# Reheating in the higher-derivative inflationary models

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In both the Starobinsky and the $R^2$ models of the inflationary universe, after the inflationary phase the Universe enters into a period in which the scalar curvature oscillates rapidly. The rapid oscillations in the geometry result in particle production when conformally noninvariant quantum fields are present. By solving the semiclassical back-reaction equations the temperature to which the Universe reheats due to the particle production is computed. It is also shown that the oscillations are damped essentially exponentially with the result that the Universe evolves into a classical radiation-dominated Friedmann phase. Both analytical and numerical solutions to the back-reaction equations are obtained, with the results in accurate agreement with each other. The numerical schemes presented here for solving the semiclassical back-reaction equations can be used for scalar quantum fields with arbitrary curvature couplings and arbitrary masses. They are expected to be useful for many future calculations of the evolution of the early Universe. The analytical analysis presented provides some insight into the coupling between the quantum field and the higher-derivative terms in the back-reaction equations. The implications of this are discussed.

## I. INTRODUCTION

Many inflationary models[1] have been proposed in which the Universe has undergone a period of superluminal expansion. In most of these models,[2] the expansion is driven by the false-vacuum energy of a Higgs field. These "standard" models have an important drawback. The effective potentials of the Higgs fields responsible for the inflation in these models have to have very specific shapes[1] in order for the models to describe our Universe, i.e., for the models to deliver sufficient inflation, a decent exit from inflation proper reheating, and reasonable density perturbations. It is not easy to come up with a particle theory in which such a Higgs field naturally exists.

In contrast, the higher-derivative inflationary models, which are based on higher-derivative terms in the effective gravitational Lagrangian, clearly have an advantage. It has been argued[3,4] that these terms would have to exist on the classical and/or quantum level. Two such models have been proposed and studied in some detail.[3–8] The Starobinsky model is based on the semiclassical back-reaction equations with conformally invariant free quantum fields. Renormalization of the stress-energy tensor for the quantum fields results in higher-derivative terms in the equations and therefore many solutions. Starobinsky showed that de Sitter space is one of these solutions and that for a certain sign of one of the regularization parameters it is unstable. This instability allows for a graceful exit from inflation. Vilenkin[8] has shown, using a wave-function analysis and the boundary condition of "tunneling from nothing," that the Universe is likely to begin close enough to the exact de Sitter solution that there will be sufficient inflation.

In the $R^2$ model the inflationary period is not exactly de Sitter but appears at a time when the Hubble parameter $H$ is linearly decreasing with respect to the proper time $t$. In Ref. 4 it was shown that such a "linear phase" is generic if a Ricci scalar squared $R^2$ term is dominating over other higher-derivative terms in the effective Lagrangian, as assumed in the model.

Since the late-time evolution of the universe in the Starobinsky model is also dominated by an $R^2$ term in the effective Lagrangian, the two models have the same qualitative behavior at late times. In both models, near the end of the inflationary phase, both the scalar curvature $R$ and the Hubble parameter $H$ decrease in magnitude. The Universe then exits from the inflationary phase and goes into an oscillation phase with $R \propto (1/t)\sin 2\omega_0 t$ and $H \propto (1/t)\cos^2\omega_0 t$. The oscillation frequency $\omega_0$ is determined by the coefficient of the $R^2$ term in the gravitational Lagrangian. The scale factor $a(t)$ goes like $t^{2/3}[1+\sin(2\omega_0 t)/(3\omega_0 t)]$ so that when averaged over several oscillations, solutions expand like classical matter-dominated Friedmann universes. Hence Starobinsky calls this the scalaron-dominated phase in his model.

The oscillations in the geometry are expected to excite the conformally *noninvariant* quantum fields living in the Universe. The resultant particle production is then expected to provide the necessary reheating.[4,8] At the same time the back reaction of these quantum fields on the spacetime geometry is expected to damp the oscillations and drive the Universe into a classical Friedmann phase. Assuming this is the case, Refs. 4 and 8 have given estimations for the reheating temperature of the Universe. These estimations are based on calculations of the Bogoliubov coefficients between the in and out vacua using the method of Zeldovich and Starobinsky.[9] The calculations are done in a fixed background spacetime. The out vacuum has been taken to be the adiabatic vacuum of the radiation-dominated Friedmann universe. The in vacuum is the usual Minkowski vacuum in an assumed static "in region." Because of the preceding inflation it is a good approximation to assume that the initial state is a Minkowski vacuum state. The energy of the particles produced is obtained from the Bogoliubov coefficients for the quantum fields. By comparing the contribution of this energy with other terms in the Einstein equations (Ref. 4), or with the energy density of the "scalarons" (Ref. 8), reheating temperatures are estimated.

There are several features which make these analyses unsatisfactory. The most serious is that a back-reaction calculation is never performed so one has to *assume* that a classical Friedmann radiation-dominated phase exists at late times. Secondly, any argument relating the energy of the "scalarons" to that of the produced particles (Ref. 8) is suspect since the energy-momentum tensors of the "scalaron" and the conformally noninvariant quantum fields are conserved *separately.* Finally vacuum-polarization effects are completely neglected by these calculations.

For these reasons it is desirable to do a more rigorous calculation of the reheating which takes the full back reaction of the quantum fields, including both vacuum polarization and particle production, directly into account. In this paper we present such a calculation and show explicitly that the phase plane oscillations are damped and that the Universe is driven into a radiation-dominated Friedmann phase. It is shown (cf. Sec. IV) that the Universe enters into the radiation-dominated phase with a temperature on the order of $10^9$ GeV, while its temperature earlier on in the oscillation phase can be as high as $10^{12}$ GeV.

Our calculation of the reheating in the models is based on the semiclassical back-reaction equations[10]

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}+\alpha\,{}^{(1)}H_{\mu\nu}+\beta\,{}^{(2)}H_{\mu\nu} = -8\pi G\langle \text{in}\,|\,T_{\mu\nu}\,|\,\text{in}\rangle^{\text{ren}} . \quad (1.1)$$

We use units such that $\hbar=c=k_B=1$, with $l_{\text{Planck}}=\sqrt{G}$. $\alpha$ and $\beta$ are arbitrary renormalization parameters and $R_{\mu\nu}$ is the Ricci tensor. In Eq. (1.1), ${}^{(1)}H_{\mu\nu}$ is associated with the variation of an $R^2$ term in the effective gravitational Lagrangian and ${}^{(2)}H_{\mu\nu}$ is associated with the variation of an $R_{\mu\nu}R^{\mu\nu}$ term.[10] We follow the notation of Birrell and Davies[10] throughout the paper. Here, $|\text{in}\rangle$ denotes the initial state of the quantum fields living in the spacetime; and $\langle T_{\mu\nu}\rangle^{\text{ren}}$ is the renormalized expectation value of the energy-momentum tensor operator $T_{\mu\nu}$. We shall see that due to the inflation present in the models, it is essentially irrelevant what we choose the initial state to be. Throughout the paper, we consider only homogeneous and isotropic spacetimes and free scalar fields. This should be a good approximation for the observable portion of our Universe as far as post-inflation reheating calculations are concerned. The scalar fields are assumed to be massless and to have nonconformal couplings to the geometry. We consider this case, in which the particle production is due to the nonconformal coupling instead of the mass term, for two reasons. First, in order for the particle production to be effective, the mass of the field has to have a value close to that of the oscillation frequency of the geometry. We may not have such a massive field in nature. Second, the Universe will not go over to a *radiation*-dominated Friedmann phase with such a massive field. Hence, in order to see explicitly the emerging of a radiation-dominated Friedmann phase, we would have to consider the subsequent decay of these massive particles. This is an unnecessary complication of our purpose. We therefore, in the reheating calculations of Secs. III and IV, consider massless *nonconformally* coupled scalar fields.

To treat the back-reaction equations with nonconformally coupled fields we have developed two numerical schemes, the high-$k$-expansion scheme and the iteration scheme. Both schemes use adiabatic regularization.[11] The high-$k$-expansion scheme is general in the sense that it can be used for all situations where adiabatic regularization has been developed. It also has important implications which are independent of the regularization procedure. The scheme has been described in Ref. 12 and will be mentioned only very briefly in this paper. The applicability of the iteration scheme is somewhat more restricted since the iteration parameter cannot be too large (cf. Sec. II). However, it is still general enough to be useful for solving the back-reaction equations (1.1) for many interesting situations, and in particular, it can be used to calculate the reheating of the inflationary models. One nice feature of this scheme is that it explicitly demonstrates that the results obtained are self-consistent solutions to the back-reaction equations. This scheme will be described in some detail in Sec. II. The two schemes produce results which agree accurately with each other. The longer calculations reported in Sec. III are done using the high-$k$-expansion scheme since it is somewhat more efficient than the iteration scheme.

These schemes for solving the semiclassical back-reaction equations for free quantum fields with arbitrary masses and curvature couplings open the door to many interesting calculations because they solve the technical problems which have prevented these calculations from being done in their entirety. In the past various approximations have had to be made which have seriously limited the applicability of the results. Such approximations include the limits of nearly conformal coupling to the scalar curvature,[13] small anisotropy,[14] and the neglection of the higher-derivative terms in the back-reaction equations.[15,16] The first calculations without such approximations were done by one of us for the conformally coupled massive scalar field in spatially flat homogeneous and isotropic spacetimes.[17] Using our schemes it is now possible to solve the back-reaction equations for free scalar fields with arbitrary couplings and masses in homogeneous and isotropic spacetimes and, probably, in the anisotropic Bianchi type-I spacetimes as well. In fact it appears that the only limitation on our methods is that they make use of adiabatic regularization which has not been developed for other spacetimes or for other quantum fields. However, there seems to be no reason, in principle, that adiabatic regularization could not be developed for other spacetimes and other quantum fields. If this is done, then we expect that our schemes could be applied to these cases as well.

In Sec. II we set up the back-reaction problem and describe the numerical schemes discussed above. In Sec. III the results obtained in using these schemes on the reheating calculations of the higher-derivative inflationary models are reported. Section IV contains an analytical study of the reheating problem. This analytical study provides valuable insight into particle production and its effects on rapidly changing geometries. Section V consists of a brief conclusion.

## II. THE BACK-REACTION EQUATIONS AND THE NUMERICAL SCHEMES

In this section we first write down the coupled set of equations which must be solved to determine the back reaction of the quantum fields on the spacetime geometry and then describe two schemes to solve them numerically. We consider a single scalar field with arbitrary coupling to the scalar curvature and arbitrary mass. The generalization to more than one such field is straightforward.

### A. Derivation of the equations

We wish to solve Eq. (1.1) for the case of a scalar field with arbitrary coupling to the scalar curvature and arbitrary mass in a homogeneous and isotropic spacetime. The metric for such a spacetime is given by

$$ds^2=dt^2-a^2(t)\left[\frac{dr^2}{1-Kr^2}+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\right], \tag{2.1}$$

where $a(t)$ is the scalar factor and $K=0,+1,-1$ corresponds to spacetimes with zero, positive, and negative spatial curvatures, respectively.

We now write down each term in Eq. (1.1) in terms of the metric (2.1). The symmetries of homogeneity and isotropy allow one to deduce all components of each tensor in these equations from its time-time component and its trace. The first two terms on the left of (1.1) are well known with

$$R_{tt}=3(\dot H+H^2)\ , \tag{2.2a}$$

$$R=6(\dot H+2H^2+K/a^2)\ . \tag{2.2b}$$

Here an overdot denotes $d/dt$ and $H=\dot a/a$ is the Hubble parameter. Since the spacetime is conformally flat, ${}^{(2)}H_{\mu\nu}=\frac{1}{3}\,{}^{(1)}H_{\mu\nu}$, and we can set $\beta=0$ in (1.1) without loss of generality.[10] For the line element (2.1), ${}^{(1)}H_{\mu\nu}$ is given by

$${}^{(1)}H_{tt}=-6(H\dot R-\tfrac{1}{12}R^2+H^2R+a^{-2}RK)\ , \tag{2.3a}$$

$${}^{(1)}H_\mu{}^\mu=-(6\ddot R+18H\dot R)\ . \tag{2.3b}$$

To determine $\langle \text{in}\,|\,T_{\mu\nu}\,|\,\text{in}\rangle$ in (1.1) we must choose the initial state for the scalar field, solve its evolution equation, and compute the expectation value of $T_{\mu\nu}$. The evolution equation for a scalar field with mass $m$ and curvature coupling $\xi$ is

$$\Box\phi+(m^2+\xi R)\phi=0\ . \tag{2.4}$$

We decompose $\phi$ by

$$\phi(t,x)=\int d\tilde\mu(k)[\hat a_k U_k(t,x)+\hat a_k^\dagger U_k^*(t,x)]\ , \tag{2.5a}$$

$$U_k(t,x)=a^{-1}\mathcal{Y}_k(x)\psi_k(t)\ , \tag{2.5b}$$

$$\mathcal{Y}_k(x)=(2\pi)^{-3/2}e^{ik\cdot x},\quad k=(k_1,k_2,k_3)\quad (K=0)$$
$$=\Pi_{kl}^{(\pm)}(x)Y_l^m(\theta,\phi),\quad k=(k,l,m)\quad (K=\pm 1)\ .$$

Here the $Y_l^m$ are the usual spherical harmonics, the $\Pi_{kl}^{(\pm)}(x)$ are given in Chap. 5 of Ref. 10, and the measure of the integration in (2.5a) is given by

$$\int d\tilde\mu(\mathbf{k})=\int d^3k\quad (K=0) \tag{2.6a}$$

$$=\int_0^\infty dk\sum_{l,m}\quad (K=-1) \tag{2.6b}$$

$$=\sum_{k=1}^\infty\sum_{l,m}\quad (K=1)\ . \tag{2.6c}$$

The mode function $\psi_k(t)$ satisfies the normalization condition

$$\psi_k\dot\psi_k^*-\dot\psi_k^*\psi_k=-i/a\ , \tag{2.7a}$$

and the evolution equation

$$\ddot\psi_k+H\dot\psi_k+[\omega_k{}^2+(\xi-\tfrac16)R]\psi_k=0\ , \tag{2.7b}$$

with $\omega_k{}^2=k^2/a^2+m^2$.

A quantum state of the scalar field is represented by a set of mode functions $\psi_k$ by requiring that the quantum state be annihilated by $\hat a_k$ in (2.5) for all $k$. The unrenormalized expectation value of the energy-momentum tensor with respect to a certain state $|\,\text{in}\rangle$ of the quantum field is given in terms of the mode functions $\psi_k$ by

$$\langle T_{tt}\rangle\equiv\langle \text{in}\,|\,T_{tt}\,|\,\text{in}\rangle=4\pi\int\frac{d\mu(k)}{a^3}\langle T_{tt}\rangle_k\ , \tag{2.8a}$$

$$\langle T_{tt}\rangle_k=\frac{1}{16\pi^3}\left\{a\,|\,\dot\psi_k\,|^2+6(\xi-\tfrac16)aH(\psi_k\dot\psi_k^*+\psi_k^*\dot\psi_k)+\left[\frac{k^2}{a^2}+m^2-6(\xi-\tfrac16)\left[H^2-\frac{K}{a^2}\right]\right]a\,|\,\psi_k\,|^2\right\}, \tag{2.8b}$$

$$\langle T\rangle\equiv\langle \text{in}\,|\,T_\mu{}^\mu\,|\,\text{in}\rangle=4\pi\int\frac{d\mu(k)}{a^3}\langle T\rangle_k\ , \tag{2.9a}$$

$$\langle T\rangle_k=\frac{1}{4\pi^3}\Bigg[3(\xi-\tfrac16)[a\,|\,\dot\psi_k\,|^2-aH(\psi_k\dot\psi_k^*+\psi_k^*\dot\psi_k)]$$
$$+\left\{\frac{m^2}{2}-3(\xi-\tfrac16)\left[\frac{k^2}{a^2}+m^2+\frac16\left[R-6H^2-\frac{6K}{a^2}\right]+(\xi-\tfrac16)R\right]\right\}a\,|\,\psi_k\,|^2\Bigg], \tag{2.9b}$$

where

$$\int d\mu(k) \equiv \int_0^\infty dk\, k^2,\ \ K=0,-1 \tag{2.10a}$$

$$\equiv \sum_{k=1}^{\infty} k^2,\ \ K=1\ . \tag{2.10b}$$

$\langle T_{\mu\nu}\rangle$ is divergent and must be renormalized. The renormalization scheme most easily adapted to numerical computations is adiabatic regularization[10,11] in which the renormalized energy-momentum tensor is obtained by subtracting the adiabatic counterterms [obtained from the fourth-order adiabatic solutions to Eq. (2.7)] mode by mode, so that

$$\langle T_{\mu\nu}\rangle^{\rm ren} = 4\pi \int \frac{d\mu(k)}{a^3}(\langle T_{\mu\nu}\rangle_k - \langle T_{\mu\nu}\rangle_k^{\rm ad})\ . \tag{2.11}$$

$\langle T_{\mu\nu}\rangle_k^{\rm ad}$ for a Robertson-Walker spacetime has been given by Bunch.[18] It is determined solely by the geometric quantities of the spacetime. Equations (1.1) and (2.7), together with (2.2), (2.3), and (2.11) form the complete set of equations that must be solved to determine the back reaction of the quantum scalar field on the spacetime geometry.

In (1.1), again because of the symmetry of (2.1), we have only to look at the $tt$ and the trace components. Like the standard classical Friedmann equations, here the $tt$ equation is also a first integral of the trace equation. We can regard the $tt$ component as a constraint equation and evolve forward with the trace equation. This, in some cases, is computationally more convenient than integrating the $tt$ equation directly.

### B. The numerical schemes

Having written down the back-reaction equations, we next discuss a major problem which arises when one tries to solve these equations numerically. We then describe two schemes which can be used to overcome this problem.

To numerically obtain the time evolution of $\psi_k(t)$ and $a(t)$, we must first algebraically solve (2.7b) and the trace of Eq. (1.1) for $\ddot{\psi}_k$ and $\ddot{R}$, respectively, so that we have expressions for them in terms of the lower-derivative quantities. It is trivial to do so for $\ddot{\psi}_k$. $\ddot{R}$ appears in both $\alpha\,{}^{(1)}H_\mu{}^\mu$ and $\langle T\rangle_k^{\rm ad}$. When the scalar field is conformally coupled, i.e., $\xi=\frac{1}{6}$, the only $\ddot{R}$ term in $\langle T\rangle_k^{\rm ad}$ is proportional to $1/\omega^7$ which makes this term finite when the $k$ integration of (2.11) is performed. (For the $K=1$ case, it is a summation over $k$. However, for simplicity, we shall say $k$ integral for all three cases henceforth.) So, for $\xi=\frac{1}{6}$, there is no problem in solving algebraically for $\ddot{R}$. One of us has used this method to study the back reaction of conformally coupled massive scalar fields.[17]

However, when we go away from conformal coupling, i.e., $\xi\neq\frac{1}{6}$, there are two additional terms in $\langle T\rangle_k^{\rm ad}$ which contain $\ddot{R}$ (see, e.g., Ref. 18). One is finite when the integration over $k$ is performed and thus poses no problem. The other is proportional to $1/\omega^3$. This term, when considered by itself, is divergent because of the integral over $k$. It cancels the logarithmic divergence of the unrenormalized trace $\langle T\rangle$. Because of this divergent term $\ddot{R}$ cannot be solved for in terms of lower-derivative quantities. This is a serious problem and it is not even clear, *a priori*, under what conditions solutions of an equation with such a peculiar structure exist. This is the major obstacle in extending the calculations of Ref. 17 to nonconformally coupled scalar fields.

We shall present two schemes which can be used to overcome this problem. Before doing so it is useful to rewrite $\langle T_{\mu\nu}\rangle^{\rm ren}$ in a slightly different way. The idea is to split $\langle T_{\mu\nu}\rangle^{\rm ren}$ into two *separately conserved* pieces: $\langle T_{\mu\nu}\rangle^g$ and $\langle T_{\mu\nu}\rangle^\psi$. Both pieces are *finite*. $\langle T_{\mu\nu}\rangle^g$ is solely determined by the geometric quantities and does not involve the mode functions explicitly. It consists of all the terms in $\langle T_{\mu\nu}\rangle^{\rm ad}$ which are finite when the integration over $k$ is performed. $\langle T_{\mu\nu}\rangle^\psi$ involves the mode functions $\psi_k$ of the quantum field. It is the unrenormalized $\langle T_{\mu\nu}\rangle$ with the divergent parts removed via the subtraction of the divergent terms of $\langle T_{\mu\nu}\rangle^{\rm ad}$. With these definitions the $tt$ and the trace equations are

$$R_{tt}-\tfrac{1}{2}R+\alpha\,{}^{(1)}H_{tt} = -8\pi G\langle T_{tt}\rangle^{\rm ren}\ , \tag{2.12a}$$

$$\langle T_{tt}\rangle^{\rm ren} = \langle T_{tt}\rangle^g + \langle T_{tt}\rangle^\psi\ , \tag{2.12b}$$

$$\langle T_{tt}\rangle^g = \frac{1}{2880\pi^2}\left[-30m^2H^2+3H^4+{}^{(1)}H_{tt}[-\tfrac{1}{6}+10(\xi-\tfrac{1}{6})]+\frac{6H^2K}{a^2}-\frac{3K^2}{a^4} -(\xi-\tfrac{1}{6})180\left[\frac{H^2K}{a^2}-\frac{K^2}{a^4}\right]-1080(\xi-\tfrac{1}{6})^2H^2R\right],\ \ K=0,-1\ , \tag{2.12c}$$

$$\langle T_{tt}\rangle^\psi = 4\pi\int\frac{d\mu(k)}{a^3}(\langle T_{tt}\rangle_k - \langle T_{tt}\rangle_k^{\rm uv})\ , \tag{2.12d}$$

$$\langle T_{tt}\rangle_k^{\rm uv} = \frac{1}{16\pi^3}\left[\omega_k - 3(\xi-\tfrac{1}{6})\frac{1}{\omega_k}\left[H^2-\frac{K}{a^2}\right]+\frac{m^2H^2}{\omega_k{}^3}+\frac{1}{4}(\xi-\tfrac{1}{6})^2\frac{1}{\omega_k{}^3}\,{}^{(1)}H_{tt}\right]\ , \tag{2.12e}$$

$$-R+\alpha\,{}^{(1)}H_\mu{}^\mu = -8\pi G\langle T\rangle^{\rm ren}\ , \tag{2.13a}$$

$$\langle T\rangle^{\rm ren} = \langle T\rangle^g + \langle T\rangle^\psi\ , \tag{2.13b}$$

$$\langle T\rangle^{g}=\frac{1}{2880\pi^2}\left[-10m^2\left[R-\frac{6K}{a^2}\right]+{}^{(1)}H_\mu{}^\mu[-\tfrac{1}{6}+10(\xi-\tfrac{1}{6})]\right.$$
$$+2H^2R-12H^4+\frac{2RK}{a^2}-\frac{24H^2K}{a^2}-\frac{12K^2}{a^4}-(\xi-\tfrac{1}{6})\left[\frac{60RK}{a^2}-\frac{360H^2K}{a^2}-\frac{360K^2}{a^4}\right]$$
$$\left.-180(\xi-\tfrac{1}{6})^2\left[6H\dot{R}+2R^2-\frac{12RK}{a^2}\right]\right],\quad K=0,-1\,, \tag{2.13c}$$

$$\langle T\rangle^{\psi}=4\pi\int\frac{d\mu(k)}{a^3}(\langle T\rangle_k-\langle T\rangle_k^{\mathrm{uv}})\,, \tag{2.13d}$$

$$\langle T\rangle_k^{\mathrm{uv}}=\frac{1}{16\pi^3}\left\{\frac{m^2}{\omega_k}-(\xi-\tfrac{1}{6})\left[\omega_k{}^{-1}\left[R-6H^2-\frac{6K}{a^2}\right]+\frac{m^2}{\omega_k{}^3}\left[R-3H^2-\frac{3K}{a^2}\right]\right]\right.$$
$$\left.+\frac{(\xi-\frac{1}{6})^2}{4\omega_k{}^3}{}^{(1)}H_\mu{}^\mu+(\xi-\tfrac{1}{6})\frac{9m^4H^2}{\omega_k{}^5}+(\xi-\tfrac{1}{6})^2\frac{9m^2}{2\omega_k{}^5}\left[H\dot{R}-\frac{R^2}{12}+H^2R+\frac{RK}{a^2}\right]\right\}. \tag{2.13e}$$

Here the quantities $\langle T_{tt}\rangle_k$ and $\langle T\rangle_k$ are given by Eqs. (2.8b) and (2.9b). The last two terms in $\langle T\rangle_k^{\mathrm{uv}}$ are convergent in the $k$ integration of (2.13d). They are included in $\langle T\rangle_k^{\mathrm{uv}}$ so that $\langle T_{\mu\nu}\rangle^{\psi}$ is conserved. For the terms $\langle T_{tt}\rangle^{g}$ and $\langle T\rangle^{g}$ the integrations over $k$ have been performed for the $K=0,-1$ cases. The sum over $k$ for the $K=1$ case is discussed in the Appendix.

In using these formulas [(2.12) and (2.13)] for the massless field, there are some rather subtle points. First, it is important to do the integrations over $k$ *before* setting the mass equal to zero,[18,19] otherwise the trace anomaly is omitted. Second, in $\langle T_{\mu\nu}\rangle^{\mathrm{uv}}$ [Eqs. (2.12e) and (2.13e)], the term proportional to ${}^{(1)}H_{\mu\nu}$ produces an infrared divergence in the massless limit. However, as this term is proportional to ${}^{(1)}H_{\mu\nu}$, this divergence can be absorbed into the definition of the parameter $\alpha$. Thus, one can impose an infrared cutoff either by putting in a lower limit to the integral over $k$ for this term or by setting $\omega_k{}^2=k^2/a^2+M^2$ (for this term only) with $M$ an arbitrary parameter with units of mass. Different values of the cutoffs correspond to different values of the parameter $\alpha$.

Combining all of the contributions for $\alpha$, we call the net constant coefficient before ${}^{(1)}H_{\mu\nu}$ in the back-reaction equations $\epsilon$, following Ref. 5. It is the negative of the coefficient of the $R^2$ term in the effective Lagrangian in the convention of Ref. 10, which we follow in this paper. With this definition and the explicit forms of the back-reaction equations in (2.12) and (2.13), we are now ready to describe our schemes for solving them. We first note that in (2.13e), the ${}^{(1)}H^\mu{}_\mu$ term contains $\ddot{R}$ and has a coefficient $1/k^3$ for large $k$. It produces a logarithmic divergence if we perform the $k$ integration in (2.13d) for this term alone. This keeps us from solving for $\ddot{R}$ explicitly in terms of lower derivative quantities unless $\xi=\frac{1}{6}$, as discussed above. In the high-$k$-expansion scheme, this problem is solved by noting that in order for the trace of the renormalized energy-momentum tensor $\langle T\rangle^{\mathrm{ren}}$ to be finite, $\langle T\rangle_k$ of (2.9b) in the high-$k$ limit has to contain a $1/k$ term and a $1/k^3$ term having exactly the same structure as that in (2.13e). Hence, the value of $\ddot{R}$ can be obtained from the structure of the high $k$ modes of the quantum field. This scheme has been discussed in Ref. 12 and will not be repeated here.

The iteration scheme is intuitively simpler. We can symbolically represent (2.13) by

$$\ddot{R}+(\xi-\tfrac{1}{6})^2\frac{G}{\epsilon}\hat{O}\,\ddot{R}=f\,, \tag{2.14}$$

where $\hat{O}$ is the operator involving the $k$ integral; and $f$ involves only the lower derivative geometric quantities. The problem in solving for $\ddot{R}$ is that we do not know how to invert $\hat{O}$. However, we notice that when $G/\epsilon$ is not too large, we should be able to solve for $\ddot{R}$ iteratively. The procedure is as follows. First $\hat{O}$ is set to zero to obtain an approximate value for $\ddot{R}$. This value is then substituted into the second term in Eq. (2.14) and a more accurate value of $\ddot{R}$ is obtained. The iteration continues until the desired accuracy is reached.

For the reheating calculation of the higher-derivative inflationary models (Refs. 4, 7, and 8), $G/\epsilon$ is less than $10^{-10}$ and the iteration scheme is clearly appropriate. We do not have a definite criterion for the convergence of the iteration scheme, but whenever it converges it is clearly guaranteed to produce a self-consistent solution to the back-reaction equations. Since $\epsilon$ is the coefficient of the $R^2$ term in the effective gravitational Lagrangian, we may expect it to be of the Planck scale for many cases, i.e., $G/\epsilon$ would be of order 1. In this case, for $(\xi-\frac{1}{6})$ also of order 1, we have tested the iteration scheme for various initial data. It always converges and produces the same result as the high-$k$-expansion scheme. For $G/\epsilon \gtrsim 10$, the iteration scheme breaks down for the choices of initial data which we have tried.

The actual numerical calculations are done as follows. First the initial values of the geometric quantities $a$, $H$, and $R$ are chosen according to the problem at hand. Then a certain number of $k$ modes of the quantum field are chosen. These modes should be centered at a value $k=k_c$ with $a/k_c$ equal to the time scale of the changing of the geometry, and should cover a range where the back reaction of the quantum field is important. This may require some insight into the problem at hand if only a small number of modes can be used because of restrictions on the available computing time. (An explicit example of how the modes are chosen will be discussed in Sec. III A.) Then an initial state $|\,\mathrm{in}\rangle$ for the quantum field is chosen by specifying values for $\psi_k(t=0)$ and $\dot{\psi}_k(t=0)$ for each of the $k$ modes used, in such a way that they satisfy the constraint (2.7a). Equation (2.12) is then solved iteratively for the initial value of $\dot{R}$. This equation has the same structure as symbolically represented in (2.14), with $\dot{R}$ replacing $\ddot{R}$. Thus the iteration procedure for solving it is the same. The $\dot{R}(t=0)$ so obtained is substituted into Eq. (2.13) which is then solved iteratively, for $\ddot{R}(t=0)$. $\ddot{\psi}_k(t=0)$ is trivially obtained from (2.7). With this complete set of initial data $\{\psi_k,\dot{\psi}_k,\ddot{\psi}_k,a,H,R,\dot{R},\ddot{R}\}$, we evolve forward a time step $\Delta t$ and are left with $\{\psi_k,\dot{\psi}_k,a,H,R,\dot{R}\}$. To obtain $\ddot{R}$ at this later time, we start the iteration procedure by using the value of $\langle T\rangle^{\psi}$ at the previous time step. As long as $\Delta t$ is small, the first approximation for $\ddot{R}$ is already quite accurate, and for most purposes only a small number of iterations are needed, provided the iterations converge. $\ddot{\psi}_k$ at this later time is obtained from (2.7); and we do not have to use the initial constraint equation (2.12) anymore, although it serves as a nice check on the numerical accuracy. The set up of the high-$k$-expansion scheme[12] is very similar except in the way that $\ddot{R}$ is solved at each time step. It does not invoke iteration and can be used even when $G/\epsilon$ is large. These schemes are expected to be useful in many future back-reaction calculations.

Before ending this section on the discussion of the numerical schemes, it is important to point out a very useful change of variables, which is used in the calculations reported in the next section. We define the real variable $W_k$ by

$$\psi_k=\left[C_1\exp\left[-i\int\frac{W_k}{a}dt\right]+C_2\exp\left[+i\int\frac{W_k}{a}dt\right]\right]\Big/\sqrt{2W_k}\ . \tag{2.15}$$

The constants $C_1$ and $C_2$ satisfy

$$|C_1|^2-|C_2|^2=1\ , \tag{2.16}$$

as required by (2.7a). The evolution equation of $W_k$ is given by

$$\ddot{W}_k=\frac{3}{2}\frac{\dot{W}_k^{\,2}}{W_k}-H\dot{W}_k-2\frac{W_k^{\,3}}{a^2}+2W_k\left[\frac{k^2}{a^2}+(\xi-\tfrac{1}{6})R\right]\ . \tag{2.17}$$

In numerical calculations, the advantage of using $W_k$ instead of $\psi_k$ is enormous, especially for larger values of $k$. A constant $W_k$ corresponds to a rapidly oscillating $\psi_k$ which requires a much finer step size for accurate numerical integration.

## III. NUMERICAL STUDY OF THE REHEATING OF THE HIGHER-DERIVATIVE INFLATIONARY MODELS

In this section we discuss the numerical results obtained in applying the numerical schemes outlines in Sec. II to the reheating calculation of the higher-derivative inflationary models. As discussed in the introduction, the late time evolutions of the Starobinsky and the $R^2$ models are qualitatively the same. In both models, at the end of the inflationary phase, the scalar curvature and the Hubble parameter decrease to small positive values and the Universe evolves into an oscillation phase. Hence, as far as reheating is concerned, there is no distinction between the two models. In the following we will use the notation and terminology of Ref. 4.

As we are interested only in the post-inflationary evolution of the Universe, we need only consider the $K=0$ case. This is because the large amount of preceding inflation makes the geometry effectively spatially flat independent of the value of $K$.

### A. Initial conditions and the setting up of the numerical calculation

We begin with a discussion of the initial conditions. The existence of a long inflationary period implies that, at the end of the inflation, the Universe should be empty; that is, any reasonable measure of the energy density due to matter and radiation should be effectively zero, independent of how much matter or radiation was in the Universe before inflation. To incorporate this, the starting point of our numerical study is taken to be at a time not long before the end of the "linear phase" when the Hubble parameter $H$ linearly decreases with respect to proper time $t$. We henceforth call this time $t=0$. As discussed in Ref. 4, this linear phase is an inflationary period; i.e., comoving distance is expanding faster than the horizon size $1/H$. It is assumed that before $t=0$ there has been a period of inflation long enough to solve the horizon and flatness problems. Since we are considering spatially flat spacetimes, the initial scale factor $a(t=0)$ can be arbitrarily taken to be 1. The initial Hubble parameter is chosen such that $\sqrt{\epsilon}H(t=0)=0.466$, which is dimensionless, while $\epsilon R(t=0)=2.44$. These values are picked so that from $t=0$ to the very end of the linear phase, the Universe has expanded by a factor a little larger than $\exp[18\epsilon H^2(t=0)]=50$ (cf. Ref. 4). This ensures that the energy density of any radiation that was present in the Universe at $t=0$ would have red-shifted by a factor of $1/a^4\approx10^{-7}$. This means that as long as the initial state is a fourth-order (or higher) adiabatic vacuum[10] so that $\langle T_{\mu\nu}\rangle^{\psi}$ given by (2.12d) and (2.13d) has no ultraviolet divergences, it does not matter how $\psi_k(t=0)$ and $\dot{\psi}_k(t=0)$ are specified, provided the energy-momentum of the field is not orders of magnitude larger than the geometric quantities. Such a large energy-momentum is physically unreasonable at $t=0$ because of the preceding inflation. We have tested and verified this point in our numerical calculations by using different initial conditions for the lower-$k$ modes. Making use of this fact, we let the constant $C_2$ be zero in (2.15). Then (2.16) requires $C_1=1$ (up to an irrelevant constant phase). The initial conditions of the $W_k$'s for high $k$, i.e., for $a/k \ll$(changing time scale of the geometry), are given by the fourth-order adiabatic expression, which has been written down explicitly in Ref. 18. Lower $k$ modes are freely specified. In our numerical calculations, a condition frequently used to specify the $W_k(t=0)$ and $\dot{W}_k(t=0)$ for small $k$ is the vanishing of $\langle T_{tt}\rangle_k^{\psi}$ and $\langle T\rangle_k^{\psi}$.

In a typical calculation, we use 400 to 600 modes with $k$ values well covering the region where the back-reaction effects of the quantum field are important, i.e., $k_{\max}$ is chosen so that $(k_{\max}{}^2\Delta k/a^3)\langle T_{\mu\nu}\rangle_{k_{\max}}^{\psi}$ is down by a few orders of magnitude compared to the contribution from the "central" $k$ values with the same $\Delta k$. However, as the Universe is slowly expanding while oscillating with essentially the same proper frequency $\omega_0$, to keep $\omega_0$ smaller than $k_{\max}/a$, a large $k_{\max}$ is needed at late times. It is time consuming and unnecessary to carry the very high-$k$ modes right from the beginning. Their initial contribution to $\langle T_{\mu\nu}\rangle^{\psi}$ is negligible and their high frequencies call for very fine step sizes. What can be done instead is to add more high-$k$ modes as the equations are evolved forward in time. The new modes enter the calculation at a time when $k/a$ is still much larger than $\omega_0$ and hence their starting values can be given by the fourth-order adiabatic expression for $W_k$.

This ends our discussion of the initial conditions and the setting up of the numerical calculations. We now turn to the results of the calculations.

### B. Results of the back-reaction calculations

#### *1. The conformally invariant case*

In order to set the stage for the discussion of the reheating of the Universe and the damping of the oscillations, we first study the case where the quantum field used in the calculation is conformally coupled to the geometry, i.e., $\xi=\frac{1}{6}$ and $m=0$ in (2.12) and (2.13). Because of the conformal invariance, there is no particle production in this case and hence no reheating.

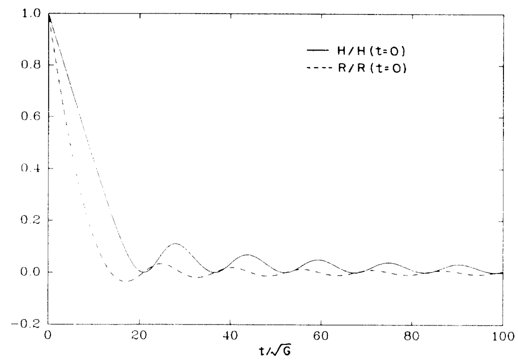


FIG. 1. The evolution of the Hubble parameter $H$ (solid line) and the scalar curvature $R$ (broken line) are shown for the case of a conformally invariant scalar field with $\epsilon=G$. On the vertical axis, $H$ is scaled by its initial value $0.466G^{-1/2}$ and $R$ by its initial value $2.44G^{-1}$. The horizontal axis is $tG^{-1/2}$. The initial values of $H$ and $R$ are chosen so that the Universe starts out close to the end of the "linear phase" of Ref. 4, in which the Universe inflates as $H$ decreases linearly. The transition from the "linear phase" to the "oscillation phase" is clearly displayed. The decay of oscillation amplitude for both $H$ and $R$ is associated with the expansion of the Universe (i.e., adiabatic decay) and is unrelated to particle production.

In Fig. 1 the evolution of $H$ (solid line) and $R$ (broken line) with respect to the proper time $t$ in units of $G$ are shown for a conformally invariant scalar field. $\sqrt{G}H$ starts at 0.466. We see that $H$ decreases linearly, approaches zero, bounces, and enters into the oscillation phase. For $\epsilon \gtrsim G$, the ${}^{(1)}H_{\mu\nu}$ term dominates the postinflationary evolution for all physically reasonable choices of the initial conditions of the *conformally* coupled field. This is because changes in the state of the field add pieces to $\langle T_{tt}\rangle$ which scale like $a^{-4}$ (see, e.g., Ref. 20). Hence, the evolutions of $H$ and $R$ shown here are essentially the same as those shown in Fig. 1 of Ref. 4 in which the calculation is done *without* any quantum field but with only the ${}^{(1)}H_{\mu\nu}$ term in the back-reaction equations. The evolution of $H$ and $R$ is given in Ref. 4 by

$$H \approx \left[\frac{3}{\omega_0}+\frac{3}{4}(t-t_{\rm os})+\frac{3}{8\omega_0}\sin 2\omega_0(t-t_{\rm os})\right]^{-1}\cos^2\omega_0(t-t_{\rm os}) \tag{3.5a}$$

$$\approx \frac{4}{3t}\cos^2\omega_0 t,\quad t-t_{\rm os} \gg \frac{1}{\omega_0}\ , \tag{3.5a'}$$

$$R \approx -6\left[\frac{3}{\omega_0}+\frac{3}{4}(t-t_{\rm os})+\frac{3}{8\omega_0}\sin 2\omega_0(t-t_{\rm os})\right]^{-1}\omega_0\sin 2\omega_0(t-t_{\rm os}) \tag{3.5b}$$

$$\approx \frac{-8\omega_0}{t}\sin 2\omega_0 t,\quad t-t_{\rm os} \gg \frac{1}{\omega_0}\ , \tag{3.5b'}$$

with $\omega_0 \approx 1/\sqrt{24\epsilon}$ and $t_{os}$ is the time the oscillation phase begins.[4] Equations (3.5a) and (3.5b) are accurate for $t - t_{os} \gtrsim 1/\omega_0$. From (3.5a'), we see that the scale factor $a(t)$ is proportional to $t^{2/3}[1+\sin(2\omega_0 t)/(3\omega_0 t)]$. Thus, when $a(t)$ is averaged over a few cycles, it increases as $t^{2/3}$, which is the same as for a matter-dominated universe. Hence, this phase is called the "scalaron-dominated" phase in the Starobinsky model.[6] The decrease in the amplitudes of the oscillations in (3.5) is associated with the expansion of the spacetime and is *unrelated* to reheating. We shall call this rate of decrease in amplitude the adiabatic rate.

### 2. *The results on reheating*

In more realistic models of the Universe, there are conformally noninvariant quantum fields which give rise to the damping of the oscillations *above* the adiabatic rate. In this section we describe our numerical results when a minimally coupled, i.e., $\xi=0$, scalar field is present.

In Fig. 2, $\sqrt{G}Rt$ is plotted against $t/\sqrt{G}$ for a typical solution when a minimally coupled field is present. The case when only a conformal field is present is also plotted (broken line) for comparison. From the figure it is clear that the back reaction of the minimally coupled field damps the oscillations in the curvature above the adiabatic rate: $Rt$ is decreasing in time. Note that the frequency of the oscillations is essentially unchanged.[21]

In Figs. 3(a) and 3(b), the $\xi=0, \epsilon=G$ case of Fig. 2 is plotted up to $t/\sqrt{G}=3000$. The vertical axes are, respectively, $Ht$ and $\sqrt{G}Rt$. In these figures, individual oscillations are barely resolvable. The oscillations are nearly completely damped at late times. In Fig. 3(a), we see that the decay of the amplitude of $Ht$ is smooth. Notice that this is in contrast with the picture of a scalaron-dominated phase,[6–8] i.e., the picture in which the Universe expands in a matter-dominated manner for some time before the produced particles dominate the expansion. A scalaron-dominated phase would imply a plateau in the time-averaged (over a few cycles) value of $Ht$ at a constant value of $\frac{2}{3}$. But from Fig. 3(a) we see that the center of the oscillations is decreasing gradually to $\frac{1}{2}$,

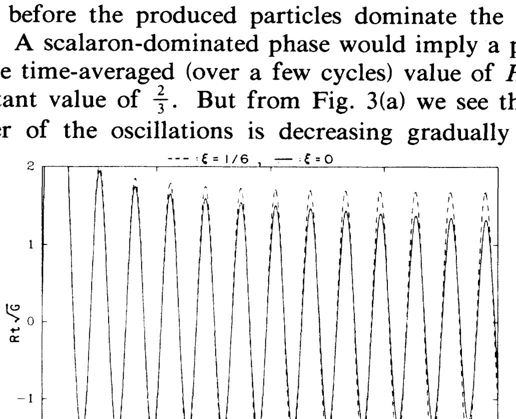


FIG. 2. The quantity $RtG^{1/2}$ is plotted against $tG^{-1/2}$. The broken line is for the case $\xi=\frac{1}{6}$ (the conformally coupled field), and the solid line is for the case $\xi=0$ (the minimally coupled field). Both solutions start with the same initial values of $R$ and $H$ as in Fig. 1. For $\xi=\frac{1}{6}$, $R$ can be described by Eq. (3.5b) and $Rt$ has a constant amplitude at late times. The damping of the oscillations above the adiabatic rate is clear for the $\xi=0$ case.

showing no plateau at $\frac{2}{3}$. At late times $Ht$ is very close to $\frac{1}{2}$, hence $a(t)\propto t^{1/2}$, and the expansion is in a radiation-dominated Friedmann manner. In Fig. 3(b) we see that $\sqrt{G}Rt$ is smoothly damped towards zero. In Fig. 3(c) we plot the logarithm of the upper envelope of the oscillations in Fig. 3(b) versus time. In a few oscillations time it is very close to a straight line, indicating that the damping in Fig. 3(b) is nearly exponential. We compute the envelope of the oscillations in Fig. 3(b) analytically in the next section. It is plotted in Fig. 3(c) as the broken line starting at $t=200\sqrt{G}$. The agreement is so precise that the two lines are barely resolvable. The rapid vanishing of $Rt$ means that at late times the evolution is following the classical Einstein equations with a trace-free energy-momentum tensor. To verify explicitly that the classical Einstein equations are satisfied, we plot $[H^2-(8\pi G/3)\langle T_{tt}\rangle^{\psi}]t^2$ vs $t/\sqrt{G}$ in Fig. 3(d). We have put in the factor of $t^2$ to take away the effect of the adiabatic decay. $\langle T_{tt}\rangle^{\psi}$ is the energy density of the scalar field apart from a vacuum-polarization piece $\langle T_{tt}\rangle^{g}$ which is given in Eq. (2.12c). We see that vacuum polarization is negligible and the classical Einstein equations are satisfied at late times. Figure 3(e) shows that $a^4\langle T_{tt}\rangle^{\psi}G^2$ is approaching a constant at late times, i.e., $\langle T_{tt}\rangle^{\psi}$ is redshifting with $a^{-4}$ as it should be for classical radiation.

In Figs. 4(a) and 4(b) the initial data for the geometry and the minimally coupled scalar field are the same as those of Figs. 3, except now $\epsilon$ is taken to be $0.2G$. In Figs. 4(a) and 4(b) we rescale all quantities by $0.2G$, i.e., the vertical axes are $Ht$ and $\sqrt{0.2G}Rt$, respectively, and the horizontal axes are $t/\sqrt{0.2G}$. We see that the oscillations are damped much more rapidly than in the previous case. This is expected since a smaller value of $\epsilon$ means that, in the effective Lagrangian, the $\epsilon R^2$ term, which is giving rise to the phase plane oscillations, is less important. This dependence on $\epsilon$ also explains why we have not shown the case of $\epsilon=10^{11}G$, which is the value proposed in Ref. 4 (the value proposed in Refs. 7 and 8 for the Starobinsky model is slightly different). The damping will be noticeable only after an extremely long time, although the damping mechanism and the final outcome will be exactly the same. The explicit dependence of the damping on $\epsilon$ will be studied in the next section.

In Fig. 4(c) we enlarge the latter part of Fig. 4(b). There is a low-frequency oscillation showing up at this later time superimposed on the higher-frequency ones. These are the same kind of oscillations which appear in Fig. 3(d) at late times. In time, these lower-frequency oscillations are damped in amplitude, and even-lower-frequency oscillations appear in a self-similar fashion. We will not study this hierarchy of lower- and even-lower-frequency oscillations in this paper; but, as we shall see in the next section, all these oscillations are exponentially damped and cannot affect the overall picture that the Universe expands in a classical radiation-dominated manner at late times.

## IV. ANALYTICAL STUDY OF THE DAMPING

The coupled system of differential equations (2.7) and (2.13) may look too complicated for analytical analysis,

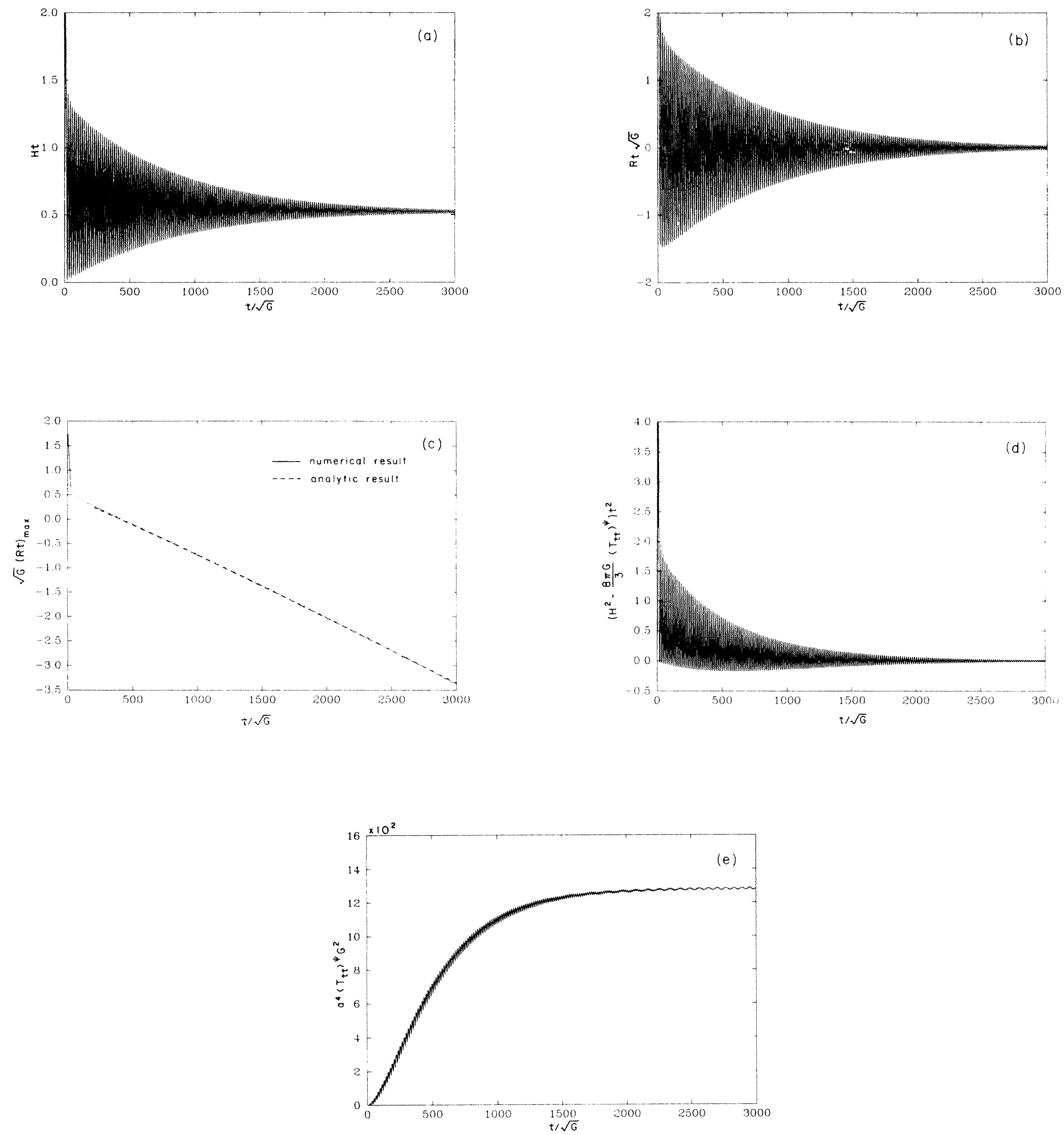


FIG. 3. (a) $Ht$ is plotted against $tG^{-1/2}$ for the $\epsilon=G,\xi=0$ case with the same initial data as in Figs. 1 and 2. Individual oscillations are barely resolvable on the scale of the plot. The amplitude of the oscillations is decreasing in time with the mean value tending smoothly to $\frac{1}{2}$, showing no plateau at $\frac{2}{3}$. At very late times, when the oscillations are well damped, $H$ evolves in a radiation-dominated Friedmann manner, i.e., $H\approx\frac{1}{2}(t-\text{const})^{-1}$. Thus, $Ht$ approaches $\frac{1}{2}$ as $t^{-1}$. (b) $RtG^{1/2}$ is plotted against $tG^{-1/2}$ for the same case as (a). The oscillations are damped towards zero, again signaling the onset of a radiation-dominated phase. (c) The quantity $\ln[(Rt)_{\max}G^{1/2}]$ is plotted against $tG^{1/2}$ as the solid line. $(Rt)_{\max}$ is the upper envelope of the oscillations in (b). The broken line starting at $tG^{-1/2}=200$ is plotted using the analytic result derived in Sec. IV [Eqs. (4.18) and (4.19)]. The agreement is so close that the two lines nearly coincide, except for small scale variations in the solid line. The lines are nearly straight for $tG^{-1/2}\gtrsim 200$ with a mild $\ln t$ modification which slightly decreases the slope of the lines. Hence the damping in (b) is essentially exponential. (d) The quantity $[H^2-(8\pi G/3)\langle T_{tt}\rangle^{\psi}]t^2$ is plotted against $tG^{-1/2}$ for the same $\xi=0,\epsilon=G$ case. This is a measure of the deviation from the classical Einstein equation. The factor of $t^2$ is introduced to counter the effects of adiabatic decay. The plot shows that, at late times, the classical Einstein equation is satisfied and the energy density of the scalar field, given by $\langle T_{tt}\rangle^{\psi}$, behaves classically. (e) The quantity $a^4\langle T_{tt}\rangle^{\psi}G^2$ is plotted against $tG^{-1/2}$ for the same $\xi=0,\epsilon=G$ case. For $t\gtrsim 2000G^{1/2}$, one sees that $\langle T_{tt}\rangle^{\psi}$ is behaving essentially classically since it is red-shifting like $1/a^4$. The initial value of $a$ has been taken to be 1.

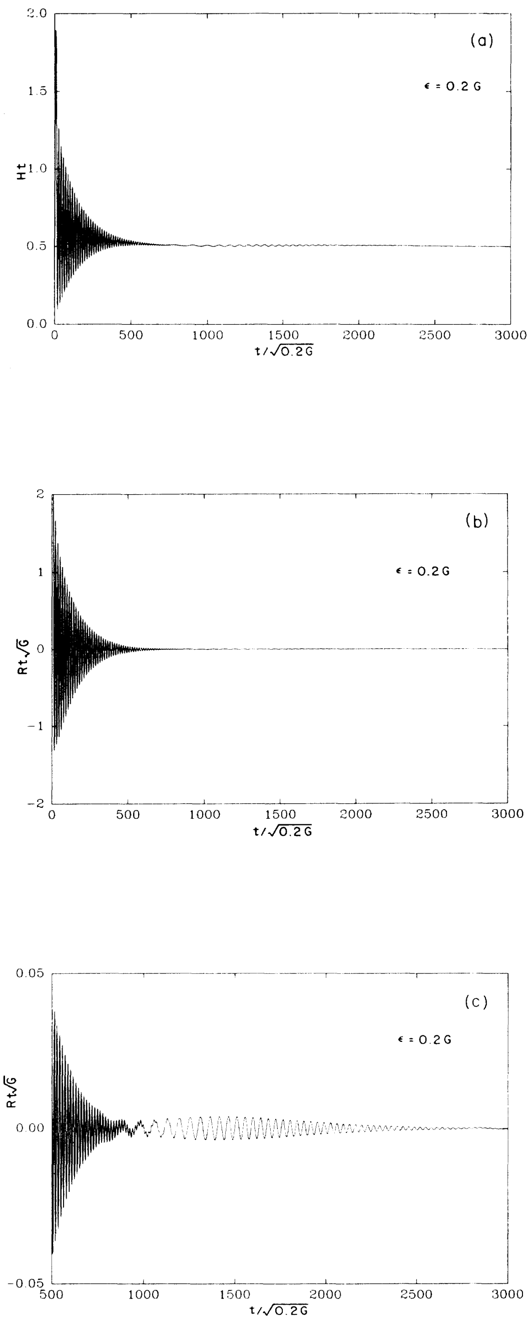


FIG. 4. (a) The quantity $Ht$ is plotted against $t(0.2G)^{-1/2}$. All initial data of the geometry and the minimally coupled field are the same as in Figs. 3, except that now $\epsilon=0.2G$ instead of $G$. The additional rescaling of 0.2 in this case makes the width of individual oscillations in the figure the same as those in Figs. 3. Clearly the oscillations are damped much more efficiently than in the $\epsilon=G$ case. The dependence of the damping rate on $\epsilon$ is given in Eq. (4.18). (b) The quantity $RtG^{1/2}$ is plotted against $t(0.2G)^{-1/2}$ for the $\epsilon=0.2G$ case as in (a). $Rt$ is rapidly damped to zero. (c) The right portion of (b) is enlarged, showing the emergence of lower-frequency oscillations superimposed on the well-damped higher-frequency oscillations. In time this new series of oscillations is damped and even longer frequency oscillations appear in a self-similar fashion.

but as we shall see, the dominant piece of physics for the damping of the oscillations is in fact quite simple. In the following analysis we consider a massless nonconformally coupled scalar field in a spatially flat ($K=0$) universe. A simplified version of this analysis is sketched in Ref. 22. 22. We first study the damping of the oscillations and then compute the reheating temperature of the Universe.

### A. The damping of the oscillations

We start with the observation that there exist two obvious time scales in the problem: the shorter time scale $t_s=1/\omega_0$ associated with the oscillation of the scalar curvature, and the longer time scale $t_l=a/\dot{a}$ associated with the expansion of the Universe. $t_l$ is also the time scale of the adiabatic damping, i.e., the decrease in amplitude of $R$ and $H$, if there are only conformally invariant fields in the Universe. In the oscillation phase of the higher-derivative inflationary models, $t_l$ is increasing in time and is orders of magnitude longer than $t_s$ except at the very beginning of the oscillation phase. We now want to find a third time scale $t_d$, hidden in the equations, associated with the damping coming from the effect of particle production when a nonconformally invariant field is present. We expect $t_d$ to be longer than or comparable to $t_l$.

We look at the evolution equations (2.13) and (2.17) at a time well into the oscillation phase, i.e., $t-t_{\text{os}} \gg t_s$. In the trace equation (2.13), we can drop all terms in $\langle T\rangle^g$ except the term proportional to ${}^{(1)}H_\mu{}^\mu$. All dropped terms are proportional to $1/t^{2n}$ with $n\geq 1$: Both $R$ and $H$ would decay faster than (or equal to) $1/t$ if there is damping above (or equal to) the adiabatic rate. It is more convenient to work with the quantity

$$Q\equiv a^{3/2}R \;, \tag{4.1}$$

and the dimensionless time

$$\tilde{t}=2\omega_0 t \;. \tag{4.2}$$

Here $\omega_0\simeq 1/\sqrt{24\epsilon}$ is the oscillation frequency of the Universe. The evolution of $Q$ is given by

$$\frac{d^2}{d\tilde{t}^2}Q+Q=\frac{\pi}{3}\frac{G}{\epsilon\omega_0{}^2}a^{3/2}\langle T\rangle^\psi \;. \tag{4.3}$$

In deriving (4.3) we have again dropped terms which are proportional to $1/\tilde{t}^{2n}$. We have also used the fact that $\langle T\rangle^g$ is dominated by the ${}^{(1)}H^\mu{}_\mu$ term in the oscillation phase (cf. Sec. III B 1), $\langle T\rangle^\psi$ is given by (2.13d), and its main contribution comes from modes centered around $k=a\omega_0$. The exact range of modes we have to sum over to obtain this main contribution is yet to be determined. [Examination of (2.9b) and (2.13d) shows the contributions of the modes with small values of $k$ in $\langle T\rangle_k$ are small due to the $k^2dk$ factor, while $\langle T\rangle_k^{uv}$ in (2.13e) will be dominated by a divergent term proportional to ${}^{(1)}H_\mu{}^\mu$, whose contribution is to be incorporated into the value of $\epsilon$ as discussed in Sec. II B. The very high-$k$ contributions are negligible due to the cancellation between $\langle T\rangle_k$ and $\langle T\rangle_k^{uv}$.] For the evolution of the quantum field, we first study the behavior of a single $k$ mode starting with a large $\omega\equiv k/a$ which gradually decreases and approaches

$\omega_0$ as $a(t)$ increases. When $\omega \gg \omega_0$, $W_k$ of (2.15) is essentially $k$. We shall use the variable

$$Z_k=\sqrt{a}\,(W_k-1)\ , \tag{4.4}$$

and keep only terms which are first order in $Z_k$. Equation (2.17) then reduces to

$$\frac{d^2}{d\tilde{t}^2}Z_k+(\omega^2/\omega_0{}^2)Z_k=\delta\sin\tilde{t}\ , \tag{4.5a}$$

with

$$\delta=\frac{\omega}{2\omega_0{}^2}(\xi-\tfrac{1}{6})Q_A\ . \tag{4.5b}$$

Here we have again dropped terms proportional to $1/\tilde{t}^{2n}$. The sinusoidal driving force in (4.5) has a slowly changing (on time scale of $t_l$) amplitude $\delta$. $Q_A$ is the magnitude of $Q$ with the $t_s$ oscillation factor taken out. $a(t)$ and hence $\omega$ also contain pieces which oscillate on the $t_s$ time scale, but they are relatively small compared to the total magnitudes of $a(t)$ and $\omega$ and we shall ignore them. From (4.5a) we see that $Z_k$ evolves like an aging oscillator driven by a sinusoidal force with fixed frequency. Starting at a time when $\omega\equiv k/a$ is much bigger than $\omega_0$, $Z_k$ is initially zero [cf. Eq. (4.4)], and remains small as long as the "spring" is still stiff. The natural frequency $\omega$ of the oscillator gradually tends to $\omega_0$ on the time scale $t_l$. When $\omega\approx\omega_0$, it can be seen from (4.5a) that there is a resonance and $Z_k$ grows with a $\pi/2$ phase lag compared to $Q$. This growing $Z_k$ gives the dominant contribution to $\langle T\rangle^{\psi}$, and hence $\langle T\rangle^{\psi}$ also has a $\pi/2$ phase lag with respect to $Q$. From (4.3) we see that this will cause $Q$ to be damped.

Quantitatively, this goes as follows. We assume that the Universe expands throughout the oscillation phase by $a(t)\approx a_i\tilde{t}^{\,r}$, with $a_i$ being constant. $r$ will go from the initial undamped value of $\frac{2}{3}$ to the radiation phase value of $\frac{1}{2}$. We shall not specify $r$ and shall find that the final result is independent of its value. Given this approximate behavior for $a(t)$, Eq. (4.5) can be solve exactly in terms of integrals, but it is more illuminating to determine the damping in the following approximate manner. When $\omega$ gets close to $\omega_0$ the driving force in (4.5a) will produce a resonance, and the amplitude of $Z_k$ will grow linearly in time such that

$$Z_k\approx-\frac{\delta}{2}\tilde{t}\cos\tilde{t}\ . \tag{4.6}$$

We denote the time when $\omega=\omega_0$ by $\tilde{t}=\tilde{t}_{\rm res}$:

$$\tilde{t}_{\rm res}=\left[\frac{k}{a_i\omega_0}\right]^{1/r}\ . \tag{4.7}$$

At this time $Z_k$ as given by (4.6) is $\pi/2$ out of phase with $Q$, which oscillates with $\sin\tilde{t}$. That is,

$$\phi_Z\,|_{\tilde{t}_{\rm res}}=\phi_Q\,|_{\tilde{t}_{\rm res}}-\frac{\pi}{2}\ . \tag{4.8}$$

The natural frequency $\omega/\omega_0$ of the oscillation of $Z_k$ will keep decreasing as $a(t)$ increases. This eventually makes $Z_k$ go out from the "resonance region." We need to determine the time it takes for $Z_k$ to go out of this resonance region as this will enable us to determine the range of modes that reside in the resonance region at a given time. The growth in (4.6) will stop at a time $\tilde{t}_{\rm end}=\tilde{t}_{\rm res}+\Delta\tilde{t}$ when the phase lag of $Z_k$ with respect to $Q$ increases from $\pi/2$ to $\pi$. This increase in phase lag comes in as the natural frequency [which is $\omega(\tilde{t})/\omega_0$ in units of $\tilde{t}$] of $Z_k$ decreases in time and differs from the frequency (which is 1 in units of $\tilde{t}$) of $Q$, cf. Eq. (4.5a). The time interval $\Delta\tilde{t}=\tilde{t}_{\rm end}-\tilde{t}_{\rm res}$ can be approximately determined from

$$\left[\pi-\frac{\pi}{2}\right]=\int_{\tilde{t}_{\rm res}}^{\tilde{t}_{\rm end}}\left[1-\frac{\omega}{\omega_0}\right]d\tilde{t} \tag{4.9}$$

which gives

$$\Delta\tilde{t}\approx\left[\frac{\pi}{r}\tilde{t}_{\rm res}\right]^{1/2}\ . \tag{4.10}$$

[The value of $\Delta\tilde{t}$ so obtained has been confirmed by numerical integration of (4.5).] We can approximate $Z_k$ in the time interval $\tilde{t}_{\rm res}$ to $\tilde{t}_{\rm end}$ by its mean magnitude in this interval

$$Z_k\approx-\frac{1}{4}\left[\frac{\pi}{r}\tilde{t}_{\rm res}\right]^{1/2}\delta\,|_{\tilde{t}_{\rm res}}\cos\tilde{t},\ \ \tilde{t}\in(\tilde{t}_{\rm res},\tilde{t}_{\rm end})\ , \tag{4.11}$$

where $\delta$ is evaluated at $\tilde{t}_{\rm res}$. From (2.13d), (2.13e), and (2.9b), $\langle T\rangle_k^{\psi}$ for this mode is given by

$$\langle T\rangle_k^{\psi}\approx\frac{3(\xi-1/6)}{4\pi^3}a\left[|\dot{\psi}_k|^2-\frac{k^2}{a^2}|\psi_k|^2\right] \tag{4.12a}$$

$$=\frac{3}{8\pi^3}(\xi-\tfrac{1}{6})\left[\frac{W_k}{a}-\frac{k^2}{aW_k}\right]\ . \tag{4.12b}$$

All the other terms in $\langle T\rangle_k^{\psi}$ are down by at least $1/\tilde{t}$. Using (4.4) and (4.11), we obtain

$$\langle T\rangle_k^{\psi}\simeq-\frac{3}{16\pi^3}(\xi-\tfrac{1}{6})a^{-3/2}\left[\frac{\pi}{r}\tilde{t}_{\rm res}\right]^{1/2}\delta\,|_{\tilde{t}_{\rm res}}\cos\tilde{t}\ . \tag{4.13}$$

Next we have to determine the range of $\omega$ in this "resonance region:"

$$\Delta\omega=\frac{k}{a^2}\Delta a\simeq\frac{k}{2a\omega_0}\overline{H}\Delta\tilde{t}\ . \tag{4.14}$$

Here, $\overline{H}$ is the Hubble parameter, $H\equiv\dot{a}/a$, averaged over a few oscillations. $\langle T\rangle^{\psi}$ which is dominated by the modes residing in the "resonance region" is hence given by

$$\langle T\rangle^{\psi}\simeq 4\pi\omega_0{}^2\Delta\omega\langle T\rangle_k^{\phi}\,|_{k=a\omega_0}\ . \tag{4.15}$$

To evaluate $\langle T\rangle_k^{\psi}$ in (4.13) and $\Delta\omega$ in (4.14) at $k=a(t)\omega_0$, we reexpress $\tilde{t}_{\rm res}$ in (4.7) by $(k\tilde{t}^{\,r}/a\omega_0)^{1/r}$. Then (4.15) becomes

$$\langle T\rangle^{\psi}=\frac{-3}{8\pi}(\xi-\tfrac{1}{6})^2\omega_0{}^2a^{-3/2}Q^+ \tag{4.16a}$$

$$=\frac{-3}{8\pi}(\xi-\tfrac{1}{6})^2\omega_0{}^2R^+\ , \tag{4.16b}$$

where $Q^+,R^+$ denotes $Q,R$ with a $\pi/2$ phase lag, and we have used the relation $\overline{H}t/r\simeq 1$. We notice the following features of the trace of the energy-momentum tensor of the quantum field as given by (4.16). (1) It is independent of $r$, i.e., independent of the rate at which the Universe expands (as long as it is slow compared to the rate of the oscillation of the curvature, i.e., $t_l \gg t_s$). (2) It is directly proportional to the scalar curvature with a $\pi/2$ phase lag. (3) It depends strongly on the oscillation frequency of the curvature.

When we insert (4.16) into the right-hand side of (4.3), we obtain

$$Q \approx C \exp\left[-\frac{\Lambda \tilde{t}}{2\omega_0}\right] \sin\tilde{t} \;, \tag{4.17}$$

with

$$\Lambda = \frac{G}{4\epsilon}(\xi - \tfrac{1}{6})^2 \omega_0 \;. \tag{4.18}$$

$C$ is a constant which is determined by matching $R$ to Eq. (3.5b'). We find

$$R = -2\frac{\omega_0{}^2 a_{\rm os}{}^{3/2}}{a^{3/2}} e^{-\Lambda t} \sin 2\omega_0 t \;. \tag{4.19}$$

This is a remarkable result. The damping of $R$ has the same pattern and the same time scale $t_d = 1/\Lambda$ through the oscillation phase and is independent of the rate of expansion of the Universe. The transition to the radiation-dominated phase is smooth; there is no plateau of "scalaron-dominated phase" as discussed in the previous section.

The result (4.19) has been plotted as the broken line in Fig. 3(c) for the case of $\xi = 0$ and $\epsilon = G$. We determine $a_{\rm os}$ by first integrating (3.5a') over a few oscillations, which gives $a = a_{\rm os}[1 + \omega_0(t - t_{\rm os})/4]^{2/3}$ and then matching it to the early part of the oscillation phase. This gives $a_{\rm os} \simeq 62$. The analytic and numeric results shown in Fig. 3(c) are in very close agreement. Given the nature of some of the approximations made in the analytic analysis, we would not have expected, *a priori*, such a close agreement between the analytic and numeric results. Notice that the $a^{-3/2}$ modification to the exponential damping in (4.19) is visible in the figure: the slopes of the lines decrease slightly with respect to time. The agreement between the analytic result and the numerical calculation for the $\epsilon = 5$ case of Fig. 4 is essentially the same; the analogous plot is not shown.

We want to emphasize that the above analysis is not restricted to the study of reheating in the inflationary models. An $R^2$ term in the effective Lagrangian introduces a scalar degree of freedom, in addition to the usual tensorial degree of freedom of gravity.[4,5] The above analysis shows the way the oscillation in this auxiliary scalar field gets damped in an expanding universe by coupling to other quantum fields. In Ref. 23, it was suggested that the energy contained in the phase plane oscillations of this auxiliary scalar field might still be large enough today to close the Universe. It was shown in Ref. 24, using our numerical iteration scheme, that this energy is dissipated too rapidly by particle production to be significant in the present Universe. The above analysis shows that, in fact, it dissipates essentially exponentially. Any significant "universal" oscillations that exist today have to have $t_d \simeq$Hubble time$\approx 10^{10}$ yr. By (4.18), this corresponds to an oscillation period of $\approx 10^4$ sec$\times(10^{11}G/\epsilon)$. $10^{11}G$ is the value for $\epsilon$ suggested in Ref. 4.

### B. Reheating temperatures of the Universe

In the following, we return to the higher-derivative inflationary models and use (4.19) to determine the reheating temperature of the Universe.

The Hubble parameter is related to $R$ by

$$6\dot{H} + 12H^2 = R \;, \tag{4.20}$$

with $R$ given by (4.19). For $t \gg t_d$, $R \approx 0$ and the solution is $H \simeq (2t + \text{const})^{-1}$. Matching this to the solution for $t \ll t_d$, i.e., (3.5a'), and using the fact that the oscillations are to be damped essentially exponentially, we find that $H$ can be approximated by

$$H \simeq \frac{1}{2t} + \frac{1}{t}(\tfrac{1}{6} + \tfrac{2}{3}\cos 2\omega_0 t) e^{-\Lambda t} \;, \tag{4.21}$$

for $t \gg t_{\rm os}$. Let us arbitrarily pick the beginning of the Friedmann phase to be the time $t_F$ when the oscillations have been damped by 2 $e$-foldings, i.e., $t_F = 2/\Lambda$. At this time, the effectively classical Einstein equation

$$H^2 = \frac{8\pi}{3} G \langle T_{tt} \rangle^{\psi} \tag{4.22}$$

describes the behavior of solutions with an error of about 20%, with $\langle T_{tt} \rangle^{\psi}$ evolving essentially like classical radiation [cf. Figs. 3(d) and 3(e)]. Hence the radiation-dominated Friedmann phase is beginning with an energy density

$$\langle T_{tt} \rangle^{\psi} |_{t_F} = \frac{3}{128\pi} \frac{\Lambda^2}{G} \;. \tag{4.23}$$

This corresponds to a Friedmann temperature $T_F$:

$$T_F \equiv (\langle T_{tt} \rangle^{\psi})^{1/4} = \left[\frac{3}{2048\pi}\right]^{1/4} |\xi - \tfrac{1}{6}| \left[\frac{G\omega_0{}^2}{\epsilon^2}\right]^{1/4} . \tag{4.24}$$

In order to define meaningfully a temperature, we have to consider interacting fields. We expect that the interacting fields which are nonconformally coupled to the geometry will be excited by the oscillations of the geometry in essentially the same manner as the free fields described above. For a realistic estimation for the $T_F$ of our Universe, we may want to include more than just one nonconformally coupled field. Let the ratio of nonconformally invariant to conformally invariant scalar degrees of freedom in the Universe be $N$. $T_F$ in (4.24) would have to be multiplied by $N^{1/4}$ and such a factor is of order 1 in, say, any grand unified theory proposed. Hence (4.24) implies that the Friedmann temperature of our Universe is approximately

$$T_F \approx 10^9 \left[ \frac{10^{11} G}{\epsilon} \right]^{3/4} \text{GeV} , \tag{4.25}$$

where we have used $\omega_0 \approx 1/\sqrt{24\epsilon}$ as in the higher-derivative inflationary models. For the inflationary models to produce small enough scalar perturbations $\epsilon$ must be larger than or on the order of $10^{11}G$ ($\epsilon$ is bounded from above by $\epsilon < 10^{12} - 10^{15}$ G by other considerations, cf. Ref. 4). Equation (4.25) coincides with the estimation given in Ref. 4. With this large value of $\epsilon$ the oscillation phase is long, containing some $10^{13}$ oscillations. (It is still very short compared to every day time scales, as $t_d \sim 10^{-24}$ sec.) Interesting physics would have happened early on in the oscillation phase, and it is important to determine the temperature, which is relevant to, say, a phase transition of a Higgs field at that time. For fields with grand-unified-theory (GUT) scale couplings, it takes only $\sim 1/(10^{15}\text{ GeV}) \sim 10^4 t_{\text{Planck}}$ to achieve thermal equilibrium through mutual interactions. This is shorter than the expansion time scale in the oscillation phase which is $1/H(t \gtrsim t_{\text{os}}) \gtrsim \sqrt{216\epsilon} \approx 5 \times 10^6 t_{\text{Planck}}$, and the oscillation time scale which is $\pi/\omega_0 \simeq 15\sqrt{\epsilon} \simeq 4 \times 10^6 t_{\text{Planck}}$. Hence fields with GUT scale couplings have time to equilibriate even early on in the oscillation phase. However, at this early time, the vacuum-polarization energy is large, as the $\langle T_{\mu\nu} \rangle^g$ given by (2.12c) and (2.13c) is comparable in magnitude to $\langle T_{\mu\nu} \rangle^\psi$. It is not clear how much, if any, of the vacuum-polarization energy is associated with excitation of the quantum fields. Hence, without going into a detailed investigation using interacting fields, we can at most make an order of magnitude estimate for the temperature at this early time:

$$T \approx (\langle T_{tt} \rangle^\phi)^{1/4} \approx (\langle T \rangle^\phi)^{1/4}$$

$$\approx \left[ \frac{3}{\pi} (\xi - \tfrac{1}{6})^2 \frac{\omega_0{}^3}{t} \right]^{1/4} , \tag{4.26}$$

cf. (3.5b′) and (4.16). The reheating temperature early on in the oscillation phase, i.e., $t \sim 1/\omega_0$, is hence on the order of

$$T_r \approx 10^{12} \text{ GeV} \left[ \frac{\epsilon}{10^{11} G} \right]^{-1/2} . \tag{4.27}$$

$T_r$ is also insensitive to the number of nonconformally coupled fields in the Universe. We have taken $(\xi - \frac{1}{6})$ to be of order 1. $T_r$ given by (4.27) again coincides with the estimate given in Ref. 4. Notice that this temperature is low enough to avoid the GUT phase transition and hence the associated monopole problem, and is high enough for standard baryogenesis to go through.[1] This is surely a healthy sign for the higher-derivative inflationary models.

## V. CONCLUSION

We have discussed two numerical schemes to handle the semiclassical back-reaction equations for quantum fields with arbitrary curvature couplings and masses, i.e., the high-$k$-expansion scheme[12] and the iteration scheme. They differ only in the way the highest-derivative geometric quantity in the back-reaction equations is solved for. The high-$k$-expansion scheme is limited only by the fact that it uses the adiabatic regularization, which has been developed for free scalar fields in Robertson-Walker and Bianchi-type-I spacetimes but not other cases. The iteration scheme also uses adiabatic regularization, and depends on the smallness of the iteration parameter. It has the nice feature of explicitly demonstrating that the results obtained are self-consistent solutions of the semiclassical back-reaction equations. We expect these schemes to be useful for many other calculations relevant to the evolution of the early Universe.

In Sec. III these schemes were applied to the reheating calculation of the higher-derivative inflationary models. The results are plotted in Figs. 1–4, showing that the oscillations in the curvature are damped and the Universe is driven into a radiation-dominated Friedmann phase.

In Sec. IV we presented an analytic study of the reheating and the damping of the oscillations using the back-reaction equations. It was shown that the oscillations in the curvature decay essentially exponentially in an expanding Universe, as long as nonconformally coupled scalar fields are present. Reheating temperatures are obtained which are in close agreement with the estimations given in Ref. 4. The analytic results are also plotted in Fig. 3(c) in comparison with the numerical results. They agree to high accuracy.

*Note added in proof.* It has recently been shown[26] that if the measure (2.10a) is used in computing the adiabatic counterterms in the case $K = 1$, then the trace anomaly is recovered and adiabatic regularization is equivalent to point splitting. Equations (2.12c) and (2.13c) are then also valid for the case $K = 1$.

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## APPENDIX

In this appendix we show the explicit form of the back-reaction equations for spacetimes with positive spatial curvatures, i.e., $K = 1$. We display $\langle T_{\mu\nu} \rangle^{\text{ren}}$ in the same way as was done in Eqs. (2.12) and (2.13) except that no sums over $k$ are performed. The back-reaction equations are the same as in (2.12a) and (2.13a) and the break-up of $\langle T_{\mu\nu} \rangle^{\text{ren}}$ into $\langle T_{\mu\nu} \rangle^g$ and $\langle T_{\mu\nu} \rangle^\psi$ is the same as in (2.12b) and (2.13b). With the definition

$$Z\left[\frac{n}{2}, m^2\right] \equiv \sum_{k=1}^{\infty} \frac{k^2}{a^n \omega_k^n} \tag{A1}$$

we find[18]

$$\langle T_{tt}\rangle^{g}=-\frac{1}{8\pi^2}\left\{\frac{m^4a^2H^2}{4}Z(\tfrac{5}{2},m^2)-\frac{m^4a^4}{64}Z(\tfrac{7}{2},m^2)\left[\tfrac{4}{3}H\dot{R}-\tfrac{1}{9}R^2+\tfrac{20}{3}H^2R-4H^4+\frac{4RK}{3a^2}-\frac{24H^2K}{a^2}-\frac{4K^2}{a^4}\right]\right.$$
$$+\frac{7m^6a^6}{16}Z(\tfrac{9}{2},m^2)\left[\tfrac{1}{3}H^2R+2H^4-\frac{2H^2K}{a^2}\right]-\frac{105m^8a^8H^4}{64}Z(\tfrac{11}{2},m^2)$$
$$+(\xi-\tfrac{1}{6})\left[\frac{m^2a^2}{8}Z(\tfrac{5}{2},m^2)\left[4H\dot{R}-\tfrac{1}{3}R^2+12H^2R-12H^4+\frac{4RK}{a^2}-\frac{24H^2K}{a^2}-\frac{12K^2}{a^4}\right]\right.$$
$$\left.-\frac{5m^4a^4}{4}Z(\tfrac{7}{2},m^2)\left[2H^2R+9H^4-\frac{9H^2K}{a^2}\right]+\tfrac{105}{4}m^6a^6H^4Z(\tfrac{9}{2},m^2)\right]$$
$$\left.+(\xi-\tfrac{1}{6})^2 9m^2a^2H^2RZ(\tfrac{5}{2},m^2)\right\}, \tag{A2}$$

$$\langle T_{tt}\rangle^{\psi}=\frac{4\pi}{a^3}\sum_{k=1}^{\infty}k^2(\langle T_{tt}\rangle_k-\langle T_{tt}\rangle_k^{\mathrm{uv}}), \tag{A3}$$

$$\langle T_{tt}\rangle_k^{\mathrm{uv}}=\frac{1}{16\pi^3}\left\{\omega_k-(\xi-\tfrac{1}{6})\left[\frac{3}{\omega_k}\left[H^2-\frac{K}{a^2}\right]+\frac{3m^2H^2}{\omega_k{}^3}\right]+\tfrac{1}{4}(\xi-\tfrac{1}{6})^2\frac{{}^{(1)}H_{tt}}{\omega_k{}^3}\right\}, \tag{A4}$$

$$\langle T\rangle^{g}=\frac{-1}{4\pi^2}\left\{\frac{m^4a^2}{8}Z(\tfrac{5}{2},m^2)\left[\tfrac{1}{3}R+2H^2-\frac{2K}{a^2}\right]-\tfrac{5}{8}m^6a^4H^2Z(\tfrac{7}{2},m^2)\right.$$
$$-\frac{m^4a^4}{32}Z(\tfrac{7}{2},m^2)\left[\frac{\ddot{R}}{3}+\frac{11}{3}H\dot{R}+\frac{1}{3}R^2+6H^2R-\frac{10}{3}\frac{RK}{a^2}-\frac{8H^2K}{a^2}+\frac{8K^2}{a^4}\right]$$
$$+\frac{m^6a^6}{128}Z(\tfrac{9}{2},m^2)\left[\frac{56}{3}H\dot{R}+\tfrac{7}{3}R^2+140H^2R+84H^4-\frac{28RK}{a^2}-\frac{616H^2K}{a^2}+\frac{84K^2}{a^4}\right]$$
$$-\frac{231m^8a^8}{256}Z(\tfrac{11}{2},m^2)\left[\frac{4}{3}H^2R+8H^4-\frac{8H^2K}{a^2}\right]+\frac{1155m^{10}a^{10}H^4}{128}Z(\tfrac{13}{2},m^2)$$
$$+(\xi-\tfrac{1}{6})\left[\frac{m^2a^2}{4}Z(\tfrac{5}{2},m^2)\left[\ddot{R}+7H\dot{R}+\tfrac{1}{2}R^2+4H^2R+6H^4-\frac{4RK}{a^2}+\frac{12H^2K}{a^2}+\frac{6K^2}{a^4}\right]\right.$$
$$-\frac{m^4a^4}{32}Z(\tfrac{7}{2},m^2)\left[80H\dot{R}+10R^2+400H^2R-120H^4-\frac{100RK}{a^2}-\frac{1320H^2K}{a^2}+\frac{240K^2}{a^4}\right]$$
$$\left.+\frac{m^6a^6}{128}Z(\tfrac{9}{2},m^2)\left[2240H^2R+8400H^4-\frac{11760H^2K}{a^2}\right]-\frac{945m^8a^8H^4}{8}Z(\tfrac{11}{2},m^2)\right]$$
$$+(\xi-\tfrac{1}{6})^2\left[\frac{m^2a^2}{32}Z(\tfrac{5}{2},m^2)\left[144H\dot{R}+48R^2+288H^2R-\frac{288RK}{a^2}\right]\right.$$
$$\left.\left.-\frac{45}{2}m^4a^4H^2RZ(\tfrac{7}{2},m^2)\right]\right\}, \tag{A5}$$

$$\langle T\rangle^{\psi}=\frac{4\pi}{a^3}\sum_{k=1}^{\infty}a^2(\langle T\rangle_k-\langle T\rangle_k^{\mathrm{uv}}), \tag{A6}$$

$$\langle T\rangle^{\mathrm{uv}}=\frac{1}{16\pi^3}\left\{\frac{m^2}{\omega_k}-(\xi-\tfrac{1}{6})\left[\frac{1}{\omega_k}\left[R-6H^2-\frac{6K}{a^2}\right]+\frac{m^2}{\omega_k{}^3}\left[R-3H^2-\frac{3K}{a^2}\right]+9\frac{m^4H^2}{\omega_k{}^5}\right.\right.$$

$$\left.\left.+\frac{(\xi-\frac{1}{6})^2}{4\omega_k{}^3}\left[{}^{(1)}H_\mu{}^\mu+18\frac{m^2}{\omega_k{}^2}\left[H\dot{R}-\frac{R^2}{12}+H^2R+\frac{RK}{a^2}\right]\right]\right]\right\}. \tag{A7}$$

The quantities $Z(n/2,m^2a^2)$ have been analyzed by Shen, Hu, and O'Conner.[25] They find

$$Z(n/2,m^2a^2)=\int_1^\infty dt\frac{t^2}{(t^2+m^2a^2)^{n/2}}+\tfrac{1}{2}(1+m^2a^2)^{-n/2}+I_{n/2}(m^2a^2)\,, \tag{A8a}$$

$$I_{n/2}(m^2a^2)\equiv i\int_0^\infty\frac{dt}{e^{2\pi t}-1}\left[\frac{(1+it)^2}{[(1+it)^2+m^2a^2]^{n/2}}-\frac{(1-it)^2}{[(1-it)^2+m^2a^2]^{n/2}}\right]. \tag{A8b}$$

When $\langle T_{\mu\nu}\rangle^g$ is evaluated in the limit $m$ goes to zero and $\xi$ goes to $\frac{1}{6}$, one sees that the usual trace anomaly does not appear. This may be an indication that the adiabatic counterterms for the $K=1$ case [Eqs. (A2), (A4), (A5), and (A7)] which were first obtained by Bunch[18] have to be modified.

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[21] For $\xi\neq\frac{1}{6}$, $\epsilon$ is in fact slowly increasing in time (hence also $1/\omega_0$, as is barely visible in the figure). The time dependence arises from the $(\xi-\frac{1}{6})^2\omega_k^{-3}\,{}^{(1)}H_{\mu\nu}$ term in $\langle T\rangle_{\mu\nu}{}_k^\psi$ [Eqs. (2.12e) and (2.13e)], which is the dominant piece of $\langle T_{\mu\nu}\rangle_k^\psi$ for the $k\to 0$ modes. The small-$k$ limit of integration (2.12d) and (2.13d) gives a $(\xi-\frac{1}{6})^2\,{}^{(1)}H_{\mu\nu}\ln a$ term and a term proportional to ${}^{(1)}H_{\mu\nu}$ which is divergent in the massless case. This divergent term is absorbed into the renormalization parameter $\alpha$ as discussed in Sec. II B. Hence $\epsilon$, the coefficient of the ${}^{(1)}H_{\mu\nu}$ term in the back-reaction equations, is given by $\epsilon=\text{const}+(G/2\pi)(\xi-\frac{1}{6})^2\ln a$. Since in all of our considerations, $\epsilon$ at $t=0$ (when $a=1$) is at least of order $G$, and $\xi$ is between 0 and $\frac{1}{6}$, $a$ has to be as large as $10^{98}$ before the value of $\epsilon$ is significantly changed. We can and will treat $\epsilon$ as a constant throughout this paper.


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