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Discontinuity cylinder model of gravitating U(1) cosmic strings

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We introduce a model for an infinite-length, straight U(1) cosmic string as a cylindrical, singular shell enclosing a region of false vacuum. The properties of the geometry for the region exterior to the string are fully determined under the assumption that changes in the scalar and gauge field variables occur only at the cylindrical shell. This is consistent with a limiting form of the scalar potential $V(\phi)$ where a minimum at $|\phi|=0$ is separated by a large barrier from a global minimum at $|\phi|=\eta\neq 0$. The introduction of an approximately singular "surface" for the string allows the definition of a δ-function stress-energy density that characterizes discontinuities in the fields. We show consistency of the model with the full coupled equations for the metric, and the scalar and gauge fields in curved space-time. It is found that for this model, in the absence of an "external" cosmological constant, the exterior geometry of the string approaches Minkowski space-time with a deficit angle, and it is shown that in the limit when the string becomes a line source, i.e., its radius vanishes, the deficit angle reduces to the well-known expression $\Delta\theta=8\pi\mu$, with μ the proper mass per unit length of the string.

The introduction of gauge theories with spontaneous symmetry breaking to describe elementary-particle physics has given rise to an intensive study of cosmic strings. In these models the Universe may have undergone a number of phase transitions since the big bang.[1] One important cosmological consequence is that these phase transitions can give rise to vacuum domain structures such as domain walls, strings, and monopoles.[2] Among these topological structures, cosmic strings have been of great interest because, in addition to possibly acting as gravitational lenses,[3] they may provide density perturbations leading to galaxy formation.[4]

One particular aspect of cosmic strings that has been intensively investigated by several authors is the gravitational effects of an infinitely long straight string. Zel'dovich *et al.*[5] and Kibble[2] estimated the gravitational field of a string in the Newtonian approximation. Later, Vilenkin[6] calculated the gravitational properties of a static, cylindrically symmetric string in the linear approximation to general relativity. He showed that the geometry of the space-time exterior to a string of uniform linear energy density μ exhibits, to first order in μ, a conical behavior with deficit angle $\Delta\theta=8\pi\mu$. In his approach, the stress energy of the string is approximated as that of an infinitely thin line with positive (δ function) energy density and equal negative pressure (tension) along the axis.

Recently, there have been several studies where a cosmic string is considered as an extended source (finite radius). Gott[7] and Hiscock[8] approximated the stress energy of a string, in analogy with Vilenkin's[6] calculations, as that of a cylinder of a finite radius with uniform energy density and equal negative pressure T_{zz} along its axis. In their model, surface stresses and energy density were not considered; that is, the surface of the string was assumed to be nonsingular. They then found the exact space-time metric representing the exterior of a static cylindrically symmetric string and also obtained that the geometry outside the string is conical, with a deficit angle $\Delta\theta=8\pi\mu$; thus, these results agree with those of Vilenkin to all orders in μ. Similarly, Linet[9] has generalized the model of a string as an extended source to include nonuniform linear energy density μ along the radial direction. In addition, he has also studied[10] a string as a line source where the space-time in which the string is immersed has a nonvanishing cosmological constant Λ. Tian[11] generalized this model to thick strings and showed that, with nonzero Λ, the stress-energy components $T_{\theta\theta}$ and T_{rr} cannot both vanish, as they can in the case $\Lambda=0$; consequently, the exterior metric for a string may or may not be conical.

To our knowledge, the most general treatment of the gravitational effects of a string has been given by Garfinkle.[12] In his approach, he solves the gravitational field equations, as well as those of a string treated as a selfinteracting scalar field minimally coupled to a U(1) gauge field, with an effective potential of the form $V(|\phi|)=\lambda(|\phi|^2-\eta^2)^2$, with λ and η constants. He points out that one of the difficulties of other approaches has to do with the scalar and gauge fields; that is, since the string is a configuration of these scalar and gauge fields, to find the gravitational field it is not sufficient to assume a stress energy, e.g., energy density equal negative pressure, and solve for the metric. In order to be consistent, one has to find the metric by solving simultaneously the coupled Einstein-scalar-gauge field equations. Garfinkle then showed that there exists a class of static, cylindrically symmetric solutions to those field equations which asymptotically approach Minkowski space-time with a deficit angle $\Delta\theta\approx 8\pi\mu$, to second order in η.

The purpose of this paper is to give a model for a gravitating cosmic string that allows, to some extent, analytical solutions for the fields and leads to the stress energy of the string as a reasonable limit of Garfinkle's general approach. Our model is based on the assumption that the values of the scalar and U(1) gauge fields do not significantly vary at the core and outside of the string; that is, any change in the values of the field variables takes place at a cylindrical shell (which defines the "surface" of the string) whose thickness is small compared with the radius of the string. This is an analogous treatment to that of vacuum bubbles in the old inflationary universe. Under that scenario, a thin-wall bubble of true vacuum is materialized within the false vacuum; Coleman[13] showed that the thin-wall approximation holds if the potential difference between the real and false vacuum is much smaller than the potential barrier between them. Thus, for a string we also require a substantial barrier in the scalar potential $V(|\phi|)$ between the false vacuum $|\phi|=0$, and the global vacuum $|\phi|=\eta$ (Fig. 1); however, in this case the symmetry is obviously cylindrical and, contrary to the case of vacuum bubbles, the region enclosed by the thin wall is the false-vacuum region.

We start by reviewing the full coupled Einstein-scalar-gauge field equations which describe a static, cylindrically symmetric string in curved space-time. We then show that the assumptions on which our model is based are consistent with the field equations; in particular, that there exist solutions for which the variations of the field may occur in a thin, cylindrical shell region. This allows

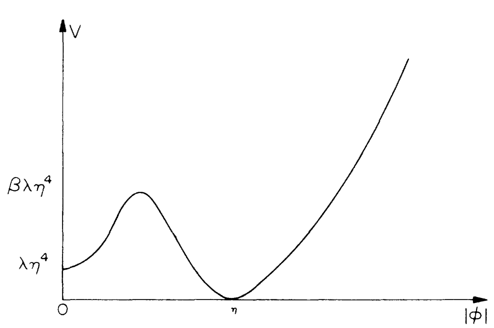

FIG. 1. The effective potential.

us to introduce an approximating δ-function singular stress energy for the string at its surface. Later, by means of Israel's jump conditions in surfaces of discontinuity,[14] we obtain the geometrical properties of the space-time interior and exterior to the string surface. Finally, we show that in the limit when the radius of the string vanishes, Vilenkin's results[6] for the stress energy and deficit angle are directly obtained.

Following Garfinkle's approach,[12] we begin by considering that the string fields are given by a vector field A_a and a complex scalar field ϕ. The total Lagrangian of these fields is

$$L=-(\nabla_a\phi+ieA_a\phi)(\nabla^a\bar{\phi}-ieA^a\bar{\phi})/2-V(\phi)$$
$$-F_{ab}F^{ab}/16\pi\ , \tag{1}$$

where $F_{ab}=\nabla_a A_b-\nabla_b A_a$ and $V(\phi)$ is the effective potential. (We use Wald's notation[15] and units where $\hbar=c=G=1$.) Varying the fields ϕ, $\bar{\phi}$, and A_a independently, one obtains

$$\nabla^a\nabla_a\phi-\partial V/\partial\bar{\phi}+ieA^a(2\nabla_a\phi+ieA_a\phi)+ie\phi\nabla^a A_a=0\ , \tag{2a}$$

$$\nabla^b F_{ba}-2\pi ie\phi(\nabla_a\bar{\phi}-ieA_a\bar{\phi})+2\pi ie\bar{\phi}(\nabla_a\phi+ieA_a\phi)=0\ . \tag{2b}$$

In addition, we have Einstein field equations, $G_{ab}=8\pi T_{ab}$, where the stress-energy tensor T_{ab} is given in terms of the fields by

$$T_{ab}=(\nabla_a\phi\nabla_b\bar{\phi}+\nabla_a\bar{\phi}\nabla_b\phi)/2$$
$$+ie(A_a\phi\nabla_b\bar{\phi}-A_b\bar{\phi}\nabla_a\phi+A_b\phi\nabla_a\bar{\phi}-A_a\bar{\phi}\nabla_b\phi)/2$$
$$+F_{ac}F_b{}^c/4\pi+e^2A_aA_b\phi\bar{\phi}+Lg_{ab}\ , \tag{3}$$

where e is a constant and L the Lagrangian defined by Eq. (1).

Since we are interested in a static infinite-length cosmic string, the configuration of the fields must possess static cylindrical symmetry; that is, the space-time has a set of three commuting Killing vector fields, one timelike and the other two spacelike, with one of them having closed orbits, such that any two are orthogonal to each other and each is hypersurface orthogonal. In addition, there exists an axis where the Killing vector with closed orbits vanishes. Normalization is chosen so that along a closed integral curve the parameter takes the values from 0 to 2π, and the spacelike and timelike Killing vector fields have norm 1 and -1, respectively, on the axis. For the coordinates t, z, r, and θ, where r is the geodesic distance from the axis in the direction orthogonal to all Killing vector fields, $(\partial/\partial t)^a$ is the timelike Killing field, $(\partial/\partial\theta)^a$ is the Killing field with closed orbits, and $(\partial/\partial z)^a$ is the Killing field along the axis. Under these assumptions the metric is given by

$$ds^2=-e^{2A}dt^2+e^{2B}dz^2+e^{2C}d\theta^2+dr^2\ , \tag{4}$$

where A, B, and C are functions of r only. From the normalization conditions, we have that $0\le\theta\le 2\pi$, with $\theta=0$ and $\theta=2\pi$ identified, and that the boundary conditions at the axis are

$$\lim_{r\to 0} A(r) = \lim_{r\to 0} B(r) = 0 \;, \tag{5a}$$

$$\lim_{r\to 0} e^{2C}/r^2 = 1 \;. \tag{5b}$$

Boundary condition (5a) arises as a consequence of the chosen normalization for the spacelike $(\partial/\partial z)^a$ and timelike $(\partial/\partial t)^a$ Killing vector fields. Condition (5b), on the other hand, states the requirement of smoothness of the metric at the axis.

We assume that the scalar field ϕ has the form $\phi = Re^{i\theta}$ where $R = R(r)$. This structure has the implicit assumption that the circulation or winding number of the string is equal to unity; it can be demonstrated that for winding numbers larger than one, for at least some potential functions $V(|\phi|)$, and choices of the gauge coupling e, the U(1) gauge string is unstable.[16] In addition, we will also assume that the gauge field A_a has the structure $eA_a = [P(r)-1]\nabla_a\theta$, and the effective potential $V(\phi)$ is axial symmetric, i.e., $V = V(R)$. With these choices, Eqs. (2) for the scalar and gauge fields become

$$R'' + (A+B+C)'R' = dV/dR + e^{-2C}P^2R \;, \tag{6a}$$

$$P'' + (A+B-C)'P'' = 4\pi e^2R^2P \;, \tag{6b}$$

where $(\;)' \equiv d/dr$. Because of the choice that we have made for the scalar and gauge fields, the stress-energy tensor (3) is diagonal. Thus, it is useful to define the following set of orthonormal vector fields:

$$\begin{aligned} &\hat{\mathbf{t}}^a \equiv e^{-A}(\partial/\partial t)^a, \;\; \hat{\mathbf{z}}^a \equiv e^{-B}(\partial/\partial z)^a \;, \\ &\hat{\boldsymbol{\theta}}^a \equiv e^{-C}(\partial/\partial\theta)^a, \;\; \hat{\mathbf{r}}^a \equiv (\partial/\partial r)^a \;. \end{aligned} \tag{7}$$

In terms of these unit vector fields, the stress-energy tensor T_{ab} for the scalar and gauge fields has the form

$$T_{ab} = \epsilon\hat{\mathbf{t}}_a\hat{\mathbf{t}}_b + P_z\hat{\mathbf{z}}_a\hat{\mathbf{z}}_b + P_\theta\hat{\boldsymbol{\theta}}_a\hat{\boldsymbol{\theta}}_b + P_r\hat{\mathbf{r}}_a\hat{\mathbf{r}}_b \;, \tag{8}$$

where

$$\begin{aligned} \epsilon = -P_z = [&(R')^2 + e^{-2C}R^2P^2 + 2V \\ &+ e^{-2C}(P')^2/4\pi e^2]/2 \;, \end{aligned} \tag{9a}$$

$$P_\theta = [-(R')^2 + e^{-2C}R^2P^2 - 2V + e^{-2C}(P')^2/4\pi e^2]/2 \;, \tag{9b}$$

$$P_r = [(R')^2 - e^{-2C}R^2P^2 - 2V + e^{-2C}(P')^2/4\pi e^2]/2 \;. \tag{9c}$$

For the given metric (4), the only nonvanishing components of Einstein field equations, $G_{ab} = 8\pi T_{ab}$, are[17]

$$G_{ab}\hat{\mathbf{r}}^a\hat{\mathbf{r}}^b = A'B' + A'C' = 8\pi P_r \;, \tag{10a}$$

$$G_{ab}\hat{\boldsymbol{\theta}}^a\hat{\boldsymbol{\theta}}^b = A'' + B'' + (A')^2 + (B')^2 + A'B' = 8\pi P_\theta \;, \tag{10b}$$

$$G_{ab}\hat{\mathbf{z}}^a\hat{\mathbf{z}}^b = A'' + C'' + (A')^2 + (C')^2 + A'C' = 8\pi P_z \;, \tag{10c}$$

$$G_{ab}\hat{\mathbf{t}}^a\hat{\mathbf{t}}^b = B'' + C'' + (B')^2 + (C')^2 + B'C' = -8\pi\epsilon \;. \tag{10d}$$

From the boundary conditions (5) at the axis, it can be shown that $A = B$ everywhere; this make Eq. (10c) and Eq. (10d) identical. If we set $K = e^{3A/2}$ and $H = e^{C+A/2}$, the metric (4) then takes the form

$$ds^2 = K^{4/3}(-dt^2 + dz^2) + H^2K^{-2/3}d\theta^2 + dr^2 \;, \tag{11}$$

where now the metric functions K and H satisfy Einstein field equations (10) given as

$$H'' + (8\pi\epsilon + 2\pi P_\theta)H = 0 \;, \tag{12a}$$

$$K'' - 6\pi P_\theta K = 0 \;, \tag{12b}$$

$$K'H' - 6\pi P_r KH = 0 \;. \tag{12c}$$

Furthermore, using Eqs. (6) for the scalar and gauge fields (which imply conservation of the stress-energy tensor) and the Bianchi identities, it is also possible to show[12] that Eq. (10a), or equivalently Eq. (12c), is superfluous. Consequently, the set of equations to solve for the metric functions (H,K) and the string fields (R,P) are Eqs. (12a) and (12b) and Eqs. (6a) and (6b), where the latter can be rewritten as

$$(KHR')' = KH(dV/dR + P^2R/H^2K^{-2/3}) \;, \tag{13a}$$

$$(K^{5/3}P'/H)' = (4\pi e^2R^2P)K^{5/3}/H \;. \tag{13b}$$

Equations (12a) and (12b) and (13a) and (13b) are the set of equations for which Garfinkle[12] showed there exists a class of static, cylindrically symmetric solutions, with effective potential $V(|\phi|) = \lambda(|\phi|^2 - \eta^2)^2$, that asymptotically (in the cylindrical spacelike direction) approach Minkowski space-time with a deficit angle, to second order in η, given as $\Delta\theta \approx 8\pi\mu$.

We will now proceed to introduce our model which will allow us to have a solution for the metric near (and also inside) the string surface. First, we will assume that, although not explicitly specified, the scalar field ϕ interacts with itself through the standard "old inflation" potential (Fig. 1); that is, $V(R)$ possesses two relative minima, at $R = \eta$ and $R = 0$, only one of which, $R = \eta$, is an absolute minimum. These should be true local minima in both cases. The barrier between the two minima is essential for the thin-wall approximation to hold. Second, we divide the space where the string is immersed into three disjoint regions. Region M_- (the core of the string) consists of the points whose geodesic distance r_- from the axis satisfies $0 \le r_- \le r_s^- - \delta/2$, where δ is an arbitrary nonnegative number and r_s is what later will become the string coordinate radius. Region M_+ (the exterior of the string) is given by $r_s^+ + \delta/2 \le r_+$, and region M_s (the "surface" of the string) is given by $r_s^+ + \delta/2 \ge r \ge r_s^- - \delta/2$. Mutually independent coordinate charts are introduced in each region; furthermore, we assume the existence of mappings at the boundary of each region that allows the relation of the components of tensorial quantities, i.e., $r_s^+ \to r_s^-$.

Finally, we will have the following set of assumptions for the metric, scalar, and gauge fields.

(1) Throughout region M_-, i.e., in the core of the string, the field variables are well approximated by $P(r_-) = 1$, $R(r_-) = 0$, and the effective potential exhibits its local minimum, $V(0) = \lambda\eta^4$, where λ is a constant.

(2) In region M_+, exterior to the string, the field variables are well approximated by $P(r_+) = 0$, $R(r_+) = \eta$, and the effective potential for this case is given by its absolute minimum $V(\eta) = 0$.

(3) At the "surface" M_s of the string, the coefficients H and K of the metric (11) do not vary significantly, and the

scalar potential exhibits a barrier of height $\beta\lambda\eta^4$. In order that the thin-wall approximation holds, $\beta \gg 1$.

Assumptions (1) and (2) give the solutions for the scalar and gauge field variables P and R in M_+ and M_-; consequently, in these regions one only has to solve from Einstein field equations for the metric variables H and K, and check for consistency. However, in region M_s, where the magnitudes of the scalar and gauge fields could vary abruptly, the average value of the magnitudes for the fields will be obtained by integrating Einstein equations in that region in the limit when $\delta \to 0$, i.e., region M_s will be squeezed to a singular cylindrical hypersurface, and only δ-function singularities will then survive. Great care should be taken whenever one speaks about the limit $\delta \to 0$. We shall see that for the purpose of analyzing the jump behavior of the field variables one takes the limit to thin-wall behavior formally to a δ function. However, the thickness of the wall is in fact finite and determined by the parameter β giving the height of the potential barrier. In particular, one cannot in every detail approximate the effective potential barrier as a δ function.

Applying assumptions (1) and (2) to the expressions (9), one obtains that the stress-energy tensor (8) is given by

$$T_{ab}=0 \quad \text{in } M_+ \ ,$$

$$T_{ab}=-\lambda\eta^4 g_{ab}^- \quad \text{in } M_- \ .$$

Thus, Einstein field equations (12) read now

$$H''+a^2H=0 \ , \tag{14a}$$

$$K''+a^2K=0 \ , \tag{14b}$$

$$H'K'+a^2HK=0 \ , \tag{14c}$$

where $a_-{}^2=6\pi\lambda\eta^4$ in M_- and $a_+{}^2=0$ in M_+. A straightforward integration of these equations yields

$$K=b\cos a(r+r_0), \quad H=(d/a)\sin a(r+r_0) \ ,$$

where b, d, and r_0 are constants of integration, and it is understood that these solutions have their respective values at each region M_+ $(a_+{}^2=0)$ and M_- $(a_-{}^2\neq 0)$. Boundary conditions at the axis and Eqs. (5a) and (5b) imply that as $r \to 0$ the space-time becomes locally Minkowskian, i.e., $K(0)=1$ and $H(0)\to r$. Thus, it is required that the constants of integration d^- and b^- to be equal to unity, and $r_0^-=r_0^+=0$. Consequently, the solutions for the field variables H and K are given as

$$\left.\begin{aligned} K&=\cos(ar^-), \\ H&=(1/a)\sin(ar^-), \end{aligned}\right\} \text{ in } M^- \tag{15a}$$

and

$$\left.\begin{aligned} K&=b^+, \\ H&=d^+r^+, \end{aligned}\right\} \text{ in } M^+ \ , \tag{15b}$$

where the constants of integration d^+ and b^+ in Eq. (15b) will be determined later by the junction conditions in the surface of the string.

It is important to notice that H and K in (15b) are not the most general solution of Eqs. (14a)–(14c) with $a=0$. The general solution to these equations is either $K=b$, $H=c+dr$ or $K=e+fr$, $H=g$, where b, c, d, e, f, and g are constants. The first solution is Minkowski space-time minus a wedge, and it can be brought to the form given in (15b) by changing the coordinate r to $r+c/d$. The second solution ($K=e+fr$, $H=g$) is a nonflat metric analogue of a Kasner metric. As is pointed out in Ref. 12, this solution can be ruled out because in this metric the length of a particular closed integral curve of $(\partial/\partial\theta)^a$ vanishes and, consequently, does not represent an isolated string.

If $a\neq 0$, which is the case in M_-, the metric (11) with the solutions (15a) has a real singularity; that is, there exists a region where the curvature scale $R_{abcd}R^{abcd}$ becomes infinite, and this is at a finite spacelike geodesic distance from the axis. The point at which the singularity occurs can be obtained from the nonvanishing components of the Weyl tensor, which are given by[10]

$$C^{rx}{}_{rx}=4a^2[v_x{}^2/4-\tfrac{2}{9}-(v_x/6)\cos(2ar)]/[\sin(2ar)]^2 \quad \text{for } x=z,\theta,t \tag{16a}$$

and

$$C^{xy}{}_{xy}=4a^2\{v_xv_y/4+\tfrac{1}{9}+[(v_x+v_y)/6]\cos(2ar)\}/[\sin(2ar)]^2 \quad \text{for } x\neq y \ , \tag{16b}$$

where $v_z=v_t=-\frac{2}{3}$ and $v_\theta=\frac{4}{3}$. Thus the invariant $C^{ab}{}_{dc}C^{cd}{}_{ab}$ of the Weyl tensor, and consequently $R^{ab}{}_{cd}R^{cd}{}_{ab}$, will have a singularity at $2ar=n\pi$, with n a nonvanishing integer. For the case $n=0$, one can show from expressions (16a) and (16b) that the Weyl tensor is regular at the axis. The existence of this singularity means that a string of the type we postulate cannot have an arbitrarily large coordinate radius r. Although the coordinate radius r is an arbitrary parameter of this model, it cannot exceed $\pi/(2a)=\pi/2(6\pi\lambda\eta^4)^{1/2}$. However, if we recall that the physical radius of the string is given by $\rho\equiv HK^{-1/3}=(1/a)\sin(ar)[\cos(ar)]^{-1/3}$, the singular case $ar=\pi/2$ corresponds to a string of infinite physical radius.

Finally, it is easy to check that the choice, from assumptions (1) and (2), for the scalar and gauge fields in M_+ and M_- is consistent with Eqs. (13a) and (13b). The only term in these equations that requires a little more careful explanation is the one involving dV/dR in Eq. (13a). One can see, however, that this term will vanish both in M_+ and in M_- if one recalls that $V(R)$ is assumed to have two relative minima, an absolute in M_+ and a local in M_-.

Up to this point, we have obtained the solution for the metric, and the scalar and gauge field variables R and P in regions M_+ and M_-. What follows is the integration of Einstein field equations in region M_s along the radial direction, in the limit when the thickness of M_s vanishes. If the limit is finite and nonzero, it will define the jump conditions for the extrinsic curvature and the stress-energy tensor S_{ab} on M_s. Let us define the surface stress-energy tensor on M_s to be the integral of T_{ab} with respect to the proper distance r, measured perpendicularly through M_s:

$$S_{ab}=\lim_{\delta\to 0}\int_{r_s-\delta/2}^{r_s+\delta/2} T_{ab}dr \ .$$

The surface stress-energy tensor S_{ab} will then only consist of "δ-function" contributions from T_{ab}, which arise from the field variables and the potential [see Eq. (9)]. The effect of the surface layer M_s on the space-time geometry is obtained by performing a "pill-box integration" of Einstein field equations

$$\lim_{\delta\to 0}\int_{r_s-\delta/2}^{r_s+\delta/2} G_{ab}dr=8\pi S_{ab} \ . \tag{17}$$

It can be shown[18] that in the absence of jump and δ-function discontinuities in the metric g_{ab} and of δ-function discontinuities in the extrinsic curvature, the Einstein field equations when integrated yield

$$[[G_{ab}\hat{\mathbf{r}}^a\hat{\mathbf{r}}^b]]=0=8\pi S_{ab}\hat{\mathbf{r}}^a\hat{\mathbf{r}}^b \ , \tag{18a}$$

$$[[G_{ab}\hat{\mathbf{x}}^a\hat{\mathbf{r}}^b]]=0=8\pi S_{ab}\hat{\mathbf{x}}^a\hat{\mathbf{r}}^b \ , \tag{18b}$$

$$[[G_{ab}\hat{\mathbf{x}}^a\hat{\mathbf{y}}^b]]=8\pi S_{ab}\hat{\mathbf{x}}^a\hat{\mathbf{y}}^b \ , \tag{18c}$$

where $\hat{\mathbf{x}}^a,\hat{\mathbf{y}}^a=\hat{\mathbf{t}}^a$, $\hat{\mathbf{z}}^a$, $\hat{\boldsymbol{\theta}}^a$, and

$$[[\ ]]\equiv\lim_{\delta\to 0}\int_{r_s-\delta/2}^{r_s+\delta/2} dr \ . \tag{19}$$

The junction conditions (18a) and (18b) have the physical meaning that the momentum flow is entirely in M_s, i.e., no momentum associated with the surface layer flows out of M_s. On the other hand, the junction condition (18c) states that the stress-energy tensor S_{ab} in M_s generates a jump discontinuity in the extrinsic curvature. The extrinsic curvature is one derivative of the metric, and the curvature is two derivatives. Hence, jumps in the metric (and δ singularities in the extrinsic curvature) would give more than δ singularities in the curvature (hence in the matter). They are thus excluded in the physically motivated jump conditions used here. Since in our model the Einstein tensor (10) and the stress energy (8) are diagonal, junction conditions (18b) hold identically. Also from expression (10a) for the $G_{ab}\hat{\mathbf{r}}^a\hat{\mathbf{r}}^b$ component of the Einstein tensor, one sees that the left-hand side of the junction condition (18a) holds identically because $G_{ab}\hat{\mathbf{r}}^a\hat{\mathbf{r}}^b$ only depends on first derivatives of the metric which, at the most, have jump discontinuities and therefore do not contribute when they are pill-box (19) integrated.

What is left to show is that the right-hand side of Eq. (18a) also vanishes; that is, from (9c) we need to show that

$$S_{ab}\hat{\mathbf{r}}^a\hat{\mathbf{r}}^b=[[T_{ab}\hat{\mathbf{r}}^a\hat{\mathbf{r}}^b]]=[[P_r]]=0 \ . \tag{20}$$

We begin by multiplying Eq. (13a) by KHR' and Eq. (13b) by $K^{5/3}P'/H$. Integration of these equations yields

$$(R')^2=2V+H^{-2}K^{2/3}\int P^2(R^2)'dr \ , \tag{21a}$$

$$(P')^2=4\pi e^2\int R^2(P^2)'dr \ , \tag{21b}$$

where we have used assumption (3) which states that the metric variables H and K remain almost constant through M_s. Dividing Eq. (21b) by $H^2K^{-2/3}$ and adding Eq. (21a), one then obtains the following approximation which holds throughout M_s:

$$(R')^2-2V+(P')^2H^{-2}K^{2/3}/4\pi e^2\approx H^{-2}K^{2/3}R^2P^2 \ . \tag{22}$$

If we recall the expression (9c) for P_r in terms of the field variables, Eq. (22) simply expresses the statement that P_r defined in M_s approximately vanishes. This approximation, when pill-box (19) integrated, becomes an equality since for that case the metric functions H and K remain strictly constant throughout M_s, and consequently one arrives at the desired result, Eq. (20). This completes the argument that, within the given assumptions (1), (2), and (3), a model for a string, as a cylindrical shell-type surface of discontinuity enclosing a region of false vacuum, is consistent with the general equations for a U(1) cosmic string.

We need now to exploit junction condition (18c), which expresses how the geometry of the space-time reacts to the presence of M_s. We begin by pill-box integrating Einstein field equations (12), where now it is safe to write the stress-energy tensor T_{ab} as

$$T_{ab}=T_{ab}^-\theta(r_s^- - r_-)+S_{ab}\delta(r_s-r)+T_{ab}^+\theta(r_+ - r_s^+) \tag{23}$$

with $S^{ab}=\sigma(\hat{\mathbf{t}}^a\hat{\mathbf{t}}^b-\hat{\mathbf{z}}^a\hat{\mathbf{z}}^b)-\tau\hat{\boldsymbol{\theta}}^a\hat{\boldsymbol{\theta}}^b$, $\sigma\equiv[[\epsilon]]$, and $\tau\equiv-[[P_\theta]]$. It is clear from Eq. (23) that only the δ-function term, and not the jump terms, will contribute to integration (19). Therefore, if we multiply Eq. (12a) by K and add it to Eq. (12c), similarly for Eq. (12b), we obtain after pill-box integration that

$$(K'/K)^+-(K'/K)^-=-6\pi\tau \ , \tag{24a}$$

$$(H'/H)^+-(H'/H)^-=8\pi\sigma+2\pi\tau \ , \tag{24b}$$

where $(\ )^+-(\ )^-$ is understood as a limit process. Equations (24a) and (24b) are the so-called jump conditions[14] that relate the jumps in the derivatives of the metric, i.e., jumps in the extrinsic curvature, to the stress-energy at the singular hypersurface. Substitution of the obtained solutions (15a) and (15b) for the metric functions K and H in M^+ and M^- allows us to rewrite Eqs. (24a) and (24b) as

$$a\tan(ar_s^-)=-6\pi\tau \ , \tag{25a}$$

$$1/r_s^+ - a\cot(ar_s^-)=-8\pi\sigma+2\pi\tau \ , \tag{25b}$$

where we have used the assumptions that $a_-^2\equiv a^2=6\pi\lambda\eta^4$, and that the absolute minimum of the effective potential $V(|\phi|)$ vanishes at M_+. The quantities σ and τ are from Eq. (23) the surface energy density and the stress in the $\hat{\theta}^a\hat{\theta}^b$ direction, respectively. On the other hand, from assumption (3) we know that the metric variables K and H are continuous throughout M_s; that is,

$$b^+ = \cos(ar_s^-) , \tag{26a}$$

$$d^+ r_s^+ = (1/a)\sin(ar_s^-) . \tag{26b}$$

The jump conditions (25a) and (25b), together with the conditions (26a) and (26b) for the continuity of the metric, form the set of equations that determine the space-time geometry outside the string.

By looking back to the exterior metric (11) in M^+, we have that the deficit angle is given in this case by $\Delta\theta/2\pi = (1 - b_+{}^{-1/3}d_+)$. Using the junction conditions (25) and (26), one obtains that

$$\Delta\theta/2\pi = 4\mu + \{[1+2\cos^2(ar_s^-)]/3[\cos(ar_s^-)]^{4/3} - 1\} , \tag{27}$$

where we have defined the mass per unit length, or linear mass density, of the string as

$$\mu = 2\pi \int_0^\infty T_{ab}\hat{t}^a\hat{t}^b H K^{-1/3} dr .$$

The expression (27) for the deficit angle can be rewritten in terms of the coordinates for the radius of the string in M_+ by means of the mapping $r_s^- \rightarrow r_s^+$ given by Eq. (26b). Equation (27) also shows that in the limit when r_s^- vanishes, i.e., when the string becomes a line source, we get the well-known result by Vilenkin for the deficit angle as given by $\Delta\theta/2\pi = 4\mu$. Furthermore, from Eq. (25a) in that limit one obtains that $\tau \equiv [[P_\theta]] = 0$. Therefore, the only nonvanishing components of the stress-energy tensor are $[[\epsilon]] = -[[P_z]]$, as it is expected for a line source string.

Finally, given the behavior of the effective potential at the string wall (M_s), one should be able to obtain explicitly the surface equation of state $\sigma = \sigma(\tau)$, and thus from the set of Eqs. (25) and (26), to determine the space-time geometry outside the string. As an example, let us consider the case when the effective potential $V(R)$ has, throughout M_s, the constant value $\beta\lambda\eta^4$, where $\beta \gg 1$ is required for the thin-wall approximation to hold. This assumption for the scalar potential allows us to rewrite the surface energy density $\sigma = [[\epsilon]]$, from Eq. (9a), as the pill-box integration (19)

$$2\sigma = [[(R')^2 + 2\beta\lambda\eta^4 + [(P')^2/\alpha\lambda + P^2R^2]/\rho^2]] , \tag{28}$$

where $\alpha \equiv 4\pi e^2/\lambda$ and ρ is the physical radius of the string defined as $\rho^2 \equiv H^2K^{-2/3}$. The thickness δ of the wall is determined by minimizing the surface energy density σ. The gradients of the field variables R and P contribute to this surface energy density as η/δ and $1/\delta$, respectively, and the "electromagnetic" term P^2R^2 contributes with $(\eta/4)^2$. Thus the pill-box integration in Eq. (28) yields

$$2\sigma = (\eta^2 + 1/\alpha\lambda\rho^2)/\delta + (2\beta\lambda\eta^4 + \eta^2/16\rho^2)\delta . \tag{29}$$

Varying with respect to δ, we obtain

$$\begin{aligned} d\sigma/d\delta &= 0 \\ &= -(\eta^2 + 1/\alpha\lambda\rho^2)/\delta^2 + (2\beta\lambda\eta^4 + \eta^2/16\rho^2) \end{aligned} \tag{30a}$$

or equivalently

$$\delta^{-2} = 2\beta\lambda\eta^2[(1 + 1/32\beta\lambda\rho^2\eta^2)/(1 + 1/\alpha\lambda\rho^2\eta^2)] . \tag{30b}$$

Aside from the factors between the brackets, which are due to the gauge field, the thickness of the string wall, as given by Eq. (30b), has essentially similar behavior to the thickness of a bubble wall in vacuum decay.[13] They will agree in the case of a global string, in which the gauge field is not present, and for the situation when β, the scale of the potential barrier, approximates $\alpha/32$. In these two cases the factor between the brackets in Eq. (30b) becomes of the order of unity, and then $\delta^{-2} \approx 2\beta\lambda\eta^2$. Notice that Eq. (30a), and consequently (30b), can also be obtained directly, although without the same physical motivation, from the approximation (22).

Following an analogous procedure to that used to obtain Eq. (29), we can also compute

$$\begin{aligned} 2\tau &\equiv -2[[P_\theta]] \\ &= (\eta^2 - 1/\alpha\lambda\rho^2)/\delta + (2\beta\lambda\eta^4 - \eta^2/16\rho^2)\delta . \end{aligned} \tag{31}$$

This equation, together with Eq. (29) for σ, allows us to write the equation of state (parametrized by the physical radius ρ) as

$$\sigma = \eta^2/\delta + 1/(\alpha\lambda\rho^2\delta) , \tag{32a}$$

$$\tau = \eta^2/\delta - \eta^2\delta/16\rho^2 . \tag{32b}$$

Equation (32b) shows again that as the physical radius ρ of the string gets comparable with its wall thickness ($4\rho \approx \delta$), $\tau \equiv -[[T_{\theta\theta}]]$ vanishes; furthermore, from Eqs. (30b) and (32a) the thickness and linear mass density of the string are now given by $\delta^{-2} \approx 2\beta\lambda\eta^2$ and $\mu \equiv \delta\sigma \approx 2\eta^2$, respectively, when $\alpha = 32\beta$. These results agree with those for a line source string.

Based upon the analogy with bubbles in the old inflationary universe scenario,[19] we have constructed a model for a U(1) cosmic string as a cylindrical shell-type singular surface which traps a region of false vacuum. The strongest assumption, upon which this model is based, is that the thickness of this singular shell, where the fields vary, is small compared with its radius in order for the thin-wall approximation to hold. We have also shown that this model, within approximations (1), (2), and (3), is consistent with the general equation[12] for a static, cylindrically symmetric U(1) cosmic string. The main motivation for introducing this model for an infinite-length straight cosmic string is that, as in the case of vacuum bubbles in inflation,[20] it is analytically more manageable. Perhaps with this model, a study of string annihilation may be done similarly to that made for bubble collisions.[21] Although our work is mathematically equivalent to the standard treatment of singular hypersurfaces, we have explicitly pill-box integrated the Einstein field equations instead of directly using the Lanczos equations and the jump conditions[14,20] (that relate the average and jump of the extrinsic curvature to the matter sources), which in some cases could obscure the treatment.

The only drawback to this model is that a particular "old inflation" form of the scalar potential $V(|\phi|)$ is required. However, one sees that the model successfully reproduces the deficit angle and the equation of state for a line source string when taking the limit of vanishing radius.

On the other hand, by looking at the numerical solutions[12] for the flat-space field variables for a U(1) string with effective potential $V(|\phi|)=\lambda(R^2-\eta^2)^2$, one sees that the asymptotic and near the origin behavior of the field variables is consistent with the assumptions here made for these fields in our model. It would then be interesting to find, from those flat-space numerical solutions, the range of values for the parameters of the theory and the barrier that should be introduced to the effective potential that allow one to confine the changes of the field variables to a thin cylindrical-shell region. Finally, a detailed study would also be important to determine whether the gravitational and/or the gauge terms in the total Lagrangian prevents a U(1) string in this model from collapsing into a "one-dimensional" string, in particular, to analyze perturbations of this static model.

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