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Relativistic bubble dynamics: From cosmic inflation to hadronic bags

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In this paper we develop the complete theory of the relativistic motion of a singular layer of matter under the influence of surface tension and volume tension. In order to account for vacuum tension effects we suggest a formalism of universal applicability: the single degree of freedom of a relativistic "bubble" is coupled in a gauge-invariant manner to a potential three-form A *in the presence of gravity*. The mathematical and physical consequences of this coupling can be summarized as follows. (i) The action functional of the theory, when written in geometric form, is formally quite similar to the Einstein-Maxwell action for the dynamics of a point charge on a Riemannian manifold. However, in comparison with electrodynamics, bubble dynamics is a highly constrained theory with a vastly different physical content: the gauge field $F=dA$ propagates no degrees of freedom and when it is coupled to gravity alone, *it gives rise to a cosmological constant of arbitrary magnitude.* (ii) We exploit this peculiar property of the gauge field to solve *exactly* some of the equations of motion of bubble dynamics. The net physical result is the nucleation of bubbles in different "vacuum phases" of the de Sitter type characterized by two effective and distinct cosmological constants, one inside and one outside the domain wall. (iii) Because of the generality of the above mechanism, the theory is applicable to a variety of different physical situations; in the case of a spherical bubble we derive the radial equation of motion and solve it explicitly in a number of cases of physical interest ranging from cosmology to particle physics. Thus, in curved spacetime we find that our action functional provides a natural basis for the so-called inflationary cosmology; in flat spacetime we find that our action functional generates the same vacuum tension advocated in the so-called "bag model" of strong interactions in order to confine quarks and gluons.

I. INTRODUCTION

In the last ten years the study of the properties of relativistic objects with finite spatial extension (strings, membranes, bags) has been an important topic of investigation among theoretical physicists, with applications ranging from particle physics to cosmology.

In particle physics the established experimental evidence, primarily the onset of scaling relations for the inelastic lepton-nucleon structure functions, led to a picture of the observed hadrons as *domains* of space to the interior of which the pointlike constituents are permanently confined. This observation, and quite independently the early recognition that the energy spectrum of the dual resonance model could be interpreted in terms of the vibration modes of a relativistic *string,* created a new trend in strong interactions which has inspired a number of phenomenological bound-state models for the "hadronization" of quarks, that is, the basic process of quark binding into the observed colorless hadrons. With the notion of spatial extension and quark confinement as an input, such models reproduce, with a varying degree of success, some of the observed features of the hadronic energy spectrum. It is hoped that this phenomenological approach will be substantiated by a deeper understanding of the hadronization phase in nonperturbative quantum chromodynamics, the *local* quantum field theory of colored quarks and gluons.[1]

In astrophysics, the analysis of the dynamics of a thin shell of dust under the influence of gravitational forces has long been a theoretical laboratory to study the end products of the gravitational collapse of massive bodies.[2] On a larger scale and in more recent times, the recognition that galaxies are distributed in a network of thick, filamentary structures (superclusters) separated by vast voids has reproposed the problem of studying the evolution of domains, domain walls, and strings on a cosmic scale.[3] Finally, bridging the gap between particle physics and cosmology, the idea of cosmic inflation[4–7] suggests that the large-scale structure of the Universe is determined by microphysics through the primordial density fluctuations of the early Universe. The "inflationary' universe scenarios, both original[5] and new,[6] assume an early epoch of exponential expansion driven by a finite vacuum energy. Guth has pointed out[5] how precisely such a large vacuum energy density would arise naturally in the SU(5) model of grand unification of the strong and electroweak interactions.[7] In order to place our work in the right perspective, it seems appropriate to recall the basic mechanism of cosmic inflation: the energy density of the Universe is usually represented as a potential function of the Higgs field ϕ postulated in grand unified theories to account for spontaneous symmetry breaking. Below a certain critical temperature $T_{\rm crit}$, the Higgs potential $V(\phi)$ is assumed to have a local minimum at $\phi=0$. This is the symmetric vacuum phase, often called the false vacuum, characterized by an energy density $\rho_v=V(0)$, and by a negative pressure $P=-\rho_v$. Thus, the primordial false vacuum is a sort of "massive" vacuum which can be represented geometrically by a portion of the de Sitter space, the maximally symmetric solution to the Einstein equations with a cosmological constant. The asymmetric vacuum phase, often called the "true" vacuum, is characterized by a certain value of the Higgs field $\phi_0\neq 0$ for which the potential attains its true minimum value $V(\phi_0)=0$ with a corresponding energy density and pressure $\rho_v=P\simeq 0$. Evidently, an energy barrier exists between the two vacua which, in classical physics, would give stability to the false vacuum.

The next stage in the original inflationary scenario is to assume a kind of first-order phase transition according to which the symmetric vacuum state decays through the formation of low-density bubbles of the broken-symmetry phase ($\rho_v\simeq 0$). Coleman has pointed out[8] that bubble formation would occur by quantum tunneling of the Higgs field from $\phi=0$ through the energy barrier between the two vacua directly to $\phi=\phi_0$. For our purposes, it is worth emphasizing that in this new configuration the geometric boundary of the bubble is regarded as a *dynamical* surface carrying most of the energy originally stored in the high-density symmetric vacuum. Mathematically, this is reflected by the existence of a kinetic term for the bubble surface in the action functional of bubble dynamics [cf. Eq. (2.3)] as well as by the equation of motion to be discussed later [cf. Eq. (5.18)].

In an updated version of the inflationary model (new inflationary cosmology) there is no energy barrier between the two vacua. Rather, the Higgs potential is shaped in such a way that the transition from the false vacuum to the broken-symmetry phase occurs by means of a slow-rollover mechanism: because of thermal or quantum fluctuation, the Higgs field is pushed away from its initial value of zero proceeding toward its true-vacuum value at $\phi=\phi_0$. Whatever mechanism is invoked for the nucleation of bubbles of the broken-symmetry phase, the wall of the newly formed low-density bubbles will be accelerated outward since the pressure inside the bubble $P\simeq 0$ is greater than the pressure in the surrounding false vacuum $P=-\rho_v$. In the new inflationary model the accelerated expansion of the Universe occurs during the early stages of the "rollover" of the Higgs field, while *the energy density remains roughly constant.* In this picture, a single bubble could grow large enough to encompass the entire observable Universe.

Regardless of the shape of the Higgs potential, the above discussion shows that the ground state of the early Universe is envisaged as a two-phase medium consisting of vacuum bubbles of the low-density evolving in the de Sitter phase of the ambient spacetime. To our mind this description is reminiscent of the hadronic vacuum pictured in the quark-bag model with surface tension.[1] In that model the hadronic vacuum is also regarded as a medium consisting of two phases separated by a closed surface. The region interior to the bag represents the hadronic phase (where quarks and gluons interact according to the fundamental laws of QCD) while the exterior region represents a different vacuum phase inaccessible to the hadronic constituents. As holes embedded in a kind of "superconducting" medium associated with the ambient spacetime, such domains of hadronic vacuum constitute very suitable traps for quarks and gluons. As long as the basic mechanisms of nonperturbative QCD remain poorly understood, the bag model provides an effective approach to hadron spectroscopy. In much the same spirit, in view of our ignorance about the *quantum* properties of the ground state of the Universe, it seems desirable to us to investigate the bubble nucleation process of the early Universe from an effective viewpoint based on the existing theories of relativistic extended objects and on general relativity. This brings us to the mathematical side of the problem: bubble dynamics has been approached so far with different techniques depending on the physical application that one has in mind. The relevant mathematical formalism for this type of problem in general relativity was developed some time ago by Israel[9] and later on by Chase.[10] Recent applications of this formalism to bubble dynamics and inflation can be found in Ref. 11. Dirac, on the other hand, used his formalism of constrained Hamiltonian systems to discuss an electrodynamic model of a conducting bubble in an attempt to resolve the electron-muon puzzle in particle physics.[12]

With this background in mind, the objective of this paper is to consider the physics of two distinct vacuum phases in a curved spacetime in terms of an action functional which is inspired by the bag model (with surface tension) of strong interactions. In particular, we discuss the cosmological evolution of a vacuum bubble in the context of such a variational approach. This approach is an extension of a geometric theory of relativistic extended objects formulated previously in Minkowski space,[13] and we will demonstrate that it is equivalent to Israel's formulation of shell dynamics.[9]

We require our action functional to be consistent with the basic postulates of general relativity; furthermore, it must also lead to a new cosmological equation that allows for the same exponential expansion rate of the Universe postulated in the new inflationary scenario on the basis of grand unified models of particle physics. In this connection, the genesis of an appreciable cosmological constant is a crucial issue in the problem of cosmic inflation. Indeed, since its birth in 1980, the theory of the inflationary universe has inspired a large number of papers[7] with specific cosmological scenarios differing from one another essentially in the mechanism advocated for the creation of the cosmological term (or, equivalently, constant vacuum energy density). One can invoke quantum gravitational effects or exploit the arbitrariness of the Higgs potential. On our part, we seek an effective theory given by an action functional that will model the existence of two distinct vacuum phases characterized by two constant but arbitrary energy densities. The existence of this two-phase medium will be seen to be a consequence of our variational formulation with no *ad hoc* assumptions concerning either the geometry of spacetime or the shape of the Higgs potential.

Remarkably, the action functional that meets all of these requirements is a straightforward generalization of the Einstein-Maxwell action: instead of a point charge coupled to the usual electromagnetic vector potential, the new action involves the timelike world hypersurface of the bubble coupled to a three-index antisymmetric potential. The corresponding "Maxwell-Einstein" equations are exactly solvable and result in the formation of two distinct vacuum phases each endowed with an effective cosmological constant of arbitrary magnitude. We will show the analogy of this mechanism with the "vacuum pressure" of quarks and gluons. For a spherical bubble, the "Lorentz-force" equation of bubble dynamics may be regarded as a new cosmological equation. It is a differential equation for which, to our knowledge, there is no *general* analytical solution. Fortunately a simple analysis based on the analogy with one-dimensional motion of a point particle in an effective potential allows us to classify solutions of the radial equation. We find (i) bubbles which initially increase in radius and either (a) reach a maximum and recontract to zero radius, (b) asymptotically approach a finite radius, or (c) expand indefinitely, (ii) bubbles which initially decrease in radius and either (a) contract to a minimum and reexpand indefinitely (de Sitter-type bubbles), (b) asymptotically contract to a finite radius, or (c) contract to zero radius, and (iii) bubbles which remain static at a fixed radius for a particular choice of initial data.

We discuss the implications of some of these solutions in astrophysics and in cosmology. In the limit of zero curvature, our radial equation correctly reproduces the Lorentz-force equation previously formulated in Minkowski spacetime.[14] Explicit solutions are exhibited in this special case and they are shown to reproduce the confinement mechanism postulated in the bag model of strong interactions. We will also comment briefly on a procedure for solving this "Lorentz-force" equation in general.

The outline of this paper is as follows. In Sec. II we give both the geometric and the coordinate-dependent definitions of our action functional, discussing the coupling between curvature, the bubble's degree of freedom, and the generalized Maxwell potential. We comment on the gauge principle associated with dynamics of the bubble. Far from being a triviality, it is necessary to enforce both locality and positivity of energy. In Sec. III we derive the field equations and the equations of motion of the bubble. In Sec. IV we deduce some exact consequences of the field equations, and in Sec. V we establish the radial equation governing the motion of a spherical bubble. In Sec. VI we discuss some special analytical solutions which may be of either cosmological or astrophysical interest. In Sec. VII we study the limiting case of flat spacetime and its connection with the bag model of strong interactions. We close with a few concluding remarks and two appendixes which discuss some technical details of our equations and compare our results with those of other authors.

II. THE ACTION FUNCTIONAL

We begin by extending the geometric definition of the action functional, previously given in Minkowski space,[13,14] to a spacetime manifold M endowed with a Lorentzian metric g (and a measure μ_g). Thus, the object that we consider (a closed membrane) is a connected orientable manifold K of dimension 3 plus an embedding x of K as a timelike submanifold in M. Therefore, the metric γ induced on $x(K)$ is also Lorentzian and a well-defined measure μ_γ exists on K. If t^a $(a=0,1,2)$ are local coordinates on K then the induced metric is explicitly given by $\gamma=\gamma_{ab}dt^a \otimes dt^b$ where

$$\gamma_{ab}=g_{\mu\nu}\frac{\partial x^\mu}{\partial t^a}\frac{\partial x^\nu}{\partial t^b} . \tag{2.1}$$

In addition, we introduce a field $F=dA$ which is a four-form on M invariant under the group of generalized gauge transformations $A\rightarrow A+d\Lambda$ where Λ is a two-form gauge function.

The action functional that we propose is then a straightforward generalization of the Einstein-Maxwell action

$$S=\frac{1}{16\pi}\int_M R\mu_g-\rho\int_K \mu_\gamma - c\int_{x(K)} A \;-\tfrac{1}{2}\int_M ||F||^2\mu_g \tag{2.2}$$

which is obviously invariant under a general coordinate transformation preserving the orientation of the manifold K as well as under the group of generalized gauge transformations $A\rightarrow A+d\Lambda$ mentioned previously. In view of our subsequent calculations it is useful to exhibit the explicit, coordinate-dependent form of the action S. In terms of local coordinates x^μ $(\mu=0,1,2,3)$ on M and t^a $(a=0,1,2)$ on K our geometric definition of S gives

$$\begin{aligned} S=&\frac{1}{16\pi}\int_M d^4x\sqrt{-g}R-\rho\int_K d^3t\sqrt{-\gamma}\\ &-\frac{c}{3!}\int_M d^4x\sqrt{-g}J^{\mu\nu\rho}(x)A_{\mu\nu\rho}(x)\\ &-\frac{1}{2\times 4!}\int_M d^4x\sqrt{-g}F^{\mu\nu\rho\sigma}(x)F_{\mu\nu\rho\sigma}(x) . \end{aligned} \tag{2.3}$$

Here g and γ stand as usual for $\det g_{\mu\nu}$ and $\det\gamma_{ab}$, respectively. The surface tension $\rho(>0)$ and the coupling constant c are given real constants with dimensions L^{-1} in geometric units (speed of light and gravitational constant are set to unity). Moreover, just as in ordinary electrodynamics, the meaning of the interaction term in Eq. (2.3) follows from our geometric definitions:

$$\int_{x(K)} A \equiv \frac{1}{3!} \int A_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$$
$$\equiv \frac{1}{3!} \int_K A_{\mu\nu\rho} \dot{x}^{\mu\nu\rho} d^3 t$$
$$= \frac{1}{3!} \int_M J^{\mu\nu\rho}(y) A_{\mu\nu\rho}(y) \sqrt{-g}\, d^4 y \;, \tag{2.4}$$

where

$$\dot{x}^{\mu\nu\rho} = \frac{1}{3!} \frac{\partial x^\mu}{\partial t^a} \frac{\partial x^\nu}{\partial t^b} \frac{\partial x^\rho}{\partial t^c} \epsilon^{abc} \sqrt{-\gamma}$$
$$= \frac{\partial x^\mu}{\partial t^0} \wedge \frac{\partial x^\nu}{\partial t^1} \wedge \frac{\partial x^\rho}{\partial t^2} \tag{2.5}$$

represents the timelike tangent three-vector to the world tube of the object.[15] Accordingly, the three-vector current density of bubble dynamics which is a generalization of the usual current in electrodynamics is explicitly given by the distribution

$$J^{\mu\nu\rho}(y) = \int_K \frac{\dot{x}^{\mu\nu\rho}}{\sqrt{-g}} \delta(y - x(t)) d^3 t \tag{2.6}$$

and acts as a source of the F field. For our purposes it is also convenient to consider the dual current to $J^{\mu\nu\rho}(x)$, i.e., the one-form distribution

$${}^*J_\alpha(x) \equiv \frac{1}{3!} \epsilon_{\alpha\mu\nu\rho} J^{\mu\nu\rho}(x) \;, \tag{2.7}$$

which is, by construction, perpendicular to the timelike tangent element $\dot{x}^{\mu\nu\rho}$ at each point on the embedded manifold $x(K)$. The requirement that the current (2.6) be covariantly conserved is consistent with the invariance of the action functional under general reparametrization of the coordinates on the spacetime manifold as well as on the embedded submanifold $x(K)$. Manifest covariance and the gauge invariance of the action functional demand in turn that the current density $J^{\mu\nu\rho}(x)$ be coupled to a totally antisymmetric, rank-three tensor potential $A_{\mu\nu\rho}$ with the associated field-strength tensor

$$F_{\mu\nu\rho\sigma} \equiv \nabla_{[\mu} A_{\nu\rho\sigma]} \;. \tag{2.8}$$

The new "Maxwell field" $F_{\mu\nu\rho\sigma}$ is manifestly invariant under the generalized gauge transformation

$$A_{\mu\nu\rho} \rightarrow A_{\mu\nu\rho} + \nabla_{[\mu} \Lambda_{\nu\rho]} \;. \tag{2.9}$$

Therefore, summing up our previous discussion, the promised generalization of the Einstein-Maxwell action to the case of relativistic bubble dynamics is given by Eq. (2.3) in coordinate-dependent form or by Eq. (2.2), in geometrical form. Finally, we close this section with a mathematical note in connection with the geometric aspects of bubble dynamics. Equations (2.4)–(2.6) define a de Rham current[16] of dimension 3 and degree 1 corresponding to the boundary ∂U of the open subset U associated with the interior of the closed membrane. The current satisfies the basic relation[16]

$$dJ_U = J_{\partial U} \tag{2.10}$$

which holds true for any submanifold $U \subset M$. Equation (2.10) is the key relation we shall use in Sec. IV to solve the generalized Maxwell equations in our theory.

At this point it is worth remarking that the gauge principle $A \rightarrow A + d\Lambda$ (under which our action is invariant) is not merely a triviality that will permit us to solve the field equation in Sec. IV but in fact is necessarily connected with the dynamics of the bubble. One way of seeing this is to consider the interaction of several bubbles by means of action-at-a-distance forces. It has previously been shown by Kalb and Ramond[17] in the case of charged point particles and strings (either open or closed) that such interactions necessarily imply gauge-invariant couplings of the form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for particles and $F_{\mu\nu\lambda} = \partial_{[\mu} \phi_{\nu\lambda]}$ for strings, where A_μ ($\phi_{[\mu\nu]}$) are the "potentials" due to the particles (strings) in the theory.

Elevating A_μ ($\phi_{[\mu\nu]}$) to the status of dynamical fields restores locality and implies the gauge principles ($A_\mu \rightarrow A_\mu + \partial_\mu \lambda$) ($\phi_{\mu\nu} \rightarrow \phi_{\mu\nu} + \partial_{[\mu} \xi_{\nu]}$). A straightforward extension in the case of bubbles (using the free action $\rho \int_K \mu_\gamma$) yields analogous results: interactions between bubbles necessarily imply gauge-invariant couplings of the form $F_{\mu\nu\lambda\sigma} = \partial_{[\mu} A_{\nu\lambda\sigma]}$ where $A_{[\nu\lambda\sigma]}$ is the "potential" due to the bubbles. Restoring locality by considering $A_{[\nu\lambda\sigma]}$ as an independent dynamical field then gives the gauge principle $A_{\nu\lambda\sigma} \rightarrow A_{\nu\lambda\sigma} + \partial_{[\nu} \Lambda_{\lambda\sigma]}$ on which our action (2.2) is based. A consideration of the dynamics of bubbles necessarily leads us to the gauge-invariant action (2.2).

This gauge principle is also necessary to preserve positivity of energy. Indeed, it is possible to show that an arbitrary kinetic energy functional of $A_{\mu\nu\lambda}$ will excite ghost degrees of freedom in the flat-space limit. The gauge principle removes these unphysical degrees of freedom, leaving us (as we shall see) with a positive- (static-) energy functional in the flat-space limit.

III. THE LORENTZ-MAXWELL-EINSTEIN EQUATIONS

The objective of this section is to derive the field equations and the equation of motion governing the time evolution of a relativistic bubble and its effect on the spacetime geometry.

It is instructive to outline the main steps in the variation of the action functional with respect to the embedding x, as it leads to an interesting generalization of the Lorentz-force equation of classical electrodynamics to the case of relativistic bubble dynamics. First the kinetic term of the membrane: the variation with respect to the embedding x is given by

$$\delta_x S_{\rm mem} = -\rho \int_K \tfrac{1}{2} \gamma^{ab} \sqrt{-\gamma} \frac{\delta \gamma_{ab}}{\delta x^\mu} \delta x^\mu d^3 t \;.$$

Using the explicit expression (2.1) for γ_{ab}, we find

$$\delta_x S_{\rm mem} = -\rho \int \sqrt{-\gamma}\gamma^{ab} \left[g_{\mu\nu} \frac{\partial x^\nu}{\partial t^b} \frac{\partial \delta x^\mu}{\partial t^a} + \tfrac{1}{2} g_{\rho\sigma,\mu} \frac{\partial x^\rho}{\partial t^a} \frac{\partial x^\sigma}{\partial t^b} \delta x^\mu \right] d^3t$$

$$= \rho \int \left[\frac{1}{\sqrt{-\gamma}} \frac{\partial}{\partial t^a} \left[\sqrt{-\gamma}\gamma^{ab} g_{\mu\nu} \frac{\partial x^\nu}{\partial t^b} \right] - \tfrac{1}{2}\gamma^{ab} g_{\rho\sigma,\mu} \frac{\partial x^\rho}{\partial t^a} \frac{\partial x^\sigma}{\partial t^b} \right] \sqrt{-\gamma}\delta x^\mu d^3t$$

$$= \rho \int g_{\mu\nu} \left[\Box_\gamma x^\nu + (\Gamma_g)^\nu_{\rho\sigma} \gamma^{ab} \frac{\partial x^\rho}{\partial t^a} \frac{\partial x^\sigma}{\partial t^b} \right] \sqrt{-\gamma}\delta x^\mu d^3t \ ,$$

where $\Box_\gamma$ stands for the Laplace-Beltrami operator with metric γ and Γ_g stands for the connection of the metric g on M.

Next, the interaction term in S: using the defining equations (2.4) and (2.5) one finds

$$\delta_x S_{\rm int} = -\frac{c}{3!} \int \left[\partial_\mu A_{\nu\rho\sigma} \dot{x}^{\nu\rho\sigma} \delta x^\mu + \tfrac{1}{2} A_{\nu\rho\sigma} \epsilon^{abc} \frac{\partial \delta x^\nu}{\partial t^a} \frac{\partial x^\rho}{\partial t^b} \frac{\partial x^\sigma}{\partial t^c} \right] d^3t$$

$$= -\frac{c}{3!} \int \left[\partial_\mu A_{\nu\rho\sigma} - 3\partial_\nu A_{\mu\rho\sigma} \right] \dot{x}^{\nu\rho\sigma} \delta x^\mu d^3t = \frac{c}{3!} \int F_{\mu\nu\rho\sigma} \dot{x}^{\nu\rho\sigma} \delta x^\mu d^3t \ .$$

Thus, the net effect of the requirement $\delta_x S = 0$ is the "Lorentz-force" equation for the bubble:

$$\rho\sqrt{-\gamma} g_{\mu\nu} \left[\Box_\gamma x^\nu + (\Gamma_g)^\nu_{\rho\sigma} \gamma^{ab} \frac{\partial x^\rho}{\partial t^a} \frac{\partial x^\sigma}{\partial t^b} \right] = \frac{c}{3!} F_{\mu\nu\rho\sigma} \dot{x}^{\nu\rho\sigma} \ . \tag{3.1}$$

Geometrically, the term in large parentheses represents the "mean curvature vector" of the submanifold $x(K)$ and we note that the Lorentz force is perpendicular to $x(K)$, as it should be.

The variation of the action with respect to the three-form A is given by

$$\delta_A S = -\frac{1}{3!} \int (F^{\mu\nu\rho\sigma}\sqrt{-g}\, \partial_\mu \delta A_{\nu\rho\sigma} + c\sqrt{-g} J^{\nu\rho\sigma} \delta A_{\nu\rho\sigma}) d^4x$$

$$= \frac{1}{3!} \int \left[\frac{1}{\sqrt{-g}} \partial_\mu (F^{\mu\nu\rho\sigma}\sqrt{-g}) - cJ^{\nu\rho\sigma} \right] \delta A_{\nu\rho\sigma} \sqrt{-g}\, d^4x$$

and the requirement $\delta_A S = 0$ leads to the covariant Maxwell equation for $F^{\mu\nu\rho\sigma}$:

$$\partial_\mu (\sqrt{-g} F^{\mu\nu\rho\sigma}) = c\sqrt{-g} J^{\nu\rho\sigma} \ .$$

Note that the above equation can be written in the usual geometric form of Maxwell's equations, i.e.,

$$d\,{}^*F = c\,{}^*J \ , \tag{3.2}$$

as long as one keeps track of the fact that presently ${}^*F$ stands for the *zero-form* dual to F, i.e., according to our conventions[15]

$$F^{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}\, {}^*F \tag{3.3}$$

while the dual current ${}^*J$ is the *one-form* explicitly defined by Eq. (2.7). Note, in addition, that in the present case the Bianchi identities $dF = 0$ are trivially satisfied and impose no constraint on the form of F.

Finally, the requirement $\delta_g S = 0$ leads to the usual Einstein equations

$$G^{\mu\nu} = 8\pi T^{\mu\nu} = 8\pi (T_M^{\mu\nu} + T_F^{\mu\nu}) \ , \tag{3.4}$$

where the symmetric energy-momentum tensor is defined by

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = T_M^{\mu\nu} + T_F^{\mu\nu} \ .$$

The field contribution $T_F^{\mu\nu}$ is calculated as

$$T_F^{\mu\nu} = \frac{1}{3!} F^{\mu\alpha\beta\lambda} F^\nu{}_{\alpha\beta\lambda} - \frac{1}{2\times 4!} g^{\mu\nu} F^{\alpha\beta\gamma\rho} F_{\alpha\beta\gamma\rho} \tag{3.5}$$

$$= -\tfrac{1}{2} g^{\mu\nu} ({}^*F)^2 \ . \tag{3.6}$$

In order to evaluate $T_M^{\mu\nu}$ we first observe that

$$S_M = -\rho \int d^3t \sqrt{-\gamma}$$

$$= -\rho \int d^3t \left[-\frac{1}{3!} \dot{x}^{\mu\nu\rho} \dot{x}_{\mu\nu\rho} \right]^{1/2}$$

$$\equiv -\rho \int d^3t \, || \dot{x} ||$$

so that

$$-2\delta_g S_M = \int \left[\tfrac{1}{2}\rho \int \frac{\dot{x}^{\mu\alpha\beta} \dot{x}^\nu{}_{\alpha\beta}}{||\dot{x}||} \delta(y - x(t)) d^3t \right] \delta g_{\mu\nu} d^4y \ .$$

Therefore,

$$T_M^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}}$$

$$= \frac{\rho}{2} \int \frac{\dot{x}^{\mu\alpha\beta} \dot{x}^\nu{}_{\alpha\beta}}{||\dot{x}||} \frac{\delta(y - x(t))}{\sqrt{-g}} d^3t \tag{3.7}$$

in complete analogy with the electrodynamic case. This analogy is further reflected by the conservation laws: if ∇_μ stands for the covariant derivative defined by the connection Γ_g, then we have verified that

$$\nabla_\mu T^{\mu\nu}=0 \tag{3.8}$$

as a consequence of the equations of motion. Furthermore, if the spacetime manifold admits a timelike Killing vector ξ_μ, a conserved energy is defined which we record here in view of our subsequent discussion:

$$E=\int_S T^{\mu\nu}\xi_\mu dS_\nu \ , \tag{3.9}$$

where we choose S to be an orthogonal surface to ξ_μ. Finally, we note that Maxwell's equation (3.2) implies the conservation law

$$d\,{}^*J=0 \tag{3.10}$$

which reflects the gauge invariance of the action and in our geometric formulation corresponds to the fact that the membrane is spatially closed.

IV. SOME EXACT CONSEQUENCES OF THE FIELD EQUATIONS

The formal analogy between the theory formulated in the previous sections and the theory of classical electrodynamics in a curved-spacetime background is transparent from the form of the equations of motion, i.e., the "Lorentz-force" equation (3.1), "Maxwell's equations" (3.2), and the usual Einstein equations (3.4). Of course, the physical content of the two theories is vastly different. Elsewhere we have analyzed the Lorentz-force equation in the limiting case of "flat" Minkowski spacetime in which the generalized Maxwell field F is a given *external* field.[13]

Unlike the usual theory of electrodynamics, the Einstein-Maxwell equations of bubble dynamics can be solved exactly. To begin with, in the present theory there is no radiation field: Maxwell's equation *in vacuo*, $d\,{}^*F=0$, implies that ${}^*F$, a zero-form, is constant everywhere. However, the field $F=dA$ is a *gauge* field; it is endowed with an energy-momentum tensor [cf. Eq. (3.6)] and couples to Einstein's tensor. Thus, even though there is no propagating field, there is a static effect: in the absence of matter ($\rho=c=0$), the action

$$S=\frac{1}{16\pi}\int_M R\mu_g-\tfrac{1}{2}\int_M ||F||^2\mu_g \tag{4.1}$$

leads to Einstein's equations with a cosmological term

$$G+4\pi({}^*F)^2=0 \ . \tag{4.2}$$

This is the simple and fundamental property of the F field which has generated a number of interesting applications[18] especially in the framework of the Kaluza-Klein theory of supergravity.[19] Conceivably, one could use the basic property (4.2) of the F field to balance out the cosmological term appearing in the Hilbert-Einstein action of general relativity. Even though, by general consensus, such a fine-tuning between the two parameters is highly improbable on the basis of our current understanding of particle physics and cosmology, some authors[20] have advocated the use of the potential three-form A in an attempt to settle the long-standing puzzle of the vanishingly small value of the cosmological constant at the present epoch. However, this is not our concern here. From our vantage point, what is significant is the fact that the new Maxwell field F is associated with a vacuum energy density of *arbitrary* magnitude: it represents a property of the ground state of the Universe whose effects should manifest themselves, directly or indirectly, at all scales of length in the physical world, from microphysics to astrophysics and cosmology. To emphasize this point we recall that even in the presence of coupling, the Maxwell equations are exactly solvable.[21] The solution is

$${}^*F=-c\theta_U+\alpha' \ , \tag{4.3}$$

where θ_U is the volume step function defined to be 1 inside the bubble and 0 outside the bubble; in the terminology of set theory θ_U, is the characteristic function of the open subset U of the spacetime manifold associated with the interior of the membrane. Finally, α' is the constant solution of the homogeneous equation corresponding to the cosmological constant previously mentioned in Eq. (4.2). The proof is a simple consequence of Eq. (2.10) combined with the following observations: if U is the open subset of M associated with the interior of the closed membrane, the corresponding de Rahm current $J_U^{\mu\nu\rho\sigma}$ has dimension 4 and degree zero. Moreover, if ω represents an arbitrary four-form in M with compact support, the defining equation of $J_U^{\mu\nu\rho\sigma}$ is the linear map which sends ω into [cf. Eq. (2.4)]

$$(J_U,\omega)=\int_U \omega\equiv\int_M J_U^{\mu\nu\rho\sigma}\omega_{\mu\nu\rho\sigma}d^4x \ . \tag{4.4}$$

Therefore, the current $J_U^{\mu\nu\rho\sigma}$ is necessarily of the form

$$J_U^{\mu\nu\rho\sigma}=\epsilon^{\mu\nu\rho\sigma}\theta_U \tag{4.5}$$

and the equivalent zero-form is

$${}^*J_U=-\theta_U \ , \tag{4.6}$$

where θ_U is the characteristic function of $U\subset M$.

Therefore, Maxwell's equation (3.2) becomes

$$d\,{}^*F=c\,{}^*J_{\partial U}=cd\,{}^*J_U \ , \tag{4.7}$$

the last equality following from Eq. (2.10). This result leads to the general solution, Eq. (4.3).

We define the value of the field on the membrane as the arithmetical mean of the values inside and outside

$${}^*F\,|_{\partial U}\equiv-\tfrac{1}{2}c+\alpha'\equiv\alpha \ . \tag{4.8}$$

Thus, the net effect of the F field in the interacting case is to create two distinct vacuum domains in spacetime. The corresponding geometries are easily deduced by substituting the general solution (4.3) of the Maxwell equations into the expression (3.6) for $T_F^{\mu\nu}$. We obtain

$$T_F^{\mu\nu}=-\tfrac{1}{2}g^{\mu\nu}[c(c-2\alpha')\theta_U+(\alpha')^2] \ . \tag{4.9}$$

Therefore, it seems natural to introduce a global cosmological constant

$$\Lambda_+=4\pi(\alpha')^2 \tag{4.10}$$

as well as a constant vacuum energy density for the interior of the membrane

$$\Lambda_{-}=4\pi(c-\alpha')^2 \tag{4.11}$$

the relative magnitudes of Λ_+ and Λ_- depending on the relative signs of c and α'. Accordingly, since the matter contribution to the energy-momentum tensor has support only on the membrane itself, Einstein's equation reduces to two sets of vacuum equations with different cosmological constants:

$$G+\Lambda_+ g=0 \tag{4.12}$$

outside the bubble, and

$$G+\Lambda_- g=0 \tag{4.13}$$

inside the bubble.

We emphasize that all of the above is an *exact* consequence of our action functional. The geometry of the spacetime manifold is completely determined once the interior and exterior line elements are matched on the membrane itself. Precisely such junction conditions were formulated long ago by Israel.[9] However, in our formulation, the motion of the bubble is determined by the Lorentz-force equation (3.1). In addition we have at our disposal a first integral of motion in the form of the conserved total mass-energy integral expression (3.9).

In the next sections we will show that for a spherical membrane our approach is consistent with Israel's method of determining the evolution equation for the bubble. In addition we shall exhibit some explicit solutions which are of interest in cosmology or astrophysics as well as in particle physics.

V. THE RADIAL EQUATION OF MOTION

In the current literature the vacuum state of the early Universe is pictured in terms of the de Sitter solution of Einstein's equation with an apparent cosmological term arising from the negative pressure of the "vacuum."

The properties of the F field established in the previous sections are ideally suited to account for the origin of cosmic domains in the early Universe.

Once formed, the evolution of the domain wall is governed by the volume and surface tension alone. In order to progress with our discussion of bubble dynamics we shall make the simplifying but natural assumption of spherical symmetry. Thus, the intrinsic metric of the membrane is taken to be

$$dS_M{}^2=-d\tau^2+R^2(\tau)d\Omega^2 \ , \tag{5.1}$$

where $R(\tau)$ is the scale factor expressed as a function of the proper time τ and $d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2$.

Next, we parametrize the bubble in terms of spherical coordinates

$$x^\mu_\pm=(R(t),\theta,\phi,t_\pm) \tag{5.2}$$

from which the explicit form of the embedding equations is known. Note that the coordinate times $t_\pm$ are functions of the proper time τ. Moreover, it follows from the results of the previous section that the exterior and interior line elements corresponding to a "vacuum" bubble are, respectively, of the form

$$dS_\pm{}^2=-f_\pm dt_\pm{}^2+f_\pm{}^{-1}dr^2+r^2d\Omega^2 \ , \tag{5.3}$$

where

$$f_+\equiv 1-\frac{\Lambda_+}{3}r^2-\frac{2E}{r} \quad (\text{outside}) \ , \tag{5.4}$$

$$f_-\equiv 1-\frac{\Lambda_-}{3}r^2 \quad (\text{inside}) \ . \tag{5.5}$$

Following Israel[9] and Chase,[10] we choose $t(\tau)$ in such a way that the exterior and interior line elements join smoothly on the surface of the bubble where they match the intrinsic metric (5.1). It is straightforward to see that this procedure leads to the junction condition

$$f_\pm{}^{-1}\dot{R}^2-f_\pm \dot{t}_\pm{}^2=-1 \ , \tag{5.6}$$

where the overdot stands for $d/d\tau$.

Therefore the induced metric γ_{ab} is represented by

$$\gamma_{ab}=\begin{bmatrix} -1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2\sin^2\theta \end{bmatrix} . \tag{5.7}$$

Next, the connection coefficients $(\Gamma_g)^\lambda_{\mu\nu}$ are explicitly given by

$$\begin{aligned} &\Gamma^1_{11}=-\tfrac{1}{2}f^{-1}f', \quad \Gamma^1_{44}=\tfrac{1}{2}ff' \ , \\ &\Gamma^1_{22}=-rf, \quad \Gamma^1_{33}=-r\sin^2\theta f \ , \\ &\Gamma^4_{14}=\tfrac{1}{2}f^{-1}f', \quad \Gamma^2_{33}=-\sin\theta\cos\theta \ , \\ &\Gamma^2_{12}=\frac{1}{r}, \quad \Gamma^3_{13}=\frac{1}{r} \ , \end{aligned} \tag{5.8}$$

where the prime indicates d/dr and the subscripts $\pm$ have been omitted. With these data we are in a position to evalute explicitly both sides of the Lorentz-force equation. The result is

$$\frac{1}{R^2}\frac{\partial}{\partial\tau}[R^2(f^2+\dot{R}^2)^{1/2}]=\frac{c}{\rho}{}^*F\dot{R} \ . \tag{5.9}$$

Define now, according to Chase[10]

$$F_\pm\equiv(f_\pm+\dot{R}^2)^{1/2} \ . \tag{5.10}$$

The value of the field strength on the bubble is given by Eq. (4.8), so taking averages of Eq. (5.9) we find

$$\frac{1}{2}\frac{1}{R^2}\frac{\partial}{\partial\tau}[R^2(F_++F_-)]=\frac{c}{\rho}\dot{R}(-\tfrac{1}{2}c+\alpha') \tag{5.11}$$

from which, by integration and using the definitions (4.10) and (4.11), we obtain

$$\tfrac{1}{6}(\Lambda_--\Lambda_+)R^3+2\pi\rho R^2(F_++F_-)=4\pi\rho X \ , \tag{5.12}$$

where X is a constant of integration. In order to identify this constant of integration we note that the two terms on the left-hand side of this equation represent, respectively, the volume energy

$$E_V=\tfrac{1}{6}(\Lambda_--\Lambda_+)R^3 \tag{5.13}$$

and the surface energy of the bubble

$$E_S = 2\pi\rho R^2(F_+ + F_-) \equiv \tfrac{1}{2}(E_S^+ + E_S^-) . \quad (5.14)$$

This can be verified explicitly, as we show in Appendix A, by computing the total energy of the bubble defined by the first integral of motion, Eq. (3.9). The volume contribution E_V originates from the expression (4.9) for T_F while the surface contribution E_S originates from the expression (3.7) for T_M and is defined as the arithmetical mean of the surface energy contributions E_S^+ and E_S^- calculated from the exterior and interior metric, respectively. Therefore, summing up our discussion so far, with the identification

$$X = E/4\pi\rho , \quad (5.15)$$

Eq. (5.12) constitutes the first integral of motion of bubble dynamics and represents, in implicit form, the evolution equation for the spherical membrane which is consistent with the given spacetime geometries (5.3) smoothly joined together on the surface of the bubble. As a matter of fact, a direct consequence of the conservation of the total mass-energy of the system is Israel's "matching equation" (5.17) involving the coefficients of the interior and exterior line elements: indeed, in view of our definitions (5.10), an equivalent expression for the first integral of motion (5.12)–(5.15) is

$$F_-^2 - F_+^2 = 4\pi\rho R(F_- + F_+) . \quad (5.16)$$

Again taking into account our definitions (5.10), Eq. (5.16) implies

$$(f_- + \dot{R}^2)^{1/2} - (f_+ + \dot{R}^2)^{1/2} = 4\pi\rho R \quad (5.17)$$

which is Israel's equation of motion and is best understood, in our notation, as the energy balance equation mentioned in the Introduction:

$$E_S^- - E_S^+ = (4\pi\rho)^2 R^3 . \quad (5.18)$$

As a consistency check on our results, in Appendix B we derive Eq. (5.17) by a direct application of Israel's method.

VI. SOME SPECIAL SOLUTIONS OF THE RADIAL EQUATION

The radial equation of motion (5.17) is the master equation of motion of bubble dynamics in the case of spherical symmetry. It can be conveniently written in the form of a cosmological equation:

$$\left[\frac{dR}{d\tau}\right]^2 = H^2R^2 - 1 - \frac{E}{R}\left[\frac{\Lambda_- - \Lambda_+}{48\pi^2\rho^2} - 1\right] + \frac{E^2}{16\pi^2\rho^2R^4} , \quad (6.1)$$

where we have set

$$H = (24\pi\rho)^{-1}[(\Lambda_- + \Lambda_+ + 48\pi^2\rho^2)^2 - 4\Lambda_-\Lambda_+]^{1/2} \quad (6.2a)$$

and the explicit expression for E is

$$E = \tfrac{1}{6}(\Lambda_- - \Lambda_+)R_0^3 + 2\pi\rho R_0^2[2(1 - \tfrac{1}{3}\Lambda_- R_0^2 + \dot{R}_0^2)^{1/2} - 4\pi\rho R_0] , \quad (6.2b)$$

where $R_0 = R(\tau_0)$ and $\dot{R}_0 = \dot{R}(\tau_0)$.

The advantage of this reformulation is that the radial motion of the bubble is simulated by the one-dimensional motion of a fictitious, classical, unit-mass point particle with total energy $\epsilon = -\frac{1}{2}$ in the classical potential:

$$V_{\text{eff}} = \frac{1}{2}\left[-H^2R^2 + \frac{E}{R}\left[\frac{\Lambda_- - \Lambda_+}{48\pi^2\rho^2} - 1\right] - \frac{E^2}{16\pi^2\rho^2R^4}\right] . \quad (6.3)$$

Note, incidentally, that with the exception of the last term, Eq. (6.1) is similar to the equation for a matter dominated, spatially closed, Friedmann universe

$$\left[\frac{dR}{d\tau}\right]^2 = -1 + \frac{\Lambda}{3}R^2 + \frac{8\pi\rho_m R_0^3}{3R} \quad (6.4)$$

with an effective cosmological constant

$$\Lambda \equiv 3H^2 > 0 \quad (6.5)$$

and a matter density

$$\rho_m \equiv 3E\left[\frac{\Lambda_+ - \Lambda_-}{48\pi^2\rho^2} + 1\right] \Big/ 8\pi R_0^3 . \quad (6.6)$$

It seems to us that this property of the radial equation is consistent with the overall requirement that, after the period of cosmic inflation, the new cosmology must agree with the standard Friedman cosmology.

The effective potential given by Eq. (6.3) possess a single maximum and, at most, two roots. From this property we can deduce the classes of behavior for the bubble mentioned in the Introduction. In what follows, the bubble is assumed to be initially expanding from $R=0$ or a turning point at time $\tau=0$ unless otherwise stated.

(a) $\Lambda_- < \Lambda_+$. In this case the equation of motion admits a solution with $E=0$. This is the case of "vacuum bubbles" expanding from a minimum radius H^{-1} to infinity according to the de Sitter law

$$R(\tau) = H^{-1}\cosh(H\tau) . \quad (6.7)$$

It should be noted that the case $\Lambda_+ = 0$ is inconsistent with Eq. (6.7) since, in this case, the energy is positive definite; the case $\Lambda_+ < \Lambda_-$ and $E > 0$ corresponds to the evolution of the spherical domain of a false vacuum in the background of a true vacuum. The simple solution (6.7), once interpreted as a spontaneous bubble nucleation event, is the key to the success of the new inflationary cosmology: it provides the exponentially expanding phase of the early Universe needed to remove some of the inconsistencies[4,5] of the standard cosmological model. During the phase transition described in the Introduction, the false metastable vacuum decays into the true asymmetric vacuum through the formation of low-density ($\Lambda_- \approx 0$) bubbles. According to Eq. (6.7), one such bubble can materialize with vanishing total energy, and then expand to encompass the entire observable Universe. In this connection, we note that Coleman and De Luccia[8] have studied the gravitational effects on the vacuum decay process in the framework of a semiclassical scalar field theory, while Berezin, Kuzmin, and Tkachev[11] have proposed an equation of motion for the bubble based on Israel's formulation of shell dynamics. With due account of the difference in units and notation we have explicitly checked that our results based on the Lorentz-force equation are consistent with those of Refs. 8 and 11. We also wish to record here Tryon's observation[22] that a bubble with vanishing total mass energy could also emerge as a spontaneous vacuum fluctuation from an initial zero-energy state (or even from a state for which energy is not defined at all[23]) rather than from a false-vacuum phase. In this case the solution (6.7) seems to provide us with a natural bridge between de Sitter era and the (presently quite unclear) preceding quantum state of the Universe.[24] This lends support to our claim the action functional of our theory provides an effective approach to the inflationary cosmology.

(b) $H^2 \approx 0$, $E^2/16\pi\rho^2 \approx 0$. In this case the equation of motion reduces to the form

$$\left[\frac{dR}{d\tau}\right]^2 = -1 + \frac{B}{R} , \tag{6.8}$$

where

$$B \equiv E\left[\frac{\Lambda_- - \Lambda_+}{48\pi^2\rho^2} - 1\right] \tag{6.9}$$

and can be regarded as the equation of motion

$$\frac{1}{2}\left[\frac{dR}{d\tau}\right]^2 = \epsilon - V(R) \tag{6.10}$$

of a unit mass point particle moving with constant total energy $\epsilon = -\frac{1}{2}$ in the potential $V(R) = B/2R$. The motion is physically acceptable only for $\Lambda_- - \Lambda_+ < 48\pi^2\rho^2$. In this case a parametric solution is obtained by switching to the conformal time

$$d\tau = R(\eta) d\eta . \tag{6.11}$$

Integration of the resulting equation gives

$$\begin{aligned} R(\eta) &= E\left[\frac{\Lambda_+ - \Lambda_-}{48\pi^2\rho^2} - 1\right]\sin^2\frac{\eta}{2} , \\ \tau(\eta) &= \frac{E}{2}\left[\frac{\Lambda_- - \Lambda_+}{48\pi^2\rho^2} - 1\right](\eta - \sin\eta) , \end{aligned} \tag{6.12}$$

which is of the same form describing a closed, oscillating universe in the standard cosmology. In our case the oscillating bubble expands from a vanishing initial radius up to

$$R_{\max} = E\left[\frac{\Lambda_+ - \Lambda_-}{48\pi^2\rho^2} - 1\right]$$

and then recollapses, the whole cycle occurring in the time

$$\Delta\tau = \pi E\left[\frac{\Lambda_+ - \Lambda_-}{48\pi^2\rho^2} - 1\right] .$$

(c) $H^2 \approx 0$, $(\Lambda_- - \Lambda_+)/48\pi^2\rho^2 \approx 1$. In this case the equation of motion reduces to the form

$$\left[\frac{dR}{d\tau}\right]^2 = -1 + \frac{A^4}{R^4} , \tag{6.13}$$

where

$$A \equiv \left[\frac{E}{4\pi\rho}\right]^{1/2} . \tag{6.14}$$

In this case, the effective potential is of the form $V(R) = -A^4/2R^4$ and we anticipate an oscillating behavior again. Indeed, setting $d\tau = R(\eta)d\eta$ as before, one finds by integrating the resulting equation of motion

$$R(\eta) = A \sin^{1/2}(2\eta) , \tag{6.15}$$

$$\tau(\eta) = \frac{A}{\sqrt{2}}\left\{\left[F\left[\alpha, \frac{1}{\sqrt{2}}\right] - 2E\left[\alpha, \frac{1}{\sqrt{2}}\right]\right] - \left[F\left[\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right] - 2E\left[\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right]\right]\right\} , \tag{6.16}$$

where F and E are elliptic integrals of the first and second kind, respectively, with

$$\alpha = \arcsin[\sqrt{2}\sin(\pi/4 - \eta)] . \tag{6.17}$$

Therefore, the motion consists again of an expansion-recontraction cycle with a range $0 \le R \le A$ and a period

$$\begin{aligned} \Delta\tau &= A \int_{-\pi/4}^{\pi/4} dy (\cos 2y)^{1/2} \\ &= \frac{\pi A}{\sqrt{2}} \frac{1}{F\left[\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right]} \approx \frac{A}{4\sqrt{2\pi}} . \end{aligned} \tag{6.18}$$

It is worth observing that the approximate solutions described in (b) and (c) are complementary to each other: near the origin solution (b) is unreliable whereas (c) represents a good approximation to the solution of the general equation since the $1/R^4$ term dominates over the others. Moreover, as a consistency check one can easily verify that (a) and (c) above correctly reproduce the asymptotic behavior of the general solution for $R \to \infty$ and for $R \to 0$, respectively.

(d) $\dot{R} = 0$. In this case Eq. (6.2b) gives the relationship between the total energy and the radius of the bubble at rest. Thus, the energy spectrum is

$$E = \left[\frac{\Lambda_- - \Lambda_+}{6} - 8\pi^2\rho^2\right]\hat{R}^3 + 4\pi\rho\hat{R}^2\left[1 - \frac{\hat{R}^2}{R_D^2}\right]^{1/2} , \tag{6.19}$$

where we have set $(3/\Lambda_-) \equiv R_D$. We observe from Eq. (5.17) that only the horizon of the interior geometry enters this relationship so we cannot define any rest point $R > R_D$. This limitation agrees both with our choice of standard spherical coordinates, which cannot be extended across the cosmological event horizon, and with the definition of E; in fact for $R > R_D$ we can no longer define a timelike Killing vector and therefore Eq. (3.9) becomes physically unreliable. In terms of the dimensionless variables

$$x = R/R_D, \quad y = E/4\pi\rho R_D^2 , \tag{6.20}$$

and with the definition

$$a = \frac{R_D}{24\pi\rho}(\Lambda_- - \Lambda_+ - 48\pi^2\rho^2) , \tag{6.21}$$

the energy spectrum is represented by the curve

$$y = ax^3 + x^2(1-x^2)^{1/2} . \tag{6.22}$$

In the domain of definition $0 \le x \le 1$ there is a single maximum

$$\bar{x} = \frac{4+3a^2+|a|(9a^2+8)^{1/2}}{6(1+a^2)} . \tag{6.23}$$

We also note that

$$\left.\frac{dy}{dx}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{x\to 1} \to -\infty$$

and that $y = a$ when $x = 1$. Therefore the curve intercepts the x axis only when $a \le 0$. In this case the x intercept is given precisely by

$$R_D = H^{-1} \tag{6.24}$$

which is the minimum radius at the bouncing point for a bubble with $E = 0$ [cf. case (a)].

Finally, a special subcase which is worth mentioning is characterized by $\rho = 0$, $\Lambda_+ = 0$. In this case there is no kinetic term in the action functional and the energy spectrum (6.19) reduces to the simple relationship

$$E = \frac{\Lambda_-}{6} R^3 = \frac{1}{2}\frac{\hat{R}^3}{R_D^2} . \tag{6.25}$$

Thus, in the absence of surface tension the membrane is simply the geometrical boundary of a "bag" supported purely by its own volume tension. This is strictly a vacuum effect due to the gravitational interaction of the F field and whose astrophysical significance, if any, remains unclear to us. An extreme case of Eq. (6.25) is one in which the Schwarzschild radius $R_S = 2E$ and de Sitter radius $R_D = (3/\Lambda_-)$ coincide with the radius R of the bubble. This is a static configuration, even though dynamically unstable; any amount of surface energy ($\rho \neq 0$), however small, would drive the object to its collapse into the central singularity. The object represented here is the crudest model of a star; it consists of a ball of de Sitter "vacuum fluid" endowed with nonvanishing energy density and negative pressure. Interestingly enough, if one takes for Λ_- the typical energy density of nuclear matter, say $\rho \approx 10^{14}$ g/cm^3, one obtains $R \approx 10$ km which is of the order of the limiting size of a neutron star prior to its gravitational collapse into a black hole.

We close this section by remarking that there is no general analytical solution to Eq. (6.1). However it is possible to combine analytical and numerical techniques to solve this equation. As previously mentioned, this equation possesses a single maximum and at most two roots. Using this information, along with the time reflection and time translation symmetries of (6.1), it is possible to splice approximate analytical solutions (which are valid for certain ranges of R) with numerical solutions (which are employed outside these regions). In particular, the form of the equation to be solved is

$$(R')^2 = AR^2 + B\frac{1}{R} + C\frac{1}{R^4} + D \equiv g(R)$$

with A,B,C,D constants which may be computed from (6.1). For large R, small R, and near the aforementioned roots and maximum, $g(R)$ may be approximated by some function for which (6.1) has an analytical solution; elsewhere, numerical integration may be used to interpolate between such solutions. All such solutions fall into one of the four classes of behavior mentioned above.

VII. BUBBLE DYNAMICS IN MINKOWSKI SPACE

Of special interest in our theory is the "flat" spacetime limit, i.e., the limit in which the gravitational interaction is switched off and the manifold M is Minkowski's spacetime (R^4,η). In this limit our action functional describes the dynamics of a closed surface embedded in Minkowski space and coupled in a gauge-invariant way to the generalized Maxwell field F:

$$S_{\text{flat}} = -\rho\int d^3t\sqrt{-\gamma} - \frac{c}{3!}\int J^{\mu\nu\rho}(x)A_{\mu\nu\rho}(x)d^4x - \frac{1}{2\times 4!}\int d^4x\, F^{\mu\nu\rho\sigma}(x)F_{\mu\nu\rho\sigma}(x) . \tag{7.1}$$

This special form of the action acquires a special relevance in particle physics in view of the observed properties of hadronic systems. It is well known that strongly interacting particles are composite objects whose structure may be explained in terms of more fundamental entities (quarks and gluons) permanently trapped in the interior of the hadron.[1] At present there are essentially two alternative views about confinement: (i) in the low-energy region where nonperturbation effects come into play, quantum chromodynamics will eventually lead to the force law needed for quark trapping; (ii) the phenomenon of confinement is simply postulated in phenomenological models involving extended structures ("bags") by introducing a "cosmological term," such as the vacuum pressure in the action for the hadronic system.

A typical effective action for a confined quark-gluon system is

$$S_{\text{eff}} = \int_U d^4x\, L_{\text{QCD}} - \tfrac{1}{2}c^2\int_U d^4x - \rho\int_{\partial U} d^3t\sqrt{-\gamma} . \tag{7.2}$$

According to this action, the dynamics of the quark-gluon system at short distances, *within* the bag domain U is governed by the Lagrangian L_{QCD}. However, the onset of the confinement mechanism at distances of the order of the hadronic size (typically, 1 fm) is guaranteed by the "vacuum pressure" term in $S_{\rm eff}$ as well as by the kinetic term for the bag's boundary. In order to stabilize the bag against the pressure of the hadronic constituents one must provide an amount of energy $\frac{1}{2}c^2$ per unit volume and an amount of energy ρ per unit surface.

It should be noted that the existence of the vacuum tension in a *finite* domain is simply postulated. In order to extend the space integrations to be over all space in any Lorentz frame, in the literature on hadronic bag models one encounters the "volume step function" $\theta_U(x)$ which is defined by

$$\theta_U(x)=\begin{cases}1 & \text{inside the bag}\ ,\\ 0 & \text{outside the bag}\ .\end{cases} \tag{7.3}$$

It is stipulated that $\theta_U(x)$ depends only on the surface coordinates of the bag and has the property

$$\partial_\mu\theta_U(x)=-n_\mu\delta_{\partial U}(x)\ , \tag{7.4}$$

where n_μ is the unit outward spacelike normal to the surface of the bag and $\delta_{\partial U}(x)$ is the surface δ function with the property

$$\int d^4x\,\delta_{\partial U}(x)f(x)=\int (d^3x)_{\partial U}f(x)\ . \tag{7.5}$$

The integral on the right-hand side (RHS) is restricted to the volume enclosed by the bag surface.

The connection between our formulation of bubble dynamics and the confinement mechanism of the bag model can be appreciated by observing first that our Maxwell equation for the F field is a mathematically rigorous and gauge-invariant formulation of the relationship (7.4).

Indeed, by construction, the dual current ${}^*J_\alpha(x)$ is directed along the spacelike normal n_α to the world track of the bubble and, at each point on the bubble, possesses a surface δ-type singularity. Thus at each point on the surface of the bubble the Maxwell equation becomes

$$\partial_\mu{}^*F=cn_\mu\delta_{\partial U}(x) \tag{7.6}$$

which is the same form as Eq. (7.4). Hence, the solution

$${}^*F=-c\theta_U(x)\ , \tag{7.7}$$

where now we set the global constant $\alpha'=0$ on account of the translational invariance of Minkowski space. The contribution of the F field to the energy-momentum tensor is now

$$T_F^{\mu\nu}=-\tfrac{1}{2}c^2\eta^{\mu\nu}\theta_U(x)\ . \tag{7.8}$$

Thus, $T_F^{\mu\nu}$ contributes to the energy but not the momentum of the bubble and the volume contribution to the total energy of the bubble is given by

$$E_V=\tfrac{1}{2}c^2V\ , \tag{7.9}$$

where V is the volume enclosed by the bubble. Thus, the interior vacuum (or hadronic phase) is endowed with a nonvanishing volume energy density which, in a Minkowskian background, is precisely the "volume tension" advocated in the bag model in order to confine quarks and gluons. It is worth observing that in our formulation, this vacuum energy density is simply the (*finite*) self-energy of the F field.

The connection with the confinement mechanism of the bag model can be established also through the Lorentz-force equation which, in Minkowski space, simplifies considerably. Here we have two options: (i) derive the equation of motion directly from the action (7.1) and then deduce the radial equation for a spherical bubble, or, (ii) specialize the radial equation of motion previously obtained to the limit in which the gravitational interaction is switched off. We have checked that both methods lead to the same result. The first possibility was rigorously analyzed in Ref. 14; hence the computational details will not be reproduced here. We start instead directly with the expressions (5.12)–(5.15) of the first integral of motion in which we restore the explicit dependence or Newton's constant:

$$EG_N=\frac{2\pi}{3}(\epsilon_{\rm in}-\epsilon_{\rm out})R^2+2\pi\rho G_NR^2\left[\left(1-\frac{4\pi}{3}G_N\epsilon_{\rm out}R^2-\frac{2EG_N}{R}+\dot R^2\right)^{1/2}+\left(1-\frac{4\pi}{3}G_N\epsilon_{\rm in}R^2+\dot R^2\right)^{1/2}\right], \tag{7.10}$$

where we have set

$$\Lambda_\mp=4\pi G_N\epsilon_{\rm in,out} \tag{7.11}$$

and ϵ represents the energy density in the new units. In Minkowski's space: $G_N\to 0$ and, in terms of Minkowski's time coordinate

$$\dot R^2\equiv\left(\frac{dR}{d\tau}\right)^2=\left(\frac{dR}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2=\frac{\mathbf{v}^2}{1-\mathbf{v}^2}\ , \tag{7.12}$$

where $\mathbf{v}=d\mathbf{R}/dt$ is the velocity of a point on the membrane. Therefore, using Eqs. (4.10) and (4.11),

$$\epsilon_{\rm in}-\epsilon_{\rm out}=c^2-2c\alpha' \tag{7.13}$$

we obtain

$$\frac{3E}{4\pi\rho}=\frac{3R^2}{(1-\mathbf{v}^2)^{1/2}}-\lambda R^3\ , \tag{7.14}$$

where

$$\lambda\equiv\frac{c}{\rho}\left(\alpha'-\frac{c}{2}\right) \tag{7.15}$$

which agrees with the radial equation established in Ref. 14. For our immediate purpose we note the following properties. When $\alpha'=0$, as we are presently assuming and, in addition, $c=0$, there is no coupling and the bubble is simply under the influence of its own surface tension. The equation of motion (7.14) reduces to

$$(1-\mathbf{v}^2)^{1/2}=(R/R_0)^2 \tag{7.16}$$

where we have defined $R_0=\sqrt{E/4\pi\rho}$.

The solution to this equation is the following elliptic function:

$$R(t)=R_0 cn\left[\frac{\sqrt{2}t}{R_0},\frac{1}{\sqrt{2}}\right]. \tag{7.17}$$

The bubble starts from rest at time $t=0$ as a sphere of radius R_0, but collapses to its central singularity $R=0$ at the time

$$T=\frac{R_0}{\sqrt{2}}K\left[\frac{1}{\sqrt{2}}\right].$$

The qualitative behavior of this extreme case is maintained in the interesting case $c\neq 0$, $\alpha'=0$ (so that $\lambda<0$). To appreciate this point it is convenient to introduce the definition

$$\beta=\frac{3}{4}\frac{E}{\pi\rho\lambda}<0$$

so that the equation of motion becomes

$$(1-\mathbf{v}^2)^{1/2}=\frac{3}{\lambda}\frac{R^2}{R^3+\beta}. \tag{7.18}$$

The physical requirement $|\mathbf{v}|\leq 1$ is therefore equivalent to the following two conditions on the range of R for a given value of the parameter β:

$$\lambda(R^3+\beta)>0 \tag{7.19}$$

and

$$\lambda P(R)\geq 0, \tag{7.20}$$

where $P(R)$ is the polynomial

$$P(R)=R^3-\frac{3}{\lambda}R^2+\beta \tag{7.21}$$

whose positive real roots are the points of rest ($\mathbf{v}=0$) of the bubble. The polynomial $P(R)$ can be brought to canonical form

$$P(r)=r^3+3pr+2q \tag{7.22}$$

by the substitution $R=r+1/\lambda$ where

$$p=-1/\lambda^2,\quad q=\frac{\beta}{2}-\frac{1}{\lambda^3}. \tag{7.23}$$

Its discriminant Δ is thus given by

$$\Delta=-p^3-q^2=\beta\left[\frac{1}{\lambda^3}-\frac{\beta}{4}\right]. \tag{7.24}$$

Since $\beta<0$, $P(R)$ has only one positive root $\overline{R}$ given by (1) for $\beta<4/\lambda^3$ ($\Delta<0$), $\overline{R}=(1/\lambda)(1-2\cosh\theta)$ where $\cosh 3\theta=\beta\lambda^3/3-1$, (2) for $\beta=4/\lambda^3$ ($\Delta=0$), $\overline{R}=-1/\lambda$, (3) for $\beta>4/\lambda^3$ ($\Delta>0$), $\overline{R}=(1/\lambda)(1-2\cos\theta)$, where $\cos 3\theta=\beta\lambda^3/2-1$ ($0<3\theta<\pi$). It follows from Eq. (7.20) that $R\leq\overline{R}\equiv R_{\max}$. Therefore the solutions are bounded and singular as in the case $\lambda=0$.

We have therefore demonstrated that, provided $\lambda<0$, the bag collapses under the combined effect of volume and surface tensions *whatever the value of the initial radius*. This mechanism of collapse implies the confinement of quarks and gluons, since the energy of the quark-gluon system will stabilize the bag into a physical hadron.

VIII. CONCLUSIONS

We have shown that the physics of two distinct vacuum phases (characterized by two constant arbitrary energy densities) is describable by an action functional invariant under generalized gauge transformations of a rank-3 antisymmetric tensor field. Such a formalism follows, as discussed in Sec. II, from a consideration of the interactions between membranes, and is a natural generalization of the interactions between pointlike/stringlike objects. In each case a gauge principle is needed to preserve locality and positivity of energy in flat space. However, in contrast with the pointlike/stringlike cases, our action functional does not lead to propagation of disturbances by waves but instead yields the aforementioned two phase medium. The boundary between these two vacua (the bubble) evolves according to Eq. (6.1) which also follows from our formalism. No *ad hoc* assumptions concerning either the shape of a Higgs potential or the geometry of spacetime were employed; the complete dynamics of this system followed from minimization of our action functional. The equation of evolution of the bubble has been shown to agree with that derived by Israel and Chase.[9,10] We have separately considered the cases of vanishing and nonvanishing gravitational field, the latter being the Minkowski space case.

In the gravitational case we have shown that the evolution of the bubble corresponds to various cosmological scenarios: the bubble either inflates exponentially, oscillates, or remains static. In particular, our formalism provides an effective approach to inflationary cosmology as we have shown in Sec. VI. What makes inflation possible in the gravitational case is the fact that the background cosmological constant α' cannot be set to zero due to the coupling of the F field to gravity. Consequently in curved spacetime the volume energy can be negative due to the presence of two separate cosmological constants Λ_+ and Λ_-; the total energy can be zero as in the case of the expanding de Sitter bubble of inflationary cosmology. In contrast with this, in the flat-spacetime case the volume energy must be positive definite, leading to the collapse of the bubble as shown in Sec. VII. In the context of the hadronic bag model this implies the confinement of quarks and gluons, whose energy must stabilize the system against such collapse. Since the energy of the bubble increases with its size, it would cost an infinite amount of energy to isolate the hadronic constituents as free particles, thus permanently confining them to the interior of the bubble.[1] We conclude that our formalism provides an effective action for describing the physics of two phase media, whether it be for inflation or for hadronic physics.

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APPENDIX A: THE CALCULATION OF E

We wish to calculate the surface and volume energy contributions to the total mass energy of a spherical bubble according to the defining Eq. (3.9) supplemented by the specific expressions (3.7) and (4.9) of the energy-momentum tensor.

The components of the unit norm timelike Killing vector are

$$\xi_\mu = (-\sqrt{-g_{00}}, 0,0,0) . \tag{A1}$$

The surface element is

$$dS_\nu = n_\nu (g^{(3)})^{1/2} dr\, d\theta\, d\phi , \tag{A2}$$

where n_ν is the normal vector to the spacelike surface of integration S,

$$n_\nu = (g_{00}, 0,0,0) , \tag{A3}$$

and $g^{(3)}$ is the determinant of the three-dimensional metric on S:

$$(g^{(3)})^{1/2} = \sqrt{-1/g_{00}}\, r^2 \sin\theta . \tag{A4}$$

With these data and integrating over the angular variables, the surface energy of the bubble is

$$E_S = -4\pi\rho \int_0^\infty dr\, r^2 (-\gamma^{00})^{1/2} g_{00} \delta(r - R(t)) . \tag{A5}$$

The recipe to handle the singularity on the surface of the bubble is the standard formula for symmetrical integration

$$\int f(x)\delta(x) dx = \tfrac{1}{2}(f_+(0) + f_-(0)) \tag{A6}$$

which amounts to averaging out the contributions of the inner and outer geometries, as we have repeatedly done in the main text of the paper. Therefore,

$$E_S = -2\pi\rho R^2 [(-\gamma_+^{00})^{1/2} g_{00}^+ + (-\gamma^{00})^{1/2} g_{00}^-] , \tag{A7}$$

where

$$\gamma_\pm^{00} = f_\pm \left[-f_\pm{}^2 + \left[\frac{dR}{dt_\pm} \right]^2 \right]^{-1} . \tag{A8}$$

The two coordinate times $t_\pm$ are now eliminated in favor of the bubble proper time τ by matching the two induced metrics on the bubble with its own intrinsic metric. This procedure gives

$$\left[\frac{dR}{dt_\pm} \right]^2 = \frac{f_\pm \dot{R}^2}{1 + f_\pm{}^{-1} \dot{R}^2} , \tag{A9}$$

where $\dot{R} \equiv dR/d\tau$ and $R = R(\tau)$ is now understood as the scale factor of the intrinsic metric. Substituting Eq. (A9) and (A8) into (A7) gives finally

$$E_S = 2\pi\rho R^2 [(f_+ + \dot{R}^2)^{1/2} + (f_- + \dot{R}^2)^{1/2}] . \tag{A10}$$

In order to calculate the volume energy of the bubble we need only the first term in Eq. (4.9). Indeed, the volume energy E_V is measured with respect to the overall energy background of the spacetime manifold and this implies that the effective energy density inside the bubble is given by the difference $(2\pi)^{-1}(\Lambda_- - \Lambda_+)$. With this understanding the volume energy contribution is

$$\begin{aligned} E_V &= \tfrac{1}{2} c(c - 2\alpha') \int_S \theta_V r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \frac{\Lambda_- - \Lambda_+}{6} R^3 . \end{aligned} \tag{A11}$$

Equations (A10) and (A11) lead to the first integral of motion quoted in the text [cf. Eqs. (5.12)–(5.15)].

APPENDIX B: THE ISRAEL-CHASE APPROACH TO THE RADIAL EQUATION

Let $r = R(\tau)$ be the equation of Σ, the timelike hypersurface of the bubble dividing spacetime into two four-dimensional domains: U_+ and U_-. Then the intrinsic metric on Σ is given by

$$(dS^2)_\Sigma = -d\tau^2 + R^2(\tau) d\Omega^2 \quad (d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2) , \tag{B1}$$

where τ represents the proper time along the coordinate lines $\theta,\phi = \text{const}$. The Israel-Chase[9,10] approach requires the knowledge of the line elements in both U_+ and U_-. In general, the line element in both domains is reducible to the Reissner-Nordström metric, by extension of Birchoff's theorem. In particular, our action functional specifies the following form of the line elements:

$$(dS^2)_- = f_-{}^{-1} dr^2 + r^2 d\Omega^2 - f_- dt_-{}^2 , \quad r < R(\tau) , \tag{B2}$$

$$(dS^2)_+ = f_+{}^{-1} dr^2 + r^2 d\Omega^2 - f_+ dt_+{}^2 , \quad r > R(\tau) . \tag{B3}$$

where $f_\pm$ are defined in the main text by Eqs. (5.4) and (5.5). We have already shown, by proper choice of $t_\pm(\tau)$ on Σ, that the metrics induced by (B2) and (B3) on Σ, agree with the intrinsic metric (B1).

Let $\mathbf{U}$ now represent the unit velocity vector tangent to Σ and $\mathbf{n}$ represent the unit spacelike vector normal to Σ and directed, say, from U_- to U_+. Then one has

$$r_{,\alpha} U^\alpha = \dot{R}(\tau) \tag{B4}$$

and

$$\frac{\delta n^\alpha}{\delta\tau} = -U^\alpha U_\beta \frac{\delta n^\beta}{\delta\tau} = U^\alpha n_\beta \frac{\delta U^\beta}{\delta\tau} . \tag{B5}$$

Therefore,

$$\begin{aligned} \frac{d}{d\tau}(r_{,\alpha} n^\alpha) &= r_{|\alpha\beta} U^\beta n^\alpha + r_{,\alpha} \frac{\delta n^\alpha}{\delta\tau} \\ &= r_{|\alpha\beta} n^\alpha U^\beta - \dot{R} n_\beta \frac{\delta U^\beta}{\delta\tau} . \end{aligned} \tag{B6}$$

Following Chase,[10] the equation of motion of the membrane takes now the form

$$\left[n_\alpha \frac{\delta U^\alpha}{\delta\tau} \right] = 4\pi(\rho + 2P) , \tag{B7}$$

$$(\rho U^j)_{;j} + P U^j{}_{;j} = [T_{\alpha\beta} n^\alpha U^\beta] , \tag{B8}$$

where, by definition, $\alpha,\beta,\ldots = 1$–4 (coordinates $r,\theta,\phi,t_\pm$), $i,j,\ldots = 2$–4 (coordinates θ,ϕ,τ), $\rho, P =$ surface energy density and pressure; $[\Phi]\equiv\Phi_+-\Phi_-$, and a semicolon denotes intrinsic covariant differentiation with respect to the three-metric of Σ. Since $(T^\alpha_\beta)_\pm \propto \delta^\alpha_\beta$ for both Schwarzschild and de Sitter spacetimes, Eq. (B8) implies

$$\frac{dM(\tau)}{d\tau}=-P\frac{d}{d\tau}(4\pi R^2)\,, \tag{B9}$$

where

$$M(\tau)\equiv 4\pi\rho R^2 \tag{B10}$$

represents the total proper mass of the membrane. Equation (B9) tells us that the rate of increase of surface energy is equal to minus the rate of work done by the pressure in expanding the bubble. In order to arrive at our form of the radial equation, from Eq. (B6) we obtain

$$\dot{R}\left[n_\alpha\frac{\delta U^\alpha}{\delta\tau}\right]=[r_{|\alpha\beta}n^\alpha U^\beta]-\left[\frac{d}{d\tau}(r_{,\alpha}n^\alpha)\right] \tag{B11}$$

and using Eq. (B7) we arrive at

$$4\pi\dot{R}(\rho+2P)=[r_{|\alpha\beta}n^\alpha U^\beta]-\frac{d}{d\tau}[(r_{,\alpha}n^\alpha)]\,. \tag{B12}$$

The explicit calculation of the RHS of Eq. (B12) with the aid of the connection coefficients listed in the text leads to the results

$$(r_{|\alpha\beta}n^\alpha U^\beta)_\pm=0\,, \tag{B13}$$

$$r_{,\alpha}n^\alpha=F\equiv[f(R)+\dot{R}^2]^{1/2}\,, \tag{B14}$$

where we have used the parametrization

$$U^\alpha=(\dot{R},0,0,\dot{t}),\quad n_\alpha=(\dot{t},0,0,-\dot{R})$$

and the matching condition

$$-1=f^{-1}\dot{R}^2-f\dot{t}^2\,.$$

Hence, Eq. (B12) becomes

$$-\left[\frac{dF}{d\tau}\right]=4\pi\dot{R}(\rho+2P)=-\frac{d}{d\tau}\frac{M(\tau)}{R}\,, \tag{B15}$$

where the last equality follows from Eqs. (B9) and (B10).

Integrating Eq. (B15) and noting that the constant of integration is necessarily zero (cf. Chase,[10] p. 142) leads to the final form of the radial equation

$$F_- - F_+ = 4\pi\rho R \tag{B16}$$

derived in the text [cf. Eq. (5.17)] via the Lorentz-force equation.

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[15]In this paper we shall use the following notation: the totally antisymmetric symbol in n dimensions is

$$[\alpha_1\alpha_2\cdots\alpha_n]=\begin{cases}+1 & \text{even permutation of reference sequence}\,,\\ -1 & \text{odd permutation of reference sequence}\,,\\ 0 & \text{some indices equal}\,.\end{cases}$$

Accordingly we define the Levi-Civita tensor as

$$\epsilon^{\alpha_1\cdots\alpha_n}=\frac{1}{\sqrt{-g}}[\alpha_1\cdots\alpha_n]\,,$$

$$\epsilon_{\alpha_1\cdots\alpha_n}=\sqrt{-g}\,[\alpha_1\cdots\alpha_n]\,.$$

The dual of a k-rank tensor is

$$^*T_{\alpha_1\cdots\alpha_{n-k}}=\frac{1}{k!}\epsilon_{\alpha_1\cdots\alpha_{n-k+1}\cdots\alpha_n}T^{\alpha_{n-k+1}\cdots\alpha_n}\,.$$

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