Hydrogenlike atom in bosonized QED

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A hydrogenlike atom is constructed in bosonized lowest-partial-wave QED. Despite the classical treatment of the theory the charge radius of the hydrogenlike atom is shown to be the order of the Bohr radius. This indicates that the classical treatment of the bosonized theory does contain fermionic quantum effects.

In a previous paper[1] we analyzed the ground state of quantum electrodynamics (QED) around a highly charged ($Z \gtrsim 137$) nucleus. Guided by several physical arguments we were led to construct the bosonized theory of lowest-partial-wave QED. We found that the ground state undergoes a phase transition from the normal QED vacuum to a supercritical one associated with the real pair creation of electrons and positrons. The analysis of Ref. 1 was performed in the leading order of semiclassical expansion

In this paper we wish to address the question of a hydrogenlike atom in our bosonized lowest-partial-wave QED. Our particular interest in such a well-understood system may need explanation. First of all, it is of interest to see, without recourse to the one-particle theory, how the hydrogenlike atom appears in the spectrum of quantum field theory. Second, and more importantly, the study of the system provides us with a measure of how well the classical approximation of the boson theory takes care of the quantum effects of the original fermion theory. This is because the characteristic size of the hydrogenlike atom, the Bohr radius, is determined by the balance between the Coulomb attraction and the effective repulsion due to the uncertainty principle. If the classical approximation of the boson theory does not contain quantum effects the size of the hydrogenlike atom constructed by our boson theory would be unacceptably small, which may be of the order of nuclear size.

We start with briefly reviewing the bosonized lowest-partial-wave QED, the quantum field theory of $j=\frac{1}{2}$ electron and $j=0$ electromagnetic fields, developed in Ref. 1. In the course of the summary we recollect the necessary formulas for our present analysis.

It was observed in Ref. 1 that the only relevant partial wave of the electron field for the question of supercritical QED is $j=\frac{1}{2}$ for $Z \lesssim 300$. In addition, if the external source is spherically symmetric, only the s-wave electromagnetic fields communicate with the external source. Discarding all the higher partial waves we have constructed an effective two-dimensional fermion theory living in one-half-space ($0 \le r \le \infty$) and one-time dimensions. We have further converted this theory into a boson theory using the bosonization technique.

The obtained bosonized Hamiltonian has the form

$$H=\int dr\left[\sum_m \tfrac{1}{2}({\Pi_m}^2+{P_m}^2+\Phi_m'^2+Q_m'^2)+\sum_{m,\delta}\frac{1}{2\pi r^2}\left\{1-\cos\left[\sqrt{\pi}\left[\Phi_m+Q_m-\delta\int_r^\infty ds[\Pi_m(s)-P_m(s)]\right]\right]\right\}\right.$$
$$\left.+\sum_m\frac{M^2}{\pi}[2-\cos(2\sqrt{\pi}\Phi_m)-\cos(2\sqrt{\pi}Q_m)]+\frac{e^2}{8\pi r^2}\left[\left[\Phi(r,t)-\frac{1}{\sqrt{\pi}}\sum_m(\Phi_m+Q_m)\right]^2-\Phi(r,t)^2\right]\right], \quad (1)$$

where Π_m and P_m denote the canonical conjugate of the Bose fields Φ_m and Q_m, respectively. The index $m\,(=\pm\frac{1}{2})$ represents the spin (third component of the angular momentum) degeneracy and δ implies the chirality signature which takes $+1$ (-1) for right- (left-) handed fermions. The information of the external source is carried by $\Phi(r,t)$ in (1) which is related to the external charge density $\rho(r,t)$ as

$$\Phi(r,t)=4\pi Z\int_0^r dr\, r^2\rho(r,t)\ . \quad (2)$$

As usual in partial-wave field theories the boson fields in (1) are subject to the following boundary condition at the origin:

$$\Phi_m(0,t)+Q_m(0,t)=0\ . \quad (3)$$

Finally the electric charge and the angular momentum allow the following expressions in terms of Bose variables:

$$Q_{\mathrm{EM}} = -\frac{1}{\sqrt{\pi}} \sum_m [\Phi_m(r) + Q_m(r)] \,|_0^\infty \,,$$
$$J_3 = \frac{1}{\sqrt{\pi}} \sum_m m[\Phi_m(r) + Q_m(r)] \,|_0^\infty \,. \tag{4}$$

We note that the framework just summarized above is quite suited for our present purpose. Most of the nuclei may be approximated by static and uniformly charged spheres. Therefore our system (1), if properly quantized, should give rise, not only to the ground state, but also to the series of radial excitations of hydrogenlike atoms with $j=\frac{1}{2}$.

We, however, restrict ourselves to the classical analysis of the boson theory in this paper, and thereby to the ground state of the hydrogenlike atom with fairly high Z ($Z \gtrsim 50$). The last restriction is due to the following reasons. First, we do not have any good reasons for believing that the semiclassical approximation is accurate for small-Z atoms. Second, there are technical difficulties in obtaining reliable answers in our variational calculation in such systems. It should be stressed that the restriction does not lower the value of our analysis because there are big differences (factor ~ 80 for uranium) between the Bohr radius and the nuclear size.

The hydrogenlike atom in our Bose theory is nothing but the solitonlike configuration around an external source as discussed in Ref. 1. For our purpose we examine the soliton configuration with electric charge -2 and zero angular momentum, the spin-singlet two-electron atom. The obtained energy of such state via the classical analysis may be identified with that of the $1S_{1/2}$ state.

It can readily be seen by (4) that to construct spin-singlet two-electron state we have to have solitonlike configurations both for $\Phi_{+1/2}$ and $\Phi_{-1/2}$ (Ref. 2). The rest of the field variables and the canonical moments are freely varied within some suitable ansatz which are consistent with the boundary condition (3).

For the purpose of comparison with the results of Ref. 3, we take the external charge density as

$$\rho(r,t) = \frac{3}{4\pi R^3}\theta(R-r) \tag{5}$$

with

$$R = 1.2\times(0.007\,33Z^2 + 1.30Z + 63.6)^{1/3}\ \mathrm{fm}\ . \tag{6}$$

The only parameter which remains to be determined is the value of M which appeared in the mass term in (1). In principle, it can be determined by the electron mass at spatial infinity (free electrons). If we use the classical soliton mass formula, then $m_e = 4M/\pi$ with m_e being the electron mass. In the case of sine-Gordon theory in 1 + 1 dimensions, however, it does not give a credible estimation of the value of M. The one-loop radiative correction yields 50% correction to the above purely classical mass formula.[4] Lacking the estimation of a one-loop correction in our theory with a nontrivial boundary condition we shall regard M as a free parameter, and adjust it at a particular value of Z so as to reproduce the binding energy predicted by the Dirac theory.

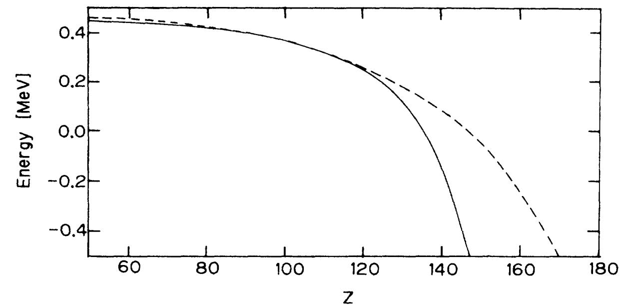

FIG. 1. The energy of hydrogenlike atom is plotted as a function of Z. The solid line shows the energy per electron calculated by our bosonized lowest-partial-wave QED, whereas the dashed line indicates the result of the Dirac theory (Ref. 3).

As mentioned earlier we employ the variational method to obtain the energies of the atomic (solitonlike) and vacuum configurations. The latter includes the effect of the polarizability of the vacuum. In Fig. 1 we plot the energy difference divided by 2 (the energy per one electron) between the atomic and the vacuum configurations as a function of Z, which should be identified as the energy of the $1S_{1/2}$ level. For comparison, the same quantity obtained by solving the Dirac equation with one-loop radiative correction[3] is also plotted. The parameter M is determined to be 0.322 MeV so that the energy of our solitonic atom agrees with that of the Dirac theory at $Z=100$. This value of M is 20% smaller than the one determined by the classical soliton mass formula which was used in Ref. 1.

As one can see in Fig. 1 the agreement between two theories is quite good in the region $50 \lesssim Z \lesssim 130$. Beyond $Z=130$ the energy of our solitonic atom becomes considerably lower than that of the Dirac atom. This result shows a remarkable consistency with our previous calculation.[1] There we have observed that, for corresponding nuclear size, the normal QED vacuum undergoes the phase transition to the supercritical one at $140 \lesssim Z \lesssim 150$ with

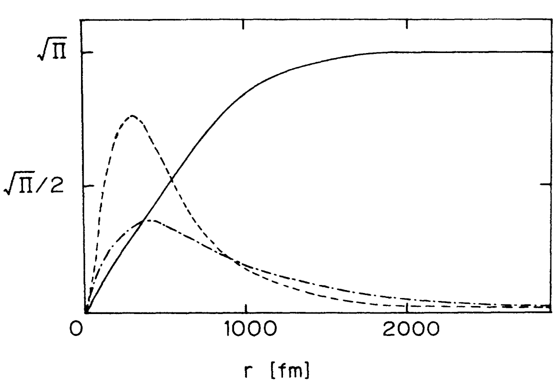

FIG. 2. The profile of the Bose fields and their canonical momenta consisting of the solitonic atom at $Z=100$. The solid line indicates $\Phi_{\pm 1/2}$, while the dashed and the dashed-dotted lines show $Q_{\pm 1/2}$ and $\Pi_{\pm 1/2} = P_{\pm 1/2}$, respectively, each multiplied by 5.

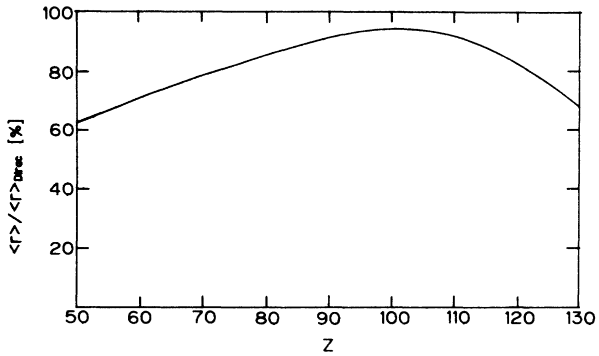

FIG. 3. The charge radius of the hydrogenlike atom normalized by the prediction (8) of the Dirac theory is depicted as a function of Z.

induced electric charges at around the nucleus and at spatial infinity. In the present calculation the critical value of Z may be given by the point where the atomic energy dives into the negative-energy continuum, which is about 145. We should, therefore, remark that beyond the phase transition point what we have computed is not the atom's energy but the ground-state energy (minus $2m_e$, actually) of the supercritical vacuum.

In Fig. 2 we show the profile of the boson fields as well as canonical momenta consisting of our solitonic atom at $Z=100$. We note that the inclusion of the nonvanishing canonical momenta is indispensable for obtaining reasonable values of the atomic energies. This is in sharp contrast to the case of the spin-parallel configuration examined in Ref. 1.

In Fig. 3 we show the charge radius of the solitonic atom normalized by the same quantity obtained by the Dirac theory. The charge radius is defined as (the prime being r derivative)

$$\langle r \rangle = Q_{\rm EM}{}^{-1} \int_0^\infty dr\, r \sum_m [\Phi'_m(r) + Q'_m(r)] \ . \tag{7}$$

This should be compared with the prediction of the Dirac equation

$$\langle r \rangle = [\tfrac{1}{2} + (1 - Z^2\alpha^2)^{1/2}](m_e Z\alpha)^{-1} \tag{8}$$

for the $1S_{1/2}$ state. In deriving (8) we have used the point-source approximation which seems to be quite good for $Z \lesssim 120$.

As can be seen in Fig. 3 the charge radius of our solitonic atom constructed by the classical Bose theory has the order of the Bohr radius over the wide range of Z. Furthermore, the agreement at the range $80 \lesssim Z \lesssim 120$ is impressive considering the crudeness of our approximation.

Skeptical readers may suspect whether the agreement occurs only at the particular values of the parameters, namely, $\alpha = (137)^{-1}$ and $m_e = 0.51$ MeV. To check this point we have performed calculations with different values of these parameters. In the region of mass parameter $\frac{1}{2} \times (0.322) \leq M \leq 2 \times (0.322)$ the deviation of the charge radius from the behavior (8) is within 8% at $Z=100$. For α it is essentially covered by the Z dependence presented above since the Coulomb interaction term in (1) plays a minor role as far as $\alpha \ll 1$.

In this paper we have constructed the hydrogenlike atoms utilizing the bosonized lowest-partial-wave QED for a relatively large ($Z \gtrsim 50$) nucleus. They are the solitonlike configurations around the nucleus in this theory. Despite the classical approximation in the boson theory the charge radius of our solitonic atom has the order of the Bohr radius. This may be interpreted as evidence that the semiclassical analysis of the Bose theory contains fermionic quantum effects, supporting our claim in Ref. 1.

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[1] Y. Hirata and H. Minakata, Phys. Rev. D **34**, 2493 (1986).

[2] We ignore in this paper the problem of the degeneracy of the Φ and the Q solitons, and thereby the question of $1S_{1/2}$-$2P_{1/2}$ splitting. To deal with it we should go beyond the classical approximation, or the coherent-state approximation.

[3] W. Pieper and W. Greiner, Z. Phys. **218**, 327 (1969).

[4] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **11**, 3424 (1975). The statement in the text refers to the result with $\sqrt{\lambda}/m = 2\sqrt{\pi}$ in this reference.