Exact implementation of baryon-number conservation in lattice gauge theory

David E. Miller
Department of Physics, Pennsylvania State University, Hazleton, Pennsylvania 18201

Krzysztof Redlich*
Fakultät für Physik, Universität Bielefeld, D-4800 Bielefeld 1, Federal Republic of Germany
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The canonical ensemble is formulated for lattice gauge theory with heavy quarks in the system. A mean-field analysis of SU(2) gauge theory is carried out within this formulation. The dependence of the Wilson loop on the thermodynamical properties is analyzed in an $SU_C(2)\times U_B(1)$ phenomenological model. This analysis can contribute to an understanding of the partial-quenching approximation relating to the vanishing of the imaginary part of the Wilson loop on a lattice with the nonzero chemical potential.

I. INTRODUCTION

It is a rather well-established fact that strongly interacting matter exhibits a behavior typical of a system with a phase transition. For sufficiently high temperature and/or baryon-number density the color is deconfined and the system undergoes a phase transition from a hadron gas to a quark-gluon plasma.[1] This critical behavior can be observed in a pure SU(3) gauge theory on a lattice as a discontinuous jump in the energy density or in the order parameter (Wilson loop) both of which suggest a first-order phase transition.[2] The situation for the full lattice gauge theory (QCD) including the dynamical fermions (quarks) is still not so well confirmed. There is, however, strong evidence[1] for the deconfinement phase transition in the full system with color sources but the order of this transition remains until the present unclear. Some of the most recent Monte Carlo results[3] suggest a second-order phase transition in the system with dynamical quarks. There are also difficulties in the Monte Carlo analysis of the model with nonzero chemical potential μ. These are mainly due to the fact that in the SU(N) gauge theory for $N>2$ the fermion contribution to the Euclidean lattice action is a complex one.[4,5] Nevertheless, the Monte Carlo investigation of the lattice QCD with $\mu\neq 0$ has been recently obtained[5] where in order to avoid the problem of the complex fermion determinant the "partial quenching" approximation[6] has been applied. The mean-field analysis of lattice QCD with $\mu\neq 0$ has been also recently studied in the literature.[7]

The contribution from a nonzero baryon number to the thermodynamics of lattice QCD has been studied up to this time strictly in the grand canonical (GC) ensemble with respect to baryon-number conservation. In the relativistic statistical thermodynamics, however, we have a choice between the GC and the canonical (C) treatments of the conservation laws. The possible differences between GC and C descriptions of the conservation laws together with the implications of the exact implementation of the charge conservation have been recently studied in the literature in different thermodynamical models.[8–11] It turns out from the above discussion that in many realistic physical situations the application of the GC ensemble with respect to the conservation laws can be questionable. This is especially true in the case when we deal with a small amount of matter enclosed in a tiny volume with a fixed but small absolute value of the quantum numbers. This situation is found in the laboratory in the central region for heavy-ion collisions with the absolute value of the baryon number $B=0$ and also for hadron-hadron collisions when B is small. In the above actual cases the C description should be preferred over the usual GC treatment of the conservation laws.[8]

One of the main purposes of this paper is the formulation of lattice QCD in the C ensemble with respect to baryon-number conservation. With the assumption that there are only heavy quarks in the system we shall find the canonical partition function in the SU(3) lattice gauge theory. As an example we use mean-field (MF) analysis of SU(2) lattice QCD to show the possible implications of the canonical formulation on the thermodynamical behavior of the system. We shall show in terms of MF approximation that in the limit of large values of baryon number and volume of the system but fixed baryon-number density the GC and C ensembles are equivalent. We shall also observe that the above analysis can give some information about the validity of the "partial quenching" approximation as it has been applied in the GC ensemble.[5]

This paper is organized as follows. In the next section we briefly summarize the canonical description with respect to internal symmetries. Then we formulate lattice QCD with the exact implementation of the baryon-number conservation. We are then able in the fourth section to compute the Wilson loop in terms of a phenomenological model with the $SU_C(2)\times U_B(1)$ internal-symmetry group formulated in the C ensemble. The mean-field analysis of SU(2) lattice gauge theory in the C ensemble in the strong-coupling limit is also present in this section. Finally we draw some conclusions about our analysis of lattice QCD in a C formulation.

II. CANONICAL DESCRIPTION OF INTERNAL SYMMETRY

The formulation of relativistic thermodynamics with the exact implementation of the conservation laws is carried out through a procedure based on group-theoretical methods.[10] For this situation the formal structure resembles the GC description with the main difference being that we define the generating function by taking the trace over all states as

$$\widetilde{Z}(g,V,\beta)=\mathrm{Tr}[e^{-\beta H}U(g)]\ , \tag{2.1}$$

where $U(g)$ is the unitary reducible representation of the symmetry group G with $g\in G$, H the Hamiltonian, V volume, and β the inverse temperature of the system. Because of the exact symmetry and the decomposition of $U(g)$ into the form $\sum_\alpha \oplus U^\alpha(g)$, one can write[10]

$$\widetilde{Z}(g,\beta,V)=\sum_\alpha \frac{\chi^\alpha(g)}{d(\alpha)}Z_\alpha(\beta,V) \tag{2.2}$$

with $Z_\alpha(\beta,V)$ the usual canonical partition function given by $\mathrm{Tr}_\alpha[\exp(-\beta H)]$ which contains exactly that value of the quantum numbers which correspond to the α representation of the symmetry group. $\chi^\alpha(g)$ and $d(\alpha)$ are the character and the dimension of the α representation of the group. By using the orthogonality properties of the group character one can find[10–13]

$$Z_\alpha(\beta,V)=\int dM(\varphi_1,\ldots,\varphi_r)\bar{\chi}^{\alpha}(\varphi_1,\ldots,\varphi_r)\times\widetilde{Z}(\beta,V,\varphi_1,\ldots,\varphi_r)\ , \tag{2.3}$$

where dM is the Haar measure over the group and $\varphi_1,\ldots,\varphi_r$ the parameters of the maximal Cartan subgroup of the symmetry group G. From (2.2) and the definition of the GC partition function one can also establish the following simple relation:

$$\widetilde{Z}(\beta,V,\varphi_1,\ldots,\varphi_r)=Z^{\mathrm{GC}}(\beta,V,\mu_1=i\varphi_1/\beta,\ldots,\mu_r=i\varphi_r/\beta)\ . \tag{2.4}$$

In our present analysis we shall apply the above formalism to the simple case of the U(1) baryon symmetry group for lattice QCD in the next section. In the following section we shall develop a simple phenomenological model with the symmetry $\mathrm{SU}(2)\times\mathrm{U}(1)$ for the exact evaluation of the thermodynamical quantities.

III. LATTICE QCD WITH BARYON-NUMBER CONSERVATION

In the lattice formulation[14] of QCD the partition function on an isotropic Euclidean lattice with N_τ (N_σ) temporal (spatial) lattice sites and a nonzero baryon chemical potential[15] μ can be found as

$$Z(\mu,N_\tau,N_\sigma,\kappa)=\int\prod_{\text{links}}dU\exp(-S_G)(\det Q)^{N_f} \tag{3.1}$$

with

$$S_G=\frac{6}{g^2}\sum_p\left[\frac{1}{N}\mathrm{Re\,Tr}UUU^\dagger U^\dagger\right] \tag{3.2}$$

being the pure gluon part. $N_f\ln\det Q$ is the quark-gluon contribution to the lattice action, which is gotten after the integration of the quark spinor fields. The fermion matrix in (3.1) has the form[16]

$$Q=1-\kappa\sum_{\nu=0}^{3}M_\nu \tag{3.3}$$

with κ the "hopping" parameter and

$$(M_\nu)_{m,n}=(1-\gamma_\nu)U_{n,m}\delta_{n,m-\hat{\nu}}+(1+\gamma_\nu)U^\dagger_{m,n}\gamma_{n,m+\hat{\nu}}\ , \tag{3.4}$$

where U is an element of the SU(N) group.

In order to bring the chemical potential into the theory, one can use the prescription of Ref. 15, which is contained in the following substitution:

$$U\to Ue^{\mu a},\quad U^\dagger\to U^\dagger e^{-\mu a}\ . \tag{3.5}$$

This is set into the $\nu=0$ term of (3.4). However, this replacement implies that the fermion matrix (3.3) is no longer Hermitian. A direct consequence of this is that the fermion contribution to the Euclidean lattice action becomes complex for $N>2$. This fact alone is the origin of the well-known difficulties in the Monte Carlo computational procedures, which require a real and positive-definite measure.

The complex contribution to the fermion determinant with $\mu\neq 0$ can also be found in a model outside of lattice QCD (Ref. 12). Thus the above feature is neither directly connected with the lattice regularization scheme nor with the way in which the chemical potential has been brought onto the lattice. Its presence has a rather general nature arising in the gauge theories which is related to the structure of the SU(N)-symmetry group with $N>2$.

After having established the thermodynamics of QCD in the GC ensemble, we now want to find a lattice partition function $Z_B(N_\tau,N_\sigma,\kappa)$ which gives an exact implementation of the baryon-number conservation. However, we should note at this point that the quantity B is actually just the difference between the number of quarks and the number of antiquarks. Then the real baryon number is simply $B/3$ for SU(3).

In the following analysis we shall restrict ourselves to the case for which there are only heavy quarks present in the system. Thus the determinant in (3.1) can be evaluated using the "hopping parameter" expansion, which for very heavy quarks and $N_\tau<4$ can be approximated by the leading term in this expansion.[16,17] With the formalism indicated in the previous section applied to the U(1) internal-symmetry group, the C partition function becomes

$$Z_B(N_\tau,N_\sigma,\kappa)=\int_0^\pi\frac{d\varphi}{\pi}\cos(B\varphi)\widetilde{Z}(\varphi,N_\tau,N_\sigma,h)\ , \tag{3.6}$$

where we have used the symmetry properties of the generating function ($\varphi\to-\varphi$) together with (3.5). In the leading order in the hopping-parameter expansion the generating function can be obtained as

$$\widetilde{Z}(\varphi,N_\tau,N_\sigma,h)=\int\prod_{\text{links}}\exp(-S_G-\widetilde{S}_F) \tag{3.7}$$

with S_G as in (3.2) and

$$\widetilde{S}_F = -h(\cos\varphi L_R - \sin\varphi L_I) , \tag{3.8}$$

where $h \equiv 4N_f(2\kappa)^{N_\tau}$ is the quark parameter and

$$L_R \equiv \sum_x \mathrm{Re}L_x \quad \text{and} \quad L_I \equiv \sum_x \mathrm{Im}L_x$$

with the Wilson loop at the spatial site x given by

$$L_x = \mathrm{Tr} \prod_{\tau=1}^{N_\tau} U_{(\bar{x},\tau),(\bar{x},\tau+1)} . \tag{3.9}$$

Thus with the generating function with (3.7) the integration over the $U_B(1)$ group can be performed exactly so that the canonical lattice partition function becomes

$$Z_B(N_\tau,N_\sigma,h) = \int \prod_{\text{links}} dU\, e^{-S_G} I_B(hy) T_B(y^{-1}L_R) \tag{3.10}$$

with $y = (L_R{}^2 + L_I{}^2)^{1/2}$, $I_B(x)$ the modified Bessel function of the first kind, and $T_B(\cos\varphi) = \cos B\varphi$ being the Chebyschev polynomial. The thermal average of any physical quantity $f(U)$ in the C ensemble can be calculated in the usual way as

$$\langle f(U) \rangle_B = \int \prod dU\, e^{-S_G} f(U) I_B(hy) T_B(y^{-1}L_R) \Big/ \int \prod dU\, e^{-S_G} I_B(hy) T_B(y^{-1}L_R) . \tag{3.11}$$

Now let us discuss the relation between the C and the GC partition function as given above in (3.10) and (3.1), respectively. In the ordinary statistical thermodynamics the C partition function is the coefficient in the cluster decomposition of the GC partition function. The same relation holds for the relativistic statistical thermodynamics where the distinction between the GC and the C ensemble are given on the level of the conservation laws.

For the case of the $U_B(1)$ symmetry this cluster decomposition has a particularly simple form:

$$Z(\mu,N_\tau,N_\sigma,h) = \sum_{B=-\infty}^{\infty} e^{\mu\beta B} Z_B(N_\tau,N_\sigma,h) \tag{3.12}$$

with $Z_B(N_\tau,N_\sigma,h)$ given by (3.10). From the relation

$$\exp\left[\tfrac{1}{2}z\left(t+\frac{1}{t}\right)\right] = \sum_{k=-\infty}^{\infty} t^k I_k(z) , \tag{3.13}$$

one can find from (3.12) that

$$Z(\mu,N_\tau,N_\sigma,h) = \int \prod dU \exp[-S_G - h\cosh(\mu\beta)L_R] \cos[h\sinh(\mu\beta)L_I] \tag{3.14}$$

which is just the GC partition function recently used in Ref. 5 for the Monte Carlo evaluation of the statistical QCD on the lattice with nonzero chemical potential. The last term $\cos[h\sinh(\mu\beta)L_I]$ together with the weight factor $\exp(-S_G - S_F)$ in (3.14) leads to the large fluctuations and thereby produce the difficulties in Monte Carlo computation. In order to avoid this problem, the "partial quenching" contained in the substitution $L_I = 0$ in (3.14) has been proposed. Unfortunately on the level of the GC ensemble one cannot test the validity of this approximation.[5]

In the C ensemble we are in general also not free from the problem concerned with these large fluctuations in the Monte Carlo computation. For large values of the baryon number the Chebyschev polynomial term in (3.10) plays a similar role to the $\cos[h\sinh(\mu\beta)L_I]$ of the GC ensemble. Nevertheless, for not too large values of the baryon number a numerical analysis in the C ensemble may be possible. Thus for instance in the central region of the heavy-ion collision where $B = 0$ as well as in the hadron-hadron scattering it could be possible to deduce the thermodynamical properties of the produced hadronic matter by using the C partition function given in (3.10) as a basis. Furthermore, one may well suspect that since the argument of the Chebyschev polynomial is proportional to $[1 + (L_I/L_R)^2]^{-1/2}$ and generally the ratio L_I/L_R is rather small, the numerical analysis can probably be performed for reasonably large values of the baryon number.[18]

Now we consider the "partial quenching" approximation on the level of the C ensemble. Because of the cluster expansion (3.12) and the relations (2.3) and (2.4) one can conclude that the approximation $L_I = 0$ in both C and GC is equivalent. As we have already pointed out in the GC formulation it is not so clear how to test the above assumption. In the C ensemble the situation is quite different. Here it is possible to check the validity of the "partial quenching" approximation by computing first the thermal average of some quantity $f(U)$ using (3.11) with some given small value of the baryon number B and then comparing it with

$$\langle f(U) \rangle_B = \int \prod dU\, e^{-S_G} f(U) I_B(hL_R) \Big/ \int \prod dU\, e^{-S_G} I_B(hL_R) . \tag{3.15}$$

Thus in this way one can deduce the possible contribution of the imaginary part of the Wilson loop L_I in the given model.[18]

IV. PHENOMENOLOGICAL MODEL AND MEAN-FIELD THEORY

Now we can use the results of the previous section to determine the thermodynamics of the system in which the absolute value of the baryon number is conserved. In the following discussion we shall restrict ourselves to the case of the SU(2)-color gauge group for which we shall analyze only the properties of the Wilson loop in the canonical ensemble.

First we consider a very simple model which consists of a gas of quarks and gluons with the exact implementation of the color and baryon-number conservation connected with the $SU_C(2)\times U_B(1)$ internal-symmetry group. This model has been recently discussed by Skagerstam[11] in relation to the finite-volume correction to the energy density which has been compared to the usual continuum limit. In our case we shall use this model in order to show the behavior of the Wilson loop on both the volume and the net baryon number of the system.

We require that only those states are allowed in the system which are color singlets so that the C partition function $Z_0(B,V,T)$ has the global color charge equal to zero with a specified baryon number B. By using (2.4) and (3.6) the C partition function takes the form[12]

$$Z_0(B,V,T)=\frac{4}{\pi}e^{-c_1/4}(1+e^{i\pi B})\int_0^1 dx\sqrt{1-x^2}e^{c_1x^2}I_B(c_2x), \tag{4.1}$$

where the constants c_1 and c_2 are, respectively, $8VT^3/\pi^2$ and $(c_1/2)(m/T)^2K_2(m/T)$ with the quark mass m and $I_B(x),K_2(x)$ the modified Bessel functions of the first and second kinds, respectively. In (4.1) the integration over the $U_B(1)$ group has already been performed and the symmetry properties of the Bessel function $I_B(x)$ have been used. Furthermore, we may readily conclude that from (4.1) the only admissible values for B are 0,2,4,. . . . This result can be interpreted to mean that all the colorless objects [SU(2) baryons] may only be constructed from even total numbers of quarks minus antiquarks.

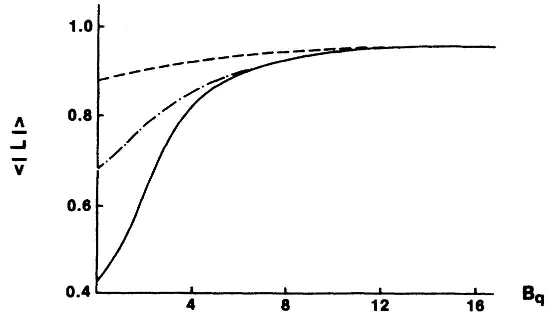

FIG. 1. The dependence of the thermal average of the Wilson loop $\langle\, |L|\,\rangle_B$ on $B_q\equiv 4VT^3$ for different values of the baryon number B as obtained in the gas model with $SU_C(2)\times U_B(1)$ internal symmetry. The lower (solid) curve represents the case $B=0$, the middle (dash-dot) curve is $B=2$, the top (dashed) curve is $B=10$.

In our previous analysis[12] we have indicated that for the SU(2) gauge group in the gauge where the zero component of the gauge field A^a is equal to $\delta_{a^3}\Phi$ with a constant Φ the Wilson loop L is the character of the fundamental representation of the $SU_C(2)$ group. Thus L becomes simply $\cos\varphi$. Starting with the partition function (4.1) the expectation value of the Wilson loop for a given B can be found as the average over the $SU_C(2)\times U_B(1)$ group given by

$$\langle\, |L|\,\rangle_B=\tfrac{1}{2}\int_0^1 dx\sqrt{1-x^2}x\,e^{c_1x^2}I_B(c_xx)\Big/\int_0^1 dx\sqrt{1-x^2}e^{c_1x^2}I_B(c_2x)\,. \tag{4.2}$$

The dependence of $\langle\, |L|\,\rangle_B$ on $B_q=4VT^3$ for a different value of the net quark number B is shown in Fig. 1. Actually in the numerical analysis we have assumed for simplicity $c_1=c_2$. The results indicate that for $B_q<4$ the Wilson loop is far from its asymptotic value and strongly depends on the value of the net charge. If we assume that the temperature in the quark-gluon plasma is around 200 MeV, which could be produced in hadron-hadron collisions, and its volume is in the range from 1 to 10 fm^3, the lower expected value of $B_q/4$ is of the order of unity. For $B_q=0$ the lowest value of the Wilson loop can be exactly determined from this model. For the net charge $B>10$ even in the case of very small B_q the Wilson loop is close to one. This agrees with the results obtained previously by Elze *et al.*[9] and by Amundsen and Skagerstam.[11]

Now let us consider a mean-field analysis of SU(2) gauge theory at finite temperature and in the strong-coupling limit formulated on the lattice in the C ensemble. Since the previous results obtained in the mean-field approximation[19,20] agree quite well with the Monte Carlo analysis, one can suspect that also in the C ensemble we can deduce some interesting features of the model in this approximation.

In the strong-coupling limit the spacelike plaquettes in the Euclidean lattice partition function (3.1) with $\kappa=0$ can be neglected.[19,20] Then the effective theory can be given by the partition function written in terms of the character expansion[20]

$$Z_{\text{eff}}=\int\prod_x dM(x)\prod_{x,l}\left[1+\sum_\nu[Z_\nu(\beta)]\right]^{N_\beta}\chi_\nu(L_x)\chi_\nu(L_{x+l}^\dagger)\,, \tag{4.3}$$

where the two products run over the lattice sites and the directed links, respectively. The expressions χ_ν and Z_ν, respectively, are the character and the character coefficient of the ν representation of the SU(N) group.

When g^{-2} is assumed to be very small, then the leading contribution to the expansion (4.3) is given by the fundamental representation. Thus

$$Z_{\rm eff}\simeq\int\prod_x dM(x)\exp\left[\beta'\sum_l {\rm Tr}L_x{\rm Tr}L^\dagger_{x+l}+{\rm c.c.}\right] \tag{4.4}$$

with $\beta'=Z^{N_\tau}_{(1,0)}$. If the quarks in the system have a very large but finite mass and at the same time there is a nonzero net average baryon number, the effective partition function is generalized[7] as

$$Z_{\rm eff}(\mu,N_\tau,N_\sigma,h)\simeq\int\prod_x dM(x)\exp\left[\beta'\sum_{x,l}{\rm Tr}L_x{\rm Tr}L^\dagger_{x+l}+{\rm c.c.}+h\cosh(\mu\beta)L_R+ih\sinh(\mu\beta)L_I\right] \tag{4.5}$$

with the same notation as in (3.14). With this partition function and the formalism presented in the previous sections we can establish the effective theory with the exact implementation of the baryon-number conservation

$$Z_{\rm eff}(B,N_\tau,N_\sigma,h)\simeq\int\prod_x dM(x)\exp\left[\beta'\sum_{x,l}{\rm Tr}L_x{\rm Tr}L^\dagger_{x+l}+{\rm c.c.}\right]I_B(hy)T_B(y^{-1}L_R) \tag{4.6}$$

following the notation of (3.10).

Now the above effective C partition function can be studied in terms of a MF approximation. In the following analysis we shall restrict our consideration only to the $SU_C(2)$ gauge group for the investigation of the effective partition function in the limit where B and N_σ are large but the ratio $B/N_\sigma{}^3$ defined as $\overline{B}$ is fixed (T limit). Then from (4.6) in this case the effective partition function is given by

$$Z^c_{\rm eff}(B,N_\sigma,B/N_\sigma{}^3,h)\cong\frac{2}{\pi}\int_{-1}^{1}\prod_x[1-({\rm Tr}Lx)^2]^{1/2}d({\rm Tr}Lx)\exp\left[\beta'\sum_{x,l}{\rm Tr}L_x{\rm Tr}L_{x+l}+\ln I_B(Ba_x)\right], \tag{4.7}$$

where the Bessel function $I_B(Ba_x)$ in the T limit can be approximated as

$$\ln I_B(Ba_x)\simeq N_\sigma{}^3\overline{B}\left[(1+a_x{}^2)^{1/2}+\ln\frac{|a_x|}{1+\sqrt{1+a_x{}^2}}\right] \tag{4.8}$$

with

$$a_x=h\overline{B}^{-1}\frac{1}{N_\sigma{}^3}\sum_x{\rm Tr}L_x\ . \tag{4.9}$$

In (4.7) the explicit expression for the Haar measure and the character formula of the fundamental representation of the $SU_C(2)$ group has been used.

The result presented in (4.7) indicates that in the C ensemble the quark parameter does not play the role of the external magnetic field which breaks the Z_N symmetry of the model. Because of the symmetry of the $I_B(x)$ function and due to the requirement that B must be an even number the Z_N symmetry is not broken even for $h\neq 0$.

At this point let us consider the MF analysis of the effective partition function as it is given in (4.7). Assuming here the thermal Wilson loop to be constant everywhere on the lattice[21] and then using the steepest-descent method for the integration over the $SU_C(2)$ group in (4.7), the leading contribution in the T limit to the canonical MF free energy can be found as

$$-F^{\rm C}_{\rm MF}/N_\sigma{}^3\simeq 3\beta' L_B{}^2+\overline{B}\left[(1+\overline{a}^{\,2})^{1/2}+\ln\frac{|\overline{a}|}{1+\sqrt{1+\overline{a}^{\,2}}}\right]+\tfrac{1}{2}\ln(1-L_B{}^2)\ , \tag{4.10}$$

where $\overline{a}$ is simply $h\overline{B}^{-1}L_{\overline{B}}$. Thus with this free energy the MF canonical value of the Wilson loop $L_{\overline{B}}$ can be found as the solution of the equation

$$\frac{\partial F^{\rm C}_{\rm MF}}{\partial L_{\overline{B}}}=6\beta' L_{\overline{B}}-\frac{L_{\overline{B}}}{1-L_B{}^2}+\frac{\overline{B}}{L_{\overline{B}}}\sqrt{1+\overline{a}^{\,2}}=0\ . \tag{4.11}$$

One could expect that in the T limit the GC method for the description of the thermodynamical properties of hadronic matter is also quite adequate. Thus in this limit the GC and C ensembles must be equivalent. In order to show this to be the case, let us consider how the $SU_C(2)$ effective theory looks in the GC description. Starting from (4.5) and taking the MF approximation, which is equivalent to the one in the C ensemble, the MF free energy in the GC ensemble becomes

$$-F^{\rm GC}_{\rm MF}/N_\sigma{}^3\simeq 3\beta' M^2+\tfrac{1}{2}\ln(1-M^2)+h\cosh(\mu N_\tau a)M\ , \tag{4.12}$$

with a the lattice spacing. The MF value of the order parameter M in GC ensemble is given as the solution of

$$\frac{\partial F_{\rm MF}^{\rm GC}}{\partial M}=6\beta' M-\frac{M}{1-M^2}+h\cosh(\mu N_\tau a)=0\ . \quad (4.13)$$

After taking the limit $\overline{B}\to 0$ in (4.10) one can immediately see that the canonical mean-field partition function is equal to the GC one (4.12) with $\mu=0$. Thus in this limit the GC and C ensembles are equivalent. However, for arbitrary nonzero values of the baryon-number density the expressions for the MF free energy in the GC and C ensembles are different which means that (4.10) and (4.12) cannot be directly compared. Thus only the physical quantities should be compared. They should be the same in both descriptions when $B\to\infty$. We can see this by considering the MF value of the Wilson loop which is obtained in both these ensembles. However, for this we still need to know the relationship between the baryon-number density and the chemical potential in the GC ensemble. With the effective partition function (4.5) this relation has a particularly simple form; namely,

$$\frac{\langle B\rangle}{N_\sigma{}^3}=hM\sinh(\mu N_\tau a)\ . \quad (4.14)$$

Now using the above result together with (4.13) we can see that if one identifies B with $\langle B\rangle$ the MF results for the Wilson loop $L_{\overline{B}}$ in the GC and those for M in the C ensemble are the same. Thus we are able to conclude that if both B and N_σ go to very large values with $B/N_\sigma{}^3$ remaining constant, then the GC and C ensembles are thermodynamically equivalent.

The above result should also be valid for the model with the $SU_C(N)$ gauge group. Nevertheless, because of the polynomial term in (3.10) for $N>2$ one comes upon considerable difficulty in the evaluation of the effective partition function for the above considered T limit from this equation. In order to find the effective theory for $N>2$ in the T limit, the application of some other method is required. This method is contained in the analytic continuation of the generating function (3.7) and the application of the Chebyschev polynomial method[8] for the evaluation of the U(1) integral in the T limit in (3.6). This analysis will be presented elsewhere.[18]

Finally we indicate the difference at finite baryon-number density between the GC and C ensembles through numerical examples shown in Fig. 2. The effective potential $V_{\rm eff}$ for the Wilson loop in the $SU_C(2)$ model is obtained from the MF free energy for the respective ensembles from (4.10) and (4.12). We notice in both cases of Fig. 2 that there is a qualitative difference between the behavior of $V_{\rm eff}$ in the GC and C ensembles. Nevertheless, the thermodynamical behavior of any physical quantity obtained in both ensembles in the T limit is the same. This we have already illustrated by the example of the Wilson loop. For any value of the baryon number in the C ensemble there is observed a singular structure of $V_{\rm eff}$ for $L_{\rm MF}$ approaching zero. In order to attain a finite value of $V_{\rm eff}$ at this point, the baryon number has to be identically zero. The structure of $V_{\rm eff}$ indicates the spontaneous breaking of the Z_2 symmetry of this model. In both ensembles there is a general quantitative sensitivity in the values of $V_{\rm eff}$ for changes in the parameters β' and $\overline{B}$. However, in the GC ensemble we note a qualitative

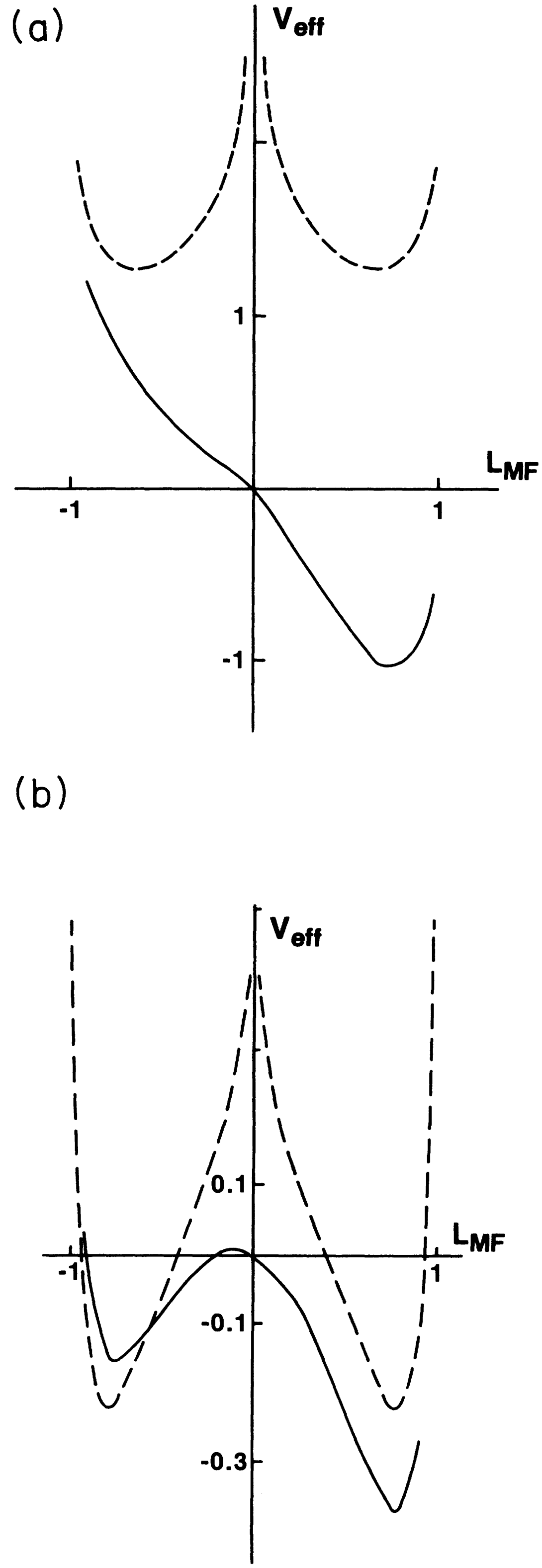

FIG. 2. The effective potential $V_{\rm eff}$ in $SU_C(2)$ model for the canonical ensemble (broken line) and the grand canonical ensemble (solid line) as a function of the mean-field values of the Wilson loop $L_{\rm MF}$ for different value of the parameters: (a) $\beta'=0.1$, $\overline{B}=0.5$, $h=0.06$; (b) $\beta'=0.4$, $\overline{B}=0.1$, $h=0.06$.

change in the structure of V_{eff} for the different values of these parameters as illustrated in Figs. 2(a) and 2(b).

V. CONCLUSIONS

Having in mind lattice QCD as the realistic theory, which can possibly give some useful information about the properties of hot hadronic matter as it might be produced from hadronic collisions, we have formulated lattice QCD in the C ensemble respecting baryon-number conservation. The obtained C partition function $Z_B(N_\tau, N_\sigma, h)$ in (3.10) can be presumably considered to be the starting point for the more detailed Monte Carlo analysis of the model.[18] It can also give some information about the validity of the "partial quenching" approximation proposed in the GC ensemble.[5]

We have also indicated that there can be significant differences between the GC and C treatment of the charge-conservation law. In particular, the quark contribution to the partition function in the C ensemble does not play the role of the external magnetic field which breaks the Z_N symmetry in our SU(2) model. However, for any finite value of the baryon number the Z_N symmetry is spontaneously broken. In the limit of large B and N_σ with fixed $B/N_\sigma{}^3$ the GC and C ensembles are thermodynamically equivalent. This has been shown by the example of the MF analysis of the $SU_C(2)$ model. In the model with $SU_C(2)\times U_B(1)$ internal symmetry we have pointed out that in the region in which one would only expect the C ensemble to be valid the behavior of the thermal average of the Wilson loop is sensitive to the absolute value of the baryon number of the system.

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*Permanent address: Institute for Theoretical Physics, University of Wroclaw, Cybulskiego, Pl-50-205 Wroclaw, Poland.

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