Bound states in quantum field theory and coherent states: A fresh look

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We consider bound-state equations in quantum field theory where the state explicitly includes radiation quanta as constituents with the number of such quanta not fixed. The fully interacting system is dealt with through equal-time commutators/anticommutators of field operators. The multiparticle channel for the radiation field is approximated through coherent-state representations.

Usually one considers bound states in field theory with a fixed number of particles. However, in field theory particle number need not be conserved, and in particular the bound state as an eigenstate of the Hamiltonian need not be an eigenstate of the number operators. In the present analysis, we shall recognize this fact for the definition of the bound state, and develop a nonperturbative framework accordingly.

To motivate, let us first consider a toy model in quantum mechanics. Let the Hamiltonian be

$$H=\epsilon c^{\dagger}c+\omega a^{\dagger}a+gc^{\dagger}c(a^{\dagger}+a)\ . \tag{1}$$

In the above, c stands for the "matter" annihilation operator, a for the "radiation" annihilation operator, and we take the usual quantum conditions $[c,c^{\dagger}]_{+}=[a,a^{\dagger}]=1$. Thus matter has been taken as fermionic and radiation as bosonic. The state $|\mathrm{vac}\rangle$ is defined through $a\,|\mathrm{vac}\rangle=c\,|\mathrm{vac}\rangle=0$. Let us now make the substitution $a=a'-(g/w)c^{\dagger}c$. We then have $[a',a'^{\dagger}]=1$. However, a' does not commute with c or $c^{\dagger}$. Equation (1) then simplifies to

$$H=\epsilon c^{\dagger}c+\omega a'^{\dagger}a'-(g^2/\omega)(c^{\dagger}c)(c^{\dagger}c)\ . \tag{2}$$

Let us next consider a state $|\,B_n\rangle=f(a^{\dagger})c^{\dagger n}\,|\mathrm{vac}\rangle$ with n $(=0,1)$ fermions and arbitrary number of radiation quanta such that $a'\,|\,B_n\rangle=0$. With $a=\delta/\delta a^{\dagger}$ this leads to the differential equation

$$[\delta/\delta a^{\dagger}+(g/\omega)n]f(a^{\dagger})=0\ , \tag{3}$$

such that, with A as a normalization constant,

$$|\,B_n\rangle=A\exp[-n(g/\omega)a^{\dagger}](c^{\dagger})^n\,|\mathrm{vac}\rangle \tag{4}$$

is an eigenstate of H with eigenvalue $n\epsilon-n^2(g^2/\omega)$. We may also construct other eigenstates of H as

$$|\,m,n\rangle=(a^{\dagger}+ng/\omega)^m\,|\,B_n\rangle\ , \tag{5}$$

where

$$H\,|\,m,n\rangle=(m\omega+n\epsilon-n^2g^2/\omega)\,|\,m,n\rangle\ .$$

We took matter as fermionic so that energy is bounded below. Here $|\,B_1\rangle$ is a coherent state[1] with $a\,|\,B_1\rangle=-(g/\omega)\,|\,B_1\rangle$. Also there are an infinity of radiation quanta with the probability for k quanta given as

$$p_k=[(g/\omega)^{2k}/k!]\exp(-g^2/\omega^2)\ . \tag{6}$$

We may here note that for adequately large g^2/ω, there is a phase transition and the single fermion state $|\,B_1\rangle$ with its radiation cloud constitutes the physical vacuum.

The above reveals the relevance of multiradiation quanta, simulating field theory, which we now proceed to consider. For illustrating the dynamics, we take a nonrelativistic Hamiltonian density at $t=0$ given as, with $\phi(\mathbf{x})=(2\omega_{\mathbf{x}})^{-1/2}[a(\mathbf{x})+a(\mathbf{x})^{\dagger}]$,

$$\begin{aligned}\mathscr{H}(\mathbf{x})=&c_1(\mathbf{x})^{\dagger}\epsilon_{1\mathbf{x}}c_1(\mathbf{x})+c_2(\mathbf{x})^{\dagger}\epsilon_{2\mathbf{x}}c_2(\mathbf{x})\\&+a(\mathbf{x})^{\dagger}\omega_{\mathbf{x}}a(\mathbf{x})\\&+[e_1c_1(\mathbf{x})^{\dagger}c_1(\mathbf{x})+e_2c_2(\mathbf{x})^{\dagger}c_2(\mathbf{x})]\phi(\mathbf{x})\ .\end{aligned} \tag{7}$$

The following notations may be noted. In the above we have expressed the Hamiltonian density in terms of the fermion and boson creation and annihilation operators with, e.g., the obvious algebra

$$\begin{aligned}[c_1(\mathbf{x}),c_1(\mathbf{y})^{\dagger}]_{+}&=[c_2(\mathbf{x}),c_2(\mathbf{y})^{\dagger}]_{+}\\&=[a(\mathbf{x}),a(\mathbf{y})^{\dagger}]=\delta(\mathbf{x}-\mathbf{y})\ .\end{aligned} \tag{8}$$

Further, $\epsilon_{1\mathbf{x}}$, $\epsilon_{2\mathbf{x}}$, and $\omega_{\mathbf{x}}$ are *differentiation operators* corresponding to the respective free-field Hamiltonians, and are defined through the Fourier-transform space. The model corresponds to two fermions interacting with a scalar meson. The spins are suppressed, but the kinetic relativistic corrections may be present through $\epsilon_{r\mathbf{x}}=(-\nabla_{\mathbf{x}}{}^2+\mu_r{}^2)^{1/2}$ and $\omega_{\mathbf{x}}=(-\nabla_{\mathbf{x}}{}^2+\mu^2)^{1/2}$. We shall also have in addition counterterms corresponding to the self-energy of the fermions.

As earlier, we define $|\mathrm{vac}\rangle$ through $c_r(\mathbf{x})\,|\mathrm{vac}\rangle=0=a(\mathbf{x})\,|\mathrm{vac}\rangle$. The bound state of two fermions and an arbitrary number of radiation quanta will be considered. This will be an eigenstate of the Hamiltonian $H=\int d\mathbf{x}\,\mathscr{H}(\mathbf{x})$ where the eigenstate $|\,B\rangle$ will have the form $|\,B_0\rangle+|\,B_1\rangle+|\,B_2\rangle+\cdots$ with $|\,B_n\rangle$ describing a state with n radiation quanta. The problem will thus involve the coupling of infinitely many channels which is not possible to solve. We shall hence use here the approximation of the multiparticle states being the coherent states, and shall proceed to construct such states.

We start with the fiducial state

$$|\,S_0(\mathbf{x},\mathbf{y})\rangle=c_1(\mathbf{x})^{\dagger}c_2(y)^{\dagger}\,|\mathrm{vac}\rangle\ .$$

We next consider the operator

$$G^\dagger = \int c_r(\mathbf{z})^\dagger A_r(\mathbf{z})^\dagger c_r(\mathbf{z}) d\mathbf{z}$$

with summation over repeated index r, and with $A_r(\mathbf{z})^\dagger = \int f_r(\mathbf{z}-\mathbf{z}')a(\mathbf{z}')^\dagger d\mathbf{z}'$. $f_r(\mathbf{z})$ $(r=1,2)$ are two functions of space coordinates $\mathbf{z}$. The justification for this choice of operators will occur later. Using Eqs. (8) we now see that

$$\begin{aligned} |S(\mathbf{x},\mathbf{y})\rangle &\equiv N\exp(G^\dagger)\,|S_0(\mathbf{x},\mathbf{y})\rangle \\ &= Nc_1(\mathbf{x})^\dagger \exp[A_1(\mathbf{x})^\dagger]\exp[A_2(\mathbf{y})^\dagger]c_2(\mathbf{y})^\dagger\,|\,\mathrm{vac}\rangle \\ &= c_1(x)^\dagger c_2(y)^\dagger\,|R(\mathbf{x},\mathbf{y})\rangle\ . \end{aligned} \tag{9}$$

In the above, N is a normalization constant which we shall determine. We may interpret that $\exp[A_1(\mathbf{x})^\dagger]$ creates the radiation quanta attached to the fermion at $\mathbf{x}$, and similarly for $\exp[A_2(\mathbf{y})^\dagger]$. Such an identification will help us to recognize the self-energy contributions of the fermions. We note that $[a(\mathbf{z}),A_r(\mathbf{x})^\dagger]=f_r(\mathbf{x}-\mathbf{z})$, which yields

$$a(\mathbf{z})\,|S(\mathbf{x},\mathbf{y})\rangle = [f_1(\mathbf{x}-\mathbf{z})+f_2(\mathbf{y}-\mathbf{z})]\,|S(\mathbf{x},\mathbf{y})\rangle\ . \tag{10}$$

It was our purpose to construct such an eigenstate.

We shall define a bound state of zero momentum in the form

$$|B(\mathbf{0})\rangle = (2\pi)^{-3/2}\int u(\mathbf{x}-\mathbf{y})\,|S(\mathbf{x},\mathbf{y})\rangle d\mathbf{x}\,d\mathbf{y}\ . \tag{11}$$

This has three arbitrary functions: $u(\mathbf{x}-\mathbf{y})$ which will correspond to the old Schrödinger wave function, and $f_r(\mathbf{x})$ $(r=1,2)$ which will decide the nature of the radiation quanta in the bound state.

We now determine the normalization constant N in Eq. (9) from the condition

$$\langle R(\mathbf{x},\mathbf{y})\,|\,R(\mathbf{x},\mathbf{y})\rangle = 1\ . \tag{12}$$

We first note that by (8)

$$\begin{aligned} [A_r(\mathbf{x}),A_s(\mathbf{y})^\dagger] &= \int f_r(\mathbf{x}-\mathbf{z})^* f_s(\mathbf{y}-\mathbf{z}) d\mathbf{z} \\ &= h_{rs}(\mathbf{x}-\mathbf{y})\ , \end{aligned} \tag{13}$$

such that (12) yields

$$N_{xy}^{\ -2} = \exp[f(\mathbf{x}-\mathbf{y})] \tag{14}$$

with

$$f(\mathbf{x}-\mathbf{y}) = h_{11}(\mathbf{0})+h_{22}(\mathbf{0})+h_{12}(\mathbf{x}-\mathbf{y})+h_{21}(\mathbf{y}-\mathbf{x})\ . \tag{15}$$

We next take the formal normalization for zero-momentum states as $\langle B(\mathbf{0})\,|\,B(\mathbf{0})\rangle = \delta(\mathbf{0})$, which yields the conventional normalization

$$\int |\,u(\mathbf{x})\,|^2 d\mathbf{x} = 1\ . \tag{16}$$

We shall now consider the expectation value of the Hamiltonian density of Eq. (7) for the state as in Eq. (11), and then minimize this to obtain the mass. We substitute

$$h[u,f_1,f_2] = t + h_M + h_i\ , \tag{17}$$

where we have taken the expectation value $(2\pi)^3\langle B(\mathbf{0})\,|\,\mathscr{H}(\mathbf{0})\,|\,B(\mathbf{0})\rangle$. The individual terms t, h_M, and h_i are, respectively, the expectation values of the fermonic kinetic part, the "meson" or radiation field part, and the interaction part of Eq. (7). These functionals are next to be evaluated.

For the evaluation of t, we first note that

$$\langle R(\mathbf{x}',\mathbf{y}')\,|\,R(\mathbf{x},\mathbf{y})\rangle = N_{x'y'}N_{xy}\exp[h_{11}(\mathbf{x}'-\mathbf{x})+h_{22}(\mathbf{y}'-\mathbf{y})+h_{12}(\mathbf{x}'-\mathbf{y})+h_{21}(\mathbf{y}'-\mathbf{x})]\ . \tag{18}$$

Using this, and with some algebra, we get

$$\begin{aligned} t &= \int u(-\mathbf{y})^* \epsilon_{1\mathbf{x}}\{\exp[h_{11}(-\mathbf{x})-h_{11}(\mathbf{0})]u(\mathbf{x}-\mathbf{y})\}_{\mathbf{x}\to 0} d\mathbf{y} \\ &\quad + \int u(\mathbf{x})^* \epsilon_{2\mathbf{y}}\{\exp[h_{22}(\mathbf{y})-h_{22}(\mathbf{0})]u(\mathbf{x}-\mathbf{y})\}_{\mathbf{y}\to 0} d\mathbf{x}\ . \end{aligned} \tag{19}$$

In case we can approximate the above limits inside the differentiation, we obtain

$$t = \int u(\mathbf{x})^*(\epsilon_{1\mathbf{x}}+\epsilon_{2\mathbf{x}})u(\mathbf{x})d\mathbf{x}\ . \tag{20}$$

Using Eqs. (10) and (12), evaluation of h_M gives

$$h_M = \int |\,u(\mathbf{x}-\mathbf{y})\,|^2 d\mathbf{x}\,d\mathbf{y}[f_1(\mathbf{x})^* + f_2(\mathbf{y})^*][\omega_{\mathbf{x}} f_1(\mathbf{x}) + \omega_{\mathbf{y}} f_2(\mathbf{y})]\ . \tag{21}$$

Next, evaluation of h_i gives

$$\begin{aligned} h_i &= \int |\,u(-\mathbf{y})\,|^2 d\mathbf{y}\{e_1[(2\omega_{\mathbf{x}})^{-1/2} f_1(\mathbf{x})^* + (2\omega_{\mathbf{y}})^{-1/2} f_2(\mathbf{y})^*]\}_{\mathbf{x}\to 0} \\ &\quad + \int |\,u(\mathbf{x})\,|^2 d\mathbf{x}\{e_2[(2\omega_{\mathbf{x}})^{-1/2} f_1(x)^* + (2\omega_{\mathbf{y}})^{-1/2} f_2(y)^*]\}_{\mathbf{y}\to 0} + \mathrm{H.c.} \end{aligned} \tag{22}$$

Here we have used $\phi(\mathbf{x}) = (2\omega_{\mathbf{x}})^{-1/2}[a(\mathbf{x})+a(\mathbf{x})^\dagger]$ and Eqs. (10) and (12).

We shall now use the limit (20) for t, and Eqs. (21) and (22), and extremize with respect to f_1 and f_2. Thus, e.g., $[\delta/\delta f_1(\mathbf{x})^*](t+h_M+h_i)=0$ yields, using (16),

$$\int |\,u(\mathbf{x}-\mathbf{y})\,|^2[\omega_{\mathbf{x}} f_1(\mathbf{x}) + \omega_{\mathbf{y}} f_2(\mathbf{y})]d\mathbf{y} + e_2(2\omega_{\mathbf{x}})^{-1/2}\,|\,u(\mathbf{x})\,|^2 + e_1(2\omega_{\mathbf{x}})^{-1/2}\delta(\mathbf{x}) = 0\ . \tag{23}$$

We can easily verify that the above as well as the other similar extremization equations for f_1 and f_2 are satisfied when

$$f_r(\mathbf{x}) = -[e_r/(\sqrt{2}\omega_x^{3/2})]\delta(\mathbf{x}) . \tag{24}$$

As mentioned earlier, all these equations are defined through Fourier transforms and are to be regularized when necessary, and, for appropriate contributions, self-energy terms are to be subtracted.

We shall now use Eq. (24) and identify the self-energy contributions. Equation (21) has, respectively, the self-energy and the potential contributions given as

$$h_M^s = \tfrac{1}{2}(e_1^2 + e_2^2)\int [\omega_x^{-3/2}\delta(\mathbf{x})][\omega_x^{-1/2}\delta(\mathbf{x})]d\mathbf{x} \tag{25}$$

and

$$h_M^v = e_1 e_2 \int |u(\mathbf{x})|^2 [\omega_x^{-2}\delta(\mathbf{x})]d\mathbf{x} . \tag{26}$$

Similarly, the self-energy and the potential contributions from Eq. (22) are, respectively, given as

$$h_i^s = -(e_1^2 + e_2^2)[\omega_x^{-2}\delta(\mathbf{x})]_{\mathbf{x}\to 0} \tag{27}$$

and

$$h_i^v = -2e_1 e_2 \int |u(\mathbf{x})|^2 [\omega_x^{-2}\delta(\mathbf{x})]d\mathbf{x} . \tag{28}$$

We assume that the terms in (25) and (27) are canceled by appropriate self-energy counterterms. This gives us the potential as

$$\begin{aligned} v &= h_M^v + h_i^v \\ &= -e_1 e_2 \int |u(\mathbf{x})|^2 [\omega_x^{-2}\delta(\mathbf{x})]d\mathbf{x} . \end{aligned} \tag{29}$$

Thus the conventional potential term is contributed both from the free radiation part as well as the interaction part in a simple manner. Now a variation with respect to $u(\mathbf{x})^*$ yields the familiar eigenvalue equation, for $\mu = 0$,

$$(\epsilon_{1x} + \epsilon_{2x})u(\mathbf{x}) - \frac{e_1 e_2}{4\pi}\frac{1}{|\mathbf{x}|}u(\mathbf{x}) = Eu(\mathbf{x}) \tag{30}$$

when we shall carry out the corresponding algebra.[2] This ends the "conventional" nature of the present theory, and we shall now proceed to show the extra physics content of the present approach.

Firstly we see that the probability for there being k radiation quanta contained in the bound state parallel to (6) is given as[3]

$$p_k = \int |u(\mathbf{x})|^2 \{[f(\mathbf{x})]^k/k!\}\exp[-f(\mathbf{x})]d\mathbf{x} . \tag{31}$$

We may also obtain the average number of radiation quanta as

$$N_R = \sum k p_k = \int |u(\mathbf{x})|^2 f(\mathbf{x})d\mathbf{x} . \tag{32}$$

We may also find the momentum squared carried by the radiation quanta. Taking the expectation value of $a(\mathbf{x})^\dagger[-\nabla_x^2 a(\mathbf{x})]_{\mathbf{x}\to 0}$ we thus formally obtain,

$$\langle P_R^2\rangle = \int |u(\mathbf{x}-\mathbf{y})|^2 |(\nabla_x f_1 + \nabla_y f_2)|^2 d\mathbf{x}\,d\mathbf{y} . \tag{33}$$

The above expression also contains divergent expressions which are to be subtracted. After doing so, we obtain the corresponding contribution as

$$\begin{aligned} \langle P_R^2\rangle &= -e_1 e_2 \int |u(\mathbf{x})|^2 [\nabla_x^2 \omega_x^{-3}\delta(\mathbf{x})]d\mathbf{x} \\ &= \frac{e_1 e_2}{(2\pi)^3}\int d\mathbf{k}\,d\mathbf{k}'\tilde{u}(\mathbf{k}')^*\tilde{u}(\mathbf{k}'-\mathbf{k})\mathbf{k}^2/[\omega(\mathbf{k})]^3 . \end{aligned} \tag{34}$$

We note that this is Yukawa coupling so that e_1 and e_2 have the same sign.

We may note that in Eq. (15) $f(\mathbf{x}-\mathbf{y})$ has a divergent contribution through $h_{11}(0)$ and $h_{22}(0)$, and the "convergent" part $f_c(\mathbf{x}-\mathbf{y})$ is given by

$$\begin{aligned} f_c(\mathbf{x}) &= h_{12}(\mathbf{x}) + h_{21}(-\mathbf{x}) \\ &= \frac{e_1 e_2}{(2\pi)^3}\int \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\omega(\mathbf{k})^3}d\mathbf{k} = \frac{e_1 e_2}{2\pi^2}K_0(\mu) . \end{aligned} \tag{35}$$

In (35), μ is the effective mass or infrared cutoff of the radiation quanta, $r = |\mathbf{x}|$ and, K_0 is the Bessel function with imaginary argument[4] such that, for small r, $K_0(\mu r) \simeq \ln(1/\mu^2 r^2)$, and $K_0 \to 0$ as $r\to\infty$. In order to get a feeling for the contribution, let us use an explicit approximation

$$f(x) = \frac{2\alpha}{\pi}\ln\left[\frac{1+\mu^2 r^2}{\mu^2 r^2}\right] , \tag{36}$$

where we have substituted $e_1 e_2 = 4\pi\alpha$, with quantum electrodynamics in mind. One then obtains

$$N_R = \frac{2\alpha}{\pi}\int |u(\mathbf{x})|^2 \ln\left[\frac{1+\mu^2 r^2}{\mu^2 r^2}\right]d\mathbf{x} . \tag{37}$$

We note that when $\mu\to 0$, (37) includes the usual infrared divergence which is to be tackled separately.[5]

The present description of radiation quanta inside the bound state has another interesting consequence. Let us define, using Eqs. (10) and (24),

$$\begin{aligned} \phi_{\rm cl}(\mathbf{z}) &\equiv \langle B(\mathbf{0})|\phi(\mathbf{z})|B(\mathbf{0})\rangle \\ &= \int |u(\mathbf{x}-\mathbf{y})|^2 d\mathbf{x}\,d\mathbf{y}[-e_1\omega_x^{-2}\delta(\mathbf{x}-\mathbf{z}) \\ &\qquad - e_2\omega_y^{-2}\delta(\mathbf{y}-\mathbf{z})] . \end{aligned} \tag{38}$$

The above correctly describes the classical potential inside the bound system when the sources are at $\mathbf{x}$ and $\mathbf{y}$. Thus, the radiation quanta inside the bound state may be recognized as the *quantum* description of the confined *classical* field. These radiation quanta, as well as the two fermions, are obviously off the mass shell. We may picture the radiation quanta to act like glue to keep the fermions together. The conclusions are nonperturbative in the sense that only equal-time commutators or anticommutators are used.

In conclusion, we may note the following.

(i) We have shown that an approximation through multiradiation quanta present in the bound state generates the conventional Schrödinger equation for the energy eigenvalues.

(ii) The momentum carried by the radiation quanta is calculable. Thus, in a hadron, all the momentum will not be carried by the quarks or antiquarks, but the gluons may also carry a substantial fraction of the momentum. This has obvious relevance for deep inelastic collisions.[6,7]

(iii) For hadronic spectroscopy, gluon number in hadrons will be nonzero, and can be large. Since gluons carry spin, there will be substantial spin corrections, and the static SU(6) models will be particularly bad regarding polarization effects.

(iv) The gluons present in the hadron will simulate the potential which may be calculable for the heavy-quarkonium system.[7] For this purpose more complicated coherent states could be relevant.[8]

In a deeper sense, the present formulation of the bound state is a "true" field theory since here we take a bound state with particle nonconservation, which is a basic feature of field theory. The fact that it has many correct hints regarding harmonic phenomenology was thus probably to be expected.

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