shown in Ref. 3, the smallest group G which includes s and $L_+^\uparrow$ must also relate the coordinates of equivalent frames. Theorem 1 states that G can be either $L_+^\uparrow\cdot T_4$ or $\mathrm{SL}(4;R)\cdot T_4$ (or discrete extensions of them). $L_+^\uparrow\cdot T_4$ is a 10-parameter group and $\mathrm{SL}(4;R)\cdot T_4$ is a 19-parameter group.

From an experimental point of view, neither $L_+^\uparrow\cdot T_4$ nor $\mathrm{SL}(4;R)\cdot T_4$ are invariance groups of physical laws thought to be well established, such as Maxwell's equations.[8] It thus seems that these groups do not represent symmetries which are realized in nature, and therefore that any extension of $L_+^\uparrow$ by means of nonlinear elements of $\overline{\mathrm{SL}}(4,R)\cdot T_4$ is ruled out. Suppose, however, that Maxwell's equations are only approximately, though very nearly, true. Would the same conclusion hold? One can readily think of two ways that small terms might be added to Maxwell's equations. The first way involves combinations of known fields (electric and magnetic fields, vector and scalar potentials) coming with small numerical factors.[9] The second way involves new fields which, in all experimental situations investigated so far, happened to have small numerical values. It turns out that in the first case, the above conclusion is probably not substantially altered. Indeed, there are coordinate transformations in $L_+^\uparrow\cdot T_4$ and $\mathrm{SL}(4,R)\cdot T_4$ which, when used in Maxwell's equations, introduce large additional terms. One can hardly see how these could be compensated by intrinsically small ones. In the second case, however, the situation is not so clear-cut. For then the correction terms, even though small in many situations, may nevertheless make the modified Maxwell's equations invariant under some larger group. (This is analogous to a situation where nearly static electric charges produce a very small magnetic field. The magnetic terms nevertheless make Maxwell's equations invariant under the Lorentz group.) The challenge would then be to devise experimental set-ups whereby large effects of these new fields could be produced. Until this is done, the relevance of groups such as $L_+^\uparrow \cdot T_4$ and $SL(4,R) \cdot T_4$ remains speculative.

In Sec. III we considered nonlinear coordinate transformations which map the world lines of points at rest to straight lines with the same fixed velocity. The most general such transformations are given by Eqs. (3.17). They make up a very large set which, however, is not a group. But the coordinate transformations between equivalent frames must form a group. Accordingly we asked ourselves whether there can be a subset of the general transformations (3.17) which would contain $L_+^\uparrow$ and form a group. Theorem 2 implies that the largest such subset is the group of linear transformations and space-time translations. Therefore, there is no viable nonlinear extension of the Lorentz group by means of coordinate transformations which map the world lines of points at rest to straight lines with the same velocity.

In Ref. 3 linear (including superluminal) extensions of the Lorentz group were analyzed from a group-theoretical point of view. This paper has extended the argument to a large class of nonlinear coordinate transformations.

ACKNOWLEDGMENT

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

---

[1]Extensive references can be found in E. Recami and R. Mignani, Riv. Nuovo Cimento **4**, 209 (1974) and in Ref. 3. Recent discussions include J. K. Kowalczynski, Int. J. Theor. Phys. **23**, 27 (1984); R. I. Sutherland and J. R. Shepanski, Phys. Rev. D **33**, 2896 (1986).

[2]P. Frank and H. Rothe, Ann. Phys. (Leipzig) **34**, 825 (1911); L. A. Pars, Philos. Mag. **42**, 249 (1921). See also Ref. 3.

[3]L. Marchildon, A. F. Antippa, and A. E. Everett, Phys. Rev. D **27**, 1740 (1983).

[4]We assume that the group of coordinate transformations between equivalent frames is a Lie group. This will also follow from the assumption that the group manifold is closed, which is probably more appealing from a physical point of view. In fact, any closed subgroup of a Lie group is itself a Lie group. [See, for instance, P. M. Cohn, *Lie Groups* (Cambridge University Press, Cambridge, 1968).] Clearly, the smallest group which contains s and $L_+^\uparrow$ is a subgroup of $SL(4;R)$ [or $\overline{SL}(4;R)$]. Therefore, if it is closed, it is a Lie group.

[5]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[6]T. Fulton, F. Rohrlich, and L. Witten, Rev. Mod. Phys. **34**, 442 (1962).

[7]V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon, London, 1964), Appendix A.

[8]We stress again that T_4 is not the group of four-dimensional space-time translations, but corresponds to the parameter w in Eq. (2.2).

[9]See, for instance, the suggestion of R. A. Lyttleton and H. Bondi, as outlined in P. Lorrain and D. R. Corson, *Electromagnetic Fields and Waves,* 2nd ed. (Freeman, San Francisco, 1970), p. 456.