String theory considered as a local gauge theory of an extended object

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In attempting to understand more about the physical origin of the so-called "chordal gauge symmetry" in string field theory it is found that one can, at least formally, consider the theory as a generalized local gauge theory. However, the fundamental object is no longer a point, as in ordinary gauge theory, but a point with a tail, and it is the motion of this tail which represents the internal gauge degree of freedom. Moreover, the differential geometry is based on the non-Abelian conformal group instead of the usual translation group.

Recent work has shown that string field theory has a very large symmetry[1-7] (called chordal symmetry by Banks and Peskin) which is parametrized by an arbitrary functional of the string. The apparent similarity this has to the gauge symmetry of ordinary gauge theories is particularly clearly exhibited in the elegant formulation due to Witten.[7] Chordal symmetry is known to include ordinary gauge invariance in the zero-slope limit, and to a large extent also the invariance under general coordinate transformations. Thus, it seems important to understand more about the physical origin of this symmetry, and to clarify the relation it has, if any, with the gauge invariance of ordinary gauge theories.

In an attempt along that general direction, it is found that one can indeed, at least formally, consider string theory as a generalized local gauge theory. However, the fundamental object is no longer simply a point as in ordinary gauge theory, but a point with a tail, and it is the motion of this tail which represents the internal degree of freedom. The string function in conventional formulations of string theory is the gauge potential in this theory and the ghost field is the differential one-form. In both of these, however, the concepts of ordinary gauge theory has to be generalized in such a way that the usual Abelian group of translations is replaced by the non-Abelian conformal group. In what follows, we shall try first to formulate a gauge-invariant theory of such an extended object, and then show that the result is equivalent, under certain important reservations to be made clear later, to Witten's formulation of the open-bosonic-string theory.[7]

By gauge theory one usually means the following. One starts with a base space X, which is usually ordinary space-time, on which wave functions $\psi(x)$ are defined: $x \in X$. $\psi(x)$ has an internal degree of freedom and thus depends, in addition to x, on an index i which gives its "phase," i.e., its direction in internal space. This "phase," however, has no absolute meaning, all physics being invariant under local gauge transformations, i.e., arbitrary rotations of the frames of reference in the internal space at each space-time point. To specify what is meant by parallel "phases" at neighboring space-time points, one introduces a gauge potential $A_\mu^{ij}(x)$, which is a matrix in the internal indices. Thus, in a given gauge, a wave function $\psi(x)$ is said to be parallel at x and $x+dx$ if the local values of ψ at x and $x+dx$, respectively, differ by $A_\mu^{ij}(x)dx^\mu\psi_j(x)$. Notice that the gauge potential A depends on x, the point in X under consideration, and on the index μ, which gives the direction of displacement to the neighboring point $x+dx$. Under an infinitesimal local gauge transformation

$$\psi_i \to \sum_j [1+\epsilon^{ij}(x)]\psi_j(x) \,, \tag{1}$$

specified by an arbitrary, gauge-algebra-valued, function $\epsilon(x)$ of the space-time point x, A changes by

$$\delta A_\mu = \partial_\mu \epsilon - [A_\mu, \epsilon] \,, \tag{2}$$

where matrix multiplication with respect to internal space indices is understood.

Suppose now, inspired by Witten,[7] we consider half-string wave functionals $\psi[X]$, where X are elements of the function space of functions $X^\mu(\sigma)$: $\sigma = 0 \to \pi/2$, which we denote by ΠX. If we like, we may choose to regard $\psi[X]$ as a function of $x = X(\pi/2)$ and a functional of the functions $\mathcal{X}(\sigma)$ for σ in the semiopen interval $[0,\pi/2)$; thus

$$\psi[X] \sim \psi[x;\mathcal{X}] \,. \tag{3}$$

Imagine now a linear transformation of the wave functional as

$$\psi[x;\mathcal{X}_1] \to \int \delta\mathcal{X}_2 (1+\epsilon[x;\mathcal{X}_1,\mathcal{X}_2])\psi[x;\mathcal{X}_2] \,, \tag{4}$$

where ϵ is an infinitesimal and real function of x and a functional of $\mathcal{X}_1$ and $\mathcal{X}_2$. Compared with the gauge transformation (1) of an ordinary gauge theory, Eq. (4) differs superficially only in replacing a sum over a discrete index j by an integral over a continuum of continuous indices $\mathcal{X}_2(\sigma)$. One may therefore think of x as a point in base space (which is just ordinary space-time) and $\mathcal{X}_1$ and $\mathcal{X}_2$ as internal space indices, and regard (4) as a gauge transformation local in x. If ϵ is infinitesimal and real, but otherwise arbitrary, the gauge group is the general linear group of transformations on the functional space over $\Pi\mathcal{X}$. Our wish now is to construct string theory as a field theory invariant under these local gauge transformations.

Notice that the physical picture so far is such that space-time is quite conventional. Only the fundamental object we deal with is not a point as in usual gauge theories, but a point with a tail (like a comma), extending from its head at $x=X(\pi/2)$ to $X(0)$. Invariance under the transformation (4) for arbitrary $\epsilon[x;\mathcal{X}_1,\mathcal{X}_2]$ would seem to say that we are allowed to wag the tail in whichever way we like, and the physics is not changed, just as unitary symmetry in ordinary gauge theories says that one can rotate the direction of (color) charge arbitrarily without changing the physics. Indeed, this freedom in wagging the tail is like an internal degree of freedom in the sense it was originally meant, namely, as being due to a genuine extension in ordinary space, not in some other space. The way it is implemented, however, looks new.

Next, we proceed to consider gauge potentials for the "comma" theory. It turns out, however, that for some reason not yet clear to us, the potential of interest as far as string theory is concerned is not that which specifies parallel "phases" at neighboring space-time points as in ordinary gauge theories, but one which requires a generalization of the concept. We note that the index μ in the ordinary gauge potential $A_\mu(x)$ may be regarded as denoting a generator of the translation group. Now this generator may in turn be regarded alternatively either as acting (actively) on the base-space point x giving a displacement of x by dx^μ, or as acting (passively) on the wave function $\psi(x)$, giving a variation $\partial_\mu\psi$. One could imagine, therefore, a generalization of the concept of a gauge potential in which the operators ∂_μ on ψ are replaced by generators of another operator algebra whose elements operate on the wave functions $\psi(x)$ but do not necessarily correspond to displacements of space-time points. (Such generalizations have of course been considered by mathematicians. Our considerations here follow closest the ideas of noncommutative differential geometry, as invented by Connes.[8]) Denoting the generators of such an algebra by D_l, one introduces then a potential $A_l(x)$ in place of $A_\mu(x)$, where $A_l(x)$ may still be regarded as specifying what is meant by parallel phases between, in some sense, "neighboring" wave functions $\psi(x)$ and $\psi(x)+D_l\psi(x)$. Under a local gauge transformation specified by $\epsilon(x)$, one finds then that A_l changes by

$$\delta A_l=[(D_l-A_l),\epsilon] \ . \tag{5}$$

Clearly, for the special case when D_l is the derivative ∂_μ, then (5) reduces to the usual gauge transformation (2) of A_μ.

For the string theory, as already remarked by Gervais,[9] the potentials of interest are those corresponding to the generators of the Virasoro algebra:

$$L_{\pm\sigma}=\left[-i\pi\frac{\delta}{\delta X^\mu(\sigma)}\pm X'^\mu(\sigma)\right]^2, \quad 0\le\sigma\le\pi/2 \ , \tag{6}$$

operating on the wave functionals $\psi[X]$. The potential will be a function of the space-time point x and depend on the index of the generator σ, $-\pi/2\le\sigma\le\pi/2$, thus $A_\sigma(x)$. It takes values in the gauge algebra, which for the "comma" theory means that $A_\sigma(x)$ can be considered as a matrix in the index $\mathcal{X}(\sigma)$, $\sigma\in[0,\pi/2)$, denoting the configuration of the comma's tail. Thus we may write

$$A_\sigma(x)\sim A_\sigma[x;\mathcal{X}_1,\mathcal{X}_2]$$

which operates on $\psi[X]$ as matrices according to the rule

$$A_\sigma\psi=\int \delta\mathcal{X}_2 A_\sigma[x;\mathcal{X}_1,\mathcal{X}_2]\psi[x;\mathcal{X}_2] \ . \tag{7}$$

Under the gauge transformation (4), A_σ transforms as

$$\delta A_\sigma=[(L_\sigma-A_\sigma),\epsilon] \ , \tag{8}$$

where matrix multiplication is again everywhere implied, e.g.,

$$A_\sigma\epsilon=\int \delta\mathcal{X}_2 A_\sigma[x;\mathcal{X}_1,\mathcal{X}_2]\epsilon[x;\mathcal{X}_2,\mathcal{X}_3] \ . \tag{9}$$

Equivalently, one can consider A_σ as a matrix in $X(\sigma)$ ($\sigma=0\to\pi/2$),

$$A_\sigma\sim A_\sigma[X_1;X_2] \ ,$$

diagonal in $X(\pi/2)$,

$$A_\sigma[X_1;X_2]\propto\delta[X_1(\pi/2)-X_2(\pi/2)] \ ,$$

a notation we shall often use in what follows.

Once given the potential, we are used in ordinary gauge theories to constructing gauge-covariant quantities such as the covariant derivative of the wave function, or the field tensor F, which is the covariant curl of the gauge potential itself. Generalizing to the potentials A_l, the covariant derivative of the wave function is straightforward:

$$\mathscr{D}_l\psi=[D_l-A_l]\psi \ . \tag{10}$$

The covariant curl of the potential, however, requires a small modification. Suppose, introducing the anticommuting variables η^l as the differentials dual to D_l, we write

$$A=A_l\eta^l \tag{11}$$

as a one-form; then in analogy to the usual exterior derivative

$$\partial=\partial_\mu dx^\mu$$

we would be tempted here to write the exterior derivative of A as (replacing ∂_μ by $[D_l,\ ]$ and dx^μ by η^l)

$$D_? A=[D_l,A_m]\eta^l\eta^m \ .$$

This does not work, since $D_?{}^2\neq0$. However, by defining instead

$$D=\left[[D_l,\ ]\eta^l+\tfrac{1}{2}C^n_{lm}\eta^m\eta^l\frac{\partial}{\partial\eta^n}\right] \ , \tag{12}$$

where C^n_{lm} is the structure constant defined by

$$[D_l,D_m]=C^n_{lm}D_n \tag{13}$$

one retrieves nilpotency. We note that if D_l commutes, as when $D_l\to\partial_\mu$, one obtains back the usual exterior derivative.

Applied to the potential A_σ, this gives for the covariant derivative of the "comma" wave function ψ as

$$\mathscr{L}_\sigma\psi=(L_\sigma-A_\sigma)\psi \tag{14}$$

and for the exterior derivative of forms

$$Q=\left[[L_\sigma,\ ]\eta^\sigma+\tfrac{1}{2}C^{\sigma_3}_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2}\frac{\delta}{\delta\eta^{\sigma_3}}\right]. \tag{15}$$

Substituting the structure constants[5]

$$C^{\sigma_3}_{\sigma_1\sigma_2}=4i\pi\delta'(\sigma_1-\sigma_2)[\delta(\sigma_3-\sigma_1)+\delta(\sigma_3-\sigma_2)] \tag{16}$$

obtained from (6) and (13), one has

$$Q=\int_{-\pi/2}^{\pi/2}d\sigma\left[[L_\sigma,\ ]\eta^\sigma+4i\pi\eta^\sigma\eta'^\sigma\frac{\delta}{\delta\eta^\sigma}\right]. \tag{17}$$

By means then of this Q and the covariant derivative $\mathscr{L}_\sigma$ in (14), one can construct gauge-covariant quantities as in ordinary gauge theories. Thus, for example, in terms of the potential one-form,

$$A=\int_{-\pi/2}^{\pi/2}d\sigma\,A_\sigma\eta^\sigma\ , \tag{18}$$

the field tensor two-form is

$$F=QA+A\cdot A\ , \tag{19}$$

where matrix multiplication according to (7) is again understood. Under the transformation (4),

$$\mathscr{L}_\sigma\psi\to(1+\epsilon)\mathscr{L}_\sigma\psi\ , \tag{20}$$

$$F\to(1+\epsilon)F(1-\epsilon)\ . \tag{21}$$

To proceed further, it is convenient at this point to introduce a matrix representation also for the differentials η^σ, which have so far been defined only abstractly as the duals to the Virasoro operators L_σ. They are one-dimensional anticommuting fields, and can therefore be "bosonized" as[10]

$$\eta^\sigma=\frac{1}{\sqrt{2\pi}}:e^{\xi(\sigma)}:\ \ \bar\eta^{\,\sigma}=\frac{1}{\sqrt{2\pi}}:e^{-\xi(\sigma)}:, \tag{22}$$

where

$$\xi(\sigma)=\int_0^\sigma d\sigma'\left[\frac{\delta}{\delta\phi(\sigma')}+i\pi\phi'(\sigma')\right]. \tag{23}$$

As such, η^σ are linear operators in the functional space $\Pi\phi$ of a bosonic variable $\phi(\sigma)$; $\sigma=0\to\pi/2$, and can be represented as matrices in the representation where ϕ is diagonal: namely, as

$$\eta^\sigma[\phi_1;\phi_2]=\int\delta\phi\delta[\phi-\phi_1]\eta^\sigma\delta[\phi-\phi_2]\ . \tag{24}$$

They operate on functionals $\psi[\phi]\in\Pi\phi$ as

$$\eta^\sigma\psi=\int\delta\phi_2\eta^\sigma[\phi_1,\phi_2]\psi[\phi_2] \tag{25}$$

and multiply each other like matrices, e.g.,

$$\eta^{\sigma_1}\eta^{\sigma_2}=\int\delta\phi_2\eta^{\sigma_1}[\phi_1,\phi_2]\eta^{\sigma_2}[\phi_2,\phi_3]\ . \tag{26}$$

We note that in this representation, η^σ and $\bar\eta^{\,\sigma}$ are both diagonal in $\phi(\sigma')$, $\sigma'\geq\sigma$: namely,

$$\eta^\sigma[\phi_1;\phi_2]\propto\prod_{\sigma'\geq\sigma}\delta(\phi_1(\sigma')-\phi_2(\sigma'))\ . \tag{27}$$

Hence, the potential one-form A in (18) can be considered as a matrix now not only in X but also in ϕ; thus,

$$A=A[X_1,\phi_1;X_2,\phi_2]\ , \tag{28}$$

which is diagonal both in X and in ϕ at $\sigma=\pi/2$.

Suppose now we wish to construct an action which is invariant under "comma" gauge transformations. Let us consider first the pure gauge theory with only the gauge potential A as field variable. One can obtain a gauge invariant by taking the trace with respect to X of gauge covariants constructed out of A. For example, inspired by Witten[7] we can take the Chern-Simons three-form

$$C=A\cdot QA+\tfrac{2}{3}A\cdot A\cdot A \tag{29}$$

and obtain the gauge invariant

$$\int\delta X_1C[X_1,\phi_1;X_1,\phi_2]\ , \tag{30}$$

which is, however, still a matrix in ϕ. To get a scalar, we can take the trace again of (30) with respect to ϕ; thus,

$$\mathscr{A}_?=\int\delta X_1\delta\phi C[X_1,\phi_1;X_1,\phi_1]\ . \tag{31}$$

Such a trace, we think, should be taken for another reason. Our present representation in terms of ϕ comes from the differentials η^σ and $\bar\eta^{\,\sigma}$ which are defined only by their commutation relations. However, if one takes a conjugation of (22) with respect to any matrix in ϕ, then

$$\eta^\sigma\to M\eta^\sigma M^{-1},\ \ \bar\eta^{\,\sigma}\to M\bar\eta^{\,\sigma}M^{-1}\ , \tag{32}$$

the commutation relations remain unchanged and thus should give an equally good representation of the η's. We argue therefore that the action should also be invariant under conjugation of η by any matrix M of ϕ, and taking the trace with respect to ϕ guarantees such an invariance.

The straight trace $\mathscr{A}_?$ taken in (31), however, is no good as an action because it is identically zero. To see this, consider again η^σ as a matrix in ϕ. From the commutation relations[10]

$$\begin{aligned}[\eta^\sigma,\phi(\sigma')]&=\eta^\sigma,\ \ \sigma'<\sigma\\&=0,\ \ \sigma'>\sigma\end{aligned} \tag{33}$$

satisfied by η^σ as kink-creation operators, one sees that if $|\phi_2\rangle$ is an eigenstate of ϕ with eigenvalue ϕ_2, then $\eta^\sigma|\phi_2\rangle$ is also an eigenstate of ϕ but with value $\phi_2(\sigma')-1$ for $\sigma'<\sigma$ and $\phi_2(\sigma')$ for $\sigma'>\sigma$. In other words, the matrix $\langle\phi_1|\eta^\sigma|\phi_2\rangle$ has no nonvanishing diagonal elements, which in turn implies that the matrix $\langle\phi_1|A|\phi_2\rangle$ for A in (18) also cannot have nonzero diagonal elements, since $\phi_2(0)-\phi_1(0)=1$ for all terms in (18). Repeating the argument for any product of three η's easily shows that for any three-form, the matrix in ϕ can have nonvanishing elements only when $\phi_2(0)-\phi_1(0)=3$, so that, in particular, the trace of $\mathscr{A}_?$ in (31) must vanish.

To obtain a nonvanishing gauge invariant we have to doctor up the trace in such a way as to remove the kinks from the forms without destroying the basic invariance which is all tied up in the multiplication rules of A as "comma" matrices. Only at the point $\sigma=\pi/2$ is A not a matrix but only a number, where we can safely make some modifications.

To do this, it is more convenient to make a functional Fourier transform in the functions $\phi(\sigma)$ as follows. Define the Fourier transform of any form B

$$\widetilde{B}[\kappa_1,\kappa_2]=\int \delta\phi_1\delta\phi_2 \exp\left[i\int_0^{\pi/2}\kappa_1(\sigma)\phi_1'(\sigma)d\sigma\right]B[\phi_1,\phi_2]\exp\left[-i\int_0^{\pi/2}\kappa_2(\sigma)\phi_2'(\sigma)d\sigma\right]. \tag{34}$$

Notice that we have $\phi'(\sigma)$ in the exponent, *not* $\phi(\sigma)$ as usual, so that $\kappa(\sigma)$ is the conjugate variable to $\phi'(\sigma)$. Suppose now we take the Chern-Simons three-form C, Fourier transform to $\widetilde{C}$, and multiply by a factor $\exp[-i3\kappa(\pi/2)]$ [we need not specify whether $\kappa(\pi/2)$ is $\kappa_1(\pi/2)$ or $\kappa_2(\pi/2)$ since $\widetilde{B}$ is also diagonal in κ at $\sigma=\pi/2$]. This has the effect of inserting in $C[\phi_1;\phi_2]$ an extra kink of -3 units at $\sigma=\pi/2$, as can be seen by taking the inverse Fourier transform of the inserted factor. In other words, if we denote

$$\widetilde{C}\,\hat{}\,[\kappa_1;\kappa_2]=\exp[-i3\kappa(\pi/2)]\widetilde{C}[\kappa_1;\kappa_2] \tag{35}$$

then its Fourier transform $C\,\hat{}\,[\phi_1;\phi_2]$ will have no more kinks and can yield diagonal elements with $\phi_1(0)=\phi_2(0)$. Hence, the trace of $C\,\hat{}$ no longer needs to vanish, and since we have not changed anything for $\sigma\neq\pi/2$, it is still invariant under "comma" gauge transformations. We may therefore take

$$\mathscr{A}=\int \delta X_1\delta\phi_1 C\,\hat{}\,[X_1,\phi_1;X_1,\phi_1] \tag{36}$$

as action. Notice that when viewed in this way, the insertion factor in (35) seems natural but we can no longer claim any uniqueness for the action (36), as Witten has claimed for his. For it would appear that taking any form (e.g., a higher Chern-Simons form), one can always insert the right number of kinks at $\sigma=\pi/2$ before taking the trace to obtain a nonvanishing gauge invariant, and use that as an alternative action.

How is the theory of (36) related to the standard string field theory as formulated by Witten?[7] We have noted already that the potential one-form A, when considered as a matrix $\widetilde{A}[X_1,\kappa_1;X_2,\kappa_2]$ in X and κ, is diagonal in both X and κ at $\sigma=\pi/2$. Hence, it may be regarded also as a functional

$$\widetilde{A}[X_1,\kappa_1;X_2,\kappa_2]\sim\widetilde{\mathbf{A}}[\mathbf{X},\boldsymbol{\kappa}] \tag{37}$$

of the full string functions $\mathbf{X}(\sigma)$ and $\boldsymbol{\kappa}(\sigma)$ defined as

$$\begin{aligned}&\mathbf{X}(\sigma)=X_1(\sigma),\quad \boldsymbol{\kappa}(\sigma)=\kappa_1(\sigma),\quad \sigma<\pi/2\,,\\ &\mathbf{X}(\sigma)=X_1(\sigma)=X_2(\sigma),\quad \boldsymbol{\kappa}(\sigma)=\kappa_1(\sigma)=\kappa_2(\sigma),\quad \sigma=\pi/2\,,\\ &\mathbf{X}(\sigma)=X_2(\pi-\sigma),\quad \boldsymbol{\kappa}(\sigma)=\kappa_2(\pi-\sigma),\quad \sigma>\pi/2\,.\end{aligned} \tag{38}$$

Suppose we define now a string functional A_W as

$$A_W=\zeta^{-1}\widetilde{\mathbf{A}}\,, \tag{39}$$

where Witten's $A\sim$our A_W, his $\phi\sim$our κ, his $C(\sigma)\sim$our $\widetilde{\boldsymbol{\eta}}^{\sigma}$, and

$$\zeta=\exp[+i\tfrac{3}{2}\boldsymbol{\kappa}(\pi/2)]\,. \tag{40}$$

One sees then that A_W is an eigenstate of the ghost number operator:

$$\widetilde{N}=-i\int_0^{\pi}d\sigma\frac{\delta}{\delta\boldsymbol{\kappa}(\sigma)} \tag{41}$$

with eigenvalue $-\frac{1}{2}$, as required for the standard string functional. Indeed, the operator $\widetilde{N}$ counts the number of kinks: $\boldsymbol{\phi}(\pi)-\boldsymbol{\phi}(0)$ in the Fourier transform of (39), which is $-\frac{1}{2}$ with $\mathbf{A}$ accounting for 1 unit, and the factor ζ^{-1} accounting for $-\frac{3}{2}$.

The multiplication rule for "comma" matrices automatically guarantees that $A\cdot A$ has one kink more than A, or that its Fourier transform corresponds to a full string functional with ghost number one unit higher. However, in view of the extra factor ζ^{-1} introduced in (39) for the standard string functional A_W, an additional factor has also to be inserted for the product to retain this property, hence the factor ζ in Witten's $*$ product, which in our notation is

$$A_W * A_W=\zeta(\zeta^{-1}\widetilde{A})(\zeta^{-1}\widetilde{A})\,. \tag{42}$$

Furthermore, the action $\mathscr{A}$ in (36) can be written as

$$\mathrm{Tr}\{\zeta^{-1}[(\zeta^{-1}\widetilde{A})\zeta Q(\zeta^{-1}\widetilde{A})+\tfrac{2}{3}(\zeta^{-1}\widetilde{A})\zeta(\zeta^{-1}\widetilde{A})\zeta(\zeta^{-1}\widetilde{A})]\}\,, \tag{43}$$

which, in terms of A_W and the $*$ product in (42), is formally the same as Witten's action with the same insertion factor $\zeta^{-1}=\exp[-i\frac{3}{2}\kappa(\pi/2)]$ that he incorporated into the definition of his integral. Indeed, if one can show that the exterior derivative Q defined in (17) as operating on "comma" matrices has the same effect as the Becchi-Rouet-Stora (BRS) charge $\mathbf{Q}$ operating on the string functional A_W, one would have shown that the two theories are equivalent.

Unfortunately, one cannot quite do that for the following reason. In the definition of Q, all operators are supposed to be normal ordered, which depends on the mode expansion and hence on the boundary conditions chosen for the string. As noted already by Witten[7] it would seem necessary first to choose boundary conditions for the half-string or the "comma" in such a way as to obtain the Fock space of the full string as a tensor product of the two half-string spaces, which one does not know how to do yet.

However, suppose one can, for some reason, ignore this implicit normal ordering. Then one finds that the operator Q in (17) is indeed the same as the BRS $\mathbf{Q}$ for the full string. One can see this as follows. The operator Q operating on the "comma" potential one-form A gives explicitly

$$QA=\int_{-\pi/2}^{\pi/2}d\sigma\int d\sigma'([L_\sigma,A_{\sigma'}]\eta^\sigma\eta^{\sigma'}+4i\pi\eta^\sigma\eta'^\sigma\{\bar\eta^\sigma,\eta^{\sigma'}\}A_{\sigma'})\ . \tag{44}$$

Consider first the commutator

$$[L_\sigma,A_{\sigma'}]=L_\sigma A_{\sigma'}-A_{\sigma'}L_\sigma\ ,$$

where both L_σ and A_σ are operators on the functional space ΠX. Thus, operating on any wave functional $\psi[X]$, we have

$$L_\sigma A_{\sigma'}\psi=\int\delta X_2\left[-i\pi\frac{\delta}{\delta X_1^\mu(\sigma)}+X_1'^\mu(\sigma)\right]^2\times A_{\sigma'}[X_1;X_2]\psi[X_2]\ , \tag{45}$$

$$A_{\sigma'}L_\sigma\psi=\int\delta X_2A_{\sigma'}[X_1;X_2]\times\left[-i\pi\frac{\delta}{\delta X_2^\mu(\sigma)}+X_2'^\mu(\sigma)\right]^2\psi[X_2]\ . \tag{46}$$

Recalling now the definition of the full string functional **A** in (37) and (38), one sees from (45) that

$$L_\sigma A_{\sigma'}=\mathbf{L}_\sigma\mathbf{A}_{\sigma'},\quad 0\le\sigma\le\pi/2\ , \tag{47}$$

where

$$\mathbf{L}_\sigma=\left[-i\pi\frac{\delta}{\delta\mathbf{X}^\mu(\sigma)}+\mathbf{X}'^\mu(\sigma)\right]^2\ . \tag{48}$$

However, for (46), on integrating by parts with respect to X_2, one obtains

$$A_{\sigma'}L_\sigma\psi=\int\delta X_2\left[i\pi\frac{\delta}{\delta X_2^\mu(\sigma)}+X_2'^\mu(\sigma)\right]^2\times A_{\sigma'}[X_1;X_2]\psi[X_2] \tag{49}$$

or, from (37) and (38),

$$A_{\sigma'}L_\sigma=\mathbf{L}_{(\pi-\sigma)}\mathbf{A}_{\sigma'}\ . \tag{50}$$

Consider next the expression

$$I_\sigma=\int d\sigma'[L_\sigma,A_{\sigma'}]\eta^\sigma\eta^{\sigma'}\ . \tag{51}$$

Using the anticommutativity of the η's, this is

$$I_\sigma=L_\sigma\eta^\sigma A+AL_\sigma\eta^\sigma\ , \tag{52}$$

where both η^σ and A are operators on functionals of ϕ, say $\psi[\phi]$, explicitly:

$$\eta^\sigma A\psi=\frac{1}{\sqrt{2\pi}}\int\delta\phi_2e^{\xi_1(\sigma)}A[\phi_1;\phi_2]\psi[\phi_2]\ , \tag{53}$$

$$A\eta^\sigma\psi=\frac{1}{\sqrt{2\pi}}\int\delta\phi_2A[\phi_1;\phi_2]e^{\xi_2(\sigma)}\psi[\phi_2]\ , \tag{54}$$

where

$$\xi_i(\sigma)=\int_0^\sigma d\sigma'\left[\frac{\delta}{\delta\phi_i(\sigma')}+i\pi\phi_i'(\sigma')\right]\ . \tag{55}$$

Again, recalling (37) and (38), one has, from (53),

$$\eta^\sigma A=\boldsymbol{\eta}^\sigma\mathbf{A},\quad 0\le\sigma<\pi/2\ , \tag{56}$$

where

$$\boldsymbol{\eta}^\sigma=\exp\left[\int_0^\sigma d\sigma'\left[\frac{\delta}{\delta\boldsymbol{\phi}(\sigma')}+i\pi\boldsymbol{\phi}'(\sigma')\right]\right]\ , \tag{57}$$

whereas from (54), on integrating by parts with respect to ϕ_2, one has

$$A\eta^\sigma=\exp\left[-\int_{\pi-\sigma}^{\pi}d\sigma'\left[\frac{\delta}{\delta\boldsymbol{\phi}(\sigma')}+i\pi\boldsymbol{\phi}'(\sigma')\right]\right]\mathbf{A}[\boldsymbol{\phi}]\ , \tag{58}$$

where $\boldsymbol{\phi}(\sigma)=\phi_2(\pi-\sigma)$ for $\sigma>\pi/2$. The operator operating on $\mathbf{A}[\boldsymbol{\phi}]$ in (58) creates a kink at $\pi-\sigma$ and may thus be defined as $\boldsymbol{\eta}^{(\pi-\sigma)}$, from which we obtain

$$I_\sigma=\mathbf{L}_\sigma\boldsymbol{\eta}^\sigma\mathbf{A}+\mathbf{L}_{(\pi-\sigma)}\boldsymbol{\eta}^{(\pi-\sigma)}\mathbf{A} \tag{59}$$

for $0\le\sigma\le\pi/2$. Similarly, repeating the same arguments, one has, for the same range of σ,

$$I_{-\sigma}=\mathbf{L}_{-\sigma}\boldsymbol{\eta}^{-\sigma}\mathbf{A}+\mathbf{L}_{-(\pi-\sigma)}\boldsymbol{\eta}^{-(\pi-\sigma)}\mathbf{A}\ . \tag{60}$$

This gives, altogether for the first term in QA,

$$\int_{-\pi/2}^{\pi/2}d\sigma\,I_\sigma=\int_{-\pi}^{\pi}d\sigma\,\mathbf{L}_\sigma\boldsymbol{\eta}^\sigma\mathbf{A}\ . \tag{61}$$

Following a similar procedure for the second term in QA, again with willful neglect of questions concerned with normal ordering, one then arrives at the conclusion that Q operating on the "comma" matrix A has indeed the same effect as the BRS operator **Q** operating on the full-string functional **A**.

The formal equivalence obtained above between the "comma" gauge theory (36) and the open-bosonic-string theory as formulated by Witten is of uncertain significance at present because of normal ordering. However, assuming that this last important reservation can be removed, e.g., by an appropriate choice of boundary conditions on the "comma," then we may have learned something about the physical origin of the mysterious "chordal symmetry" underlying string field theory.

The "comma" theory may also have some interest in itself as an example of a local gauge theory of extended objects, and may show us the way to further generalizations. First, the extension to "commas" with internal degrees of freedom of the usual type, such as color, in addition to the intrinsic internal degree of freedom of its tail, is straightforward. The wave function carries then an additional index, and matrix quantities such as A are matrices also in these indices. It will lead to a "comma" theory equivalent to a string theory with internal symmetry inserted via the usual trace factors. Second, the action in (36) corresponds only to the pure gauge theory for the "comma" theory. One can, of course, in analogy to what one did in ordinary gauge theories, go further to consider adjoining to the pure action a term corresponding to the gauge interaction of the "comma" with the potential, e.g.,

$$\mathscr{A}\to\mathscr{A}+\bar\psi\mathscr{L}_\sigma\eta^\sigma\psi\ , \tag{62}$$

using the covariant derivative $\mathscr{L}_\sigma$ defined in (14). Third, one can imagine applying similar ideas even to extended objects of higher than one dimension.

Considered as a gauge theory, the action $\mathscr{A}$ in (36) has the unusual feature of being cubic in the potential and not quartic as in ordinary Yang-Mills theory. It yields the curvature $F=0$ as the equation of motion instead of the standard Yang-Mills equation which has the divergence of curvature $F_{\mu\nu;\nu}=0$, and may thus be regarded as a somewhat unnatural generalization. However, it is perhaps worth noting that even the standard Yang-Mills theory, when formulated in loop space, has the vanishing of the loop space curvature as an equation of motion.[11,12] There it was the monopole charge which appeared as the source of loop space curvature,[12] just as the "comma" charge occurs in (62) as the source of curvature for the "comma" potential.

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[1] T. Banks and M. Peskin, Nucl. Phys. **B264**, 513 (1986).
[2] M. Kaku and J. Lykken, CUNY report, 1985 (unpublished).
[3] D. Friedan, Nucl. Phys. **B271**, 540 (1986).
[4] A. Neveu and P. C. West, Nucl. Phys. **B268**, 125 (1986); Phys. Lett. **168B**, 192 (1986).
[5] W. Siegel and B. Zwiebach, Nucl. Phys. **B263**, 105 (1986).
[6] K. Itoh, T. Kugo, H. Kunitomo, and H. Ooguri, Prog. Theor. Phys. **75**, 162 (1986).
[7] E. Witten, Nucl. Phys. **B268**, 253 (1986); **B276**, 291 (1986).
[8] A. Connes, Publ. Math. I.H.E.S. **62**, 41 (1985).
[9] J-L. Gervais, Nucl. Phys. **B276**, 349 (1986).
[10] S. Mandelstam, Phys. Rev. D **11**, 3026 (1975).
[11] A. M. Polyakov, Nucl. Phys. **B164**, 171 (1979).
[12] Chan Hong-Mo, P. Scharbach, and Tsou Sheung Tsun, Ann. Phys. (N.Y.) **166**, 396 (1986); **167**, 454 (1986).