Fixed-angle asymptotic behavior of the type-I superstring amplitude

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The asymptotic limit of the four-point function at fixed angle is analyzed in the context of type-I superstrings. The analysis shows the amplitude to fall exponentially in s, at least at the one-loop level. This rate of decrease is faster than that which is allowed by the asymptotic theorem of Cerulus and Martin, but is consistent with the bound of Chiu and Tan.

Superstring theory has cured many of the "diseases" of the early dual resonance models. Specifically, the presence of tachyons, infinities, and anomalies are problems which have been resolved in spectacular fashion. A somewhat less severe disease of the early models was its violation—up to the one-loop level—of the asymptotic theorem regarding fixed-angle scattering. We have analyzed the four-point amplitude in type-I superstring theory, and find that it too has exponential (in s) behavior at fixed angle.

Our analysis parallels quite closely that of Alessandrini, Amati, and Morel,[1] who looked at the (nonplanar) Pomeron amplitude in the Veneziano model. These authors state but do not show that the planar one-loop amplitude has the same behavior as the Born term. This amplitude is, of course, divergent in the early theory. While one would not expect the interference between planar and twisted loops that eliminates the one-loop divergence to *increase* the wide-angle behavior, one can ask whether this interference makes the behavior *worse* (i.e., more rapidly decreasing). Our analysis finds that the interference does not affect the asymptotic fixed-angle behavior of the Veneziano model.

As is well known,[2] the four-point planar amplitude in superstring theory reduces to a kinematic factor times the Veneziano model amplitude. We begin with a review of this result in the covariant formalism which we use. We derive the amplitude for the (s,t) diagram; the (s,u) and (t,u) diagrams follow readily. The reader wishing to skip this review should go to Eq. (14).

The full one-loop amplitude (including odd twists) for four external gauge bosons is given by Eq. (47) of Ref. 3 [with $D=10$ and SO(32) as the internal symmetry]:

$$L=16ig^4\int d\Omega(\{F_{\rm NS}(w)[\hat{T}_{\rm NS}(z_i,w)-\hat{T}_R(z_i,w)] +(w\to we^{2\pi i})\}-\{w\to(-w)\})\ , \quad (1)$$

where

$$\hat{T}_{\rm NS}(z_i,w)\equiv(4\pi\epsilon)^{-D/2}\left\langle 0\left|\prod_{i=1}^{4}\hat{V}_B(z_i,d,\bar{d})\right|0\right\rangle \quad (2)$$

and

$$\hat{T}_R(z_i,w)\equiv(4\pi\epsilon)^{-D/2}\left\langle 0\left|\prod_{i=1}^{4}\hat{V}_F(z_i,d,\bar{d})\right|0\right\rangle\ . \quad (3)$$

The vertex operators for gauge-boson emission from a boson (B) or fermion (F) line are

$$\hat{V}_B(z,d,\bar{d})=e^{ik\cdot\hat{Q}(z)}[\zeta\cdot\hat{P}(z)+k\cdot H(z)\zeta\cdot H(z)] \quad (4)$$

and

$$\hat{V}_F(z,d,\bar{d})=e^{ik\cdot\hat{Q}(z)}[\zeta\cdot\hat{P}(z)-\tfrac{1}{2}k\cdot\Gamma(z)\zeta\cdot\Gamma(z)]\ . \quad (5)$$

The caret above a character in these expressions is defined in Ref. 3. We have dropped the primes on the variables z,w, etc., for brevity.

The vacuum expectation values in Eq. (2) can be rewritten as

$$\left\langle 0\left|\prod_{i=1}^{4}e^{ik_i\cdot\hat{Q}}(\zeta_i\cdot\hat{P}+k_i\cdot H\zeta_i\cdot H)\right|0\right\rangle=\left\langle 0\left|\prod_{i=1}^{4}e^{ik_i\cdot\hat{Q}}\right|0\right\rangle\left\langle 0\left|\prod_{i=1}^{4}(\zeta_i\cdot\hat{P}+\zeta_i\cdot\hat{B}+k_i\cdot H\zeta_i\cdot H)\right|0\right\rangle\ , \quad (6)$$

where $\hat{B}$ is a c number arising from the commutation of $\hat{Q}$ with $\hat{P}$. An analogous expression results for Eq. (3) with $H\to\Gamma/i\sqrt{2}$. In evaluating the last expectation value of Eq. (6), we encounter terms of order H^0, H^2, H^4, H^6, and H^8 (or $\Gamma^0,\ldots$). Of these, the first trivially vanishes when taking $\hat{T}_{\rm NS}-\hat{T}_R$ in Eq. (1). The next vanishes because $k_i\cdot\zeta_i=0$. The order-H^4 and -H^6 terms cancel when combined as in Eq. (1), as can be seen from the explicit forms of the expectation values. Finally, the H^8 term, when combined with $F_{\rm NS}(w)$ and the $w\to we^{2\pi i}$ terms, yields a kinematic factor K which is independent of (z_i). This factor is given in Ref. 2.

We are left with

$$L=16iKg^4(4\pi\epsilon)^{-D/2}\int d\Omega\left\langle 0\left|\prod_{i=1}^{4}e^{ik_i\cdot\hat{Q}}\right|0\right\rangle\ . \quad (7)$$

The vacuum expectation value in Eq. (7) can be written (with $\alpha'=\frac{1}{2}$) as

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$$\left\langle 0 \left| \prod_{i=1}^{4} e^{ik_i\cdot\hat{Q}} \right| 0 \right\rangle = \prod_{i<j} e^{-k_i^\mu k_j^\nu \langle 0|\hat{Q}_\mu(\rho_i)\hat{Q}_\nu(\rho_j)|0\rangle} . \quad (8)$$

The relevant correlation function is

$$\langle 0|\hat{Q}_\mu(\rho_i)\hat{Q}_\nu(\rho_j)|0\rangle = g_{\mu\nu}\left[\frac{1}{2\epsilon} - \ln\hat{\psi}(\rho_j/\rho_i,w)\right] , \quad (9)$$

where

$$\hat{\psi}(\rho_j/\rho_i,w) = -2\pi i \frac{\theta_1(\nu_{ij}\,|\,\tau)}{\theta_1'(0\,|\,\tau)} \quad (10)$$

and

$$\nu_{ij} = \frac{\ln(\rho_j/\rho_i)}{2\pi i} , \quad (11)$$

$$\tau = \frac{\ln w}{2\pi i} . \quad (12)$$

Taking the limit $\epsilon\to 0$ gives

$$L = 16iKg^4\delta^{10}\left[\sum_{i=1}^{4} k_i\right]\int d\Omega \prod_{i<j}\hat{\psi}(\rho_j/\rho_i,w)^{k_i\cdot k_j} . \quad (13)$$

With $s=-(k_1+k_2)^2$, $t=-(k_1+k_4)^2$, our amplitude then has the form

$$L = 16i(2\pi i)^3 g^4 K\delta^{10}\left[\sum k_i\right]\int_0^1 \frac{dw}{w}\int_0^1 d\nu_4 \int_0^{\nu_4} d\nu_3 \int_0^{\nu_3} d\nu_2 [e^{-sV_\lambda/2} - (w\to -w)] , \quad (14)$$

where

$$V_\lambda \equiv V_1 - \lambda V_2 , \quad (15)$$

with

$$\lambda = -t/s = \tfrac{1}{2}(1-\cos\theta_{\text{c.m.}}) \quad (16)$$

and

$$V_1 = \ln\frac{\theta_1(\nu_{12}\,|\,\tau)\theta_1(\nu_{34}\,|\,\tau)}{\theta_1(\nu_{13}\,|\,\tau)\theta_1(\nu_{24}\,|\,\tau)} , \quad (17)$$

$$V_2 = \ln\frac{\theta_1(\nu_{14}\,|\,\tau)\theta_1(\nu_{23}\,|\,\tau)}{\theta_1(\nu_{13}\,|\,\tau)\theta_1(\nu_{24}\,|\,\tau)} . \quad (18)$$

In Eq. (16), $\theta_{\text{c.m.}}$ is the scattering angle in the center-of-momentum frame.

The asymptotic evaluation of a multiple integral is most easily achieved using the method of steepest descent. The method requires the determination of critical points, which come in two varieties. If we are evaluating

$$I = \int \prod_{i=1}^{n} dx_i e^{f(x)} , \quad (19)$$

then a critical point of the *first* kind satisfies

$$\frac{\partial f}{\partial x_i} = 0, \quad i=1,\ldots,n \quad (20)$$

for some point $\{x_i\}_0$ within the region of integration. A critical point of the second kind can exist on a boundary of the integration region. Such a point satisfies the less restrictive condition that the derivatives need vanish only in the directions tangent to the boundary:

$$\frac{\partial f}{\partial x_i} = 0, \quad i=1,\ldots,n \text{ with } x_j \text{ at an end point.} \qquad i\neq j \quad (21)$$

In our case, we need to examine the derivatives of V_λ. The conditions (20) and (21) are somewhat more easily analyzed if we introduce a new set of "angular" variables (α,β,γ). We choose these so as to symmetrize V_1 and V_2. Letting

$$\alpha = \tfrac{1}{2}(\nu_{12}-\nu_{34}) ,$$
$$\beta = \tfrac{1}{2}(\nu_{12}+\nu_{34}) , \quad (22)$$
$$\gamma = \tfrac{1}{2}(\nu_{24}+\nu_{13}) ,$$

we have (omitting the τ dependence)

$$V_1 = \ln\frac{\theta_1(\alpha+\beta)\theta_1(\beta-\alpha)}{\theta_1(\gamma+\alpha)\theta_1(\gamma-\alpha)} , \quad (23)$$

$$V_2 = \ln\frac{\theta_1(\gamma+\beta)\theta_1(\gamma-\beta)}{\theta_1(\gamma+\alpha)\theta_1(\gamma-\alpha)} . \quad (24)$$

The region of integration is

$$\tfrac{1}{2} \ge \gamma \ge \beta \ge \alpha \ge 0 , \quad (25)$$

with

$$d\nu_2 d\nu_3 d\nu_4 = 4\,d\alpha\, d\beta\, d\gamma . \quad (26)$$

(The factor of 4 comes from our use of the symmetry of the integrand about $\alpha=0$ and $\gamma=\frac{1}{2}$.) It is straightforward to analyze the case $w=0$ and see that no critical points exist within the remaining region of integration. It is also straightforward to see that the separate vanishing of the derivatives of V_1 and V_2 cannot lead to a critical point. The more general case $w\neq 0$ is not so easy to analyze. We have not found any critical points of the first kind for this case. We consider then the possibility of changing the contour of integration in the ν_i in order to encounter a critical point. Indeed, there is a critical point of the second kind which can be reached by changing the contour of integration, and we use that point to approximate our integral for large s.

Noting that

$$\ln\left[\frac{\pi\theta_1(\nu\,|\,\tau)}{\theta_1'(0\,|\,\tau)}\right] = \ln(\sin\pi\nu) + 4\sum_{n=1}^{\infty}\frac{w^n}{1-w^n}\frac{\sin^2 n\pi\nu}{n} , \quad (27)$$

we define η according to

$$V_1=\ln\eta+4\sum_{n=1}^{\infty}\frac{w^n}{1-w^n}\frac{S_n}{n}\ . \tag{28}$$

Specifically,

$$\eta=\frac{\sin\pi v_{12}\sin\pi v_{34}}{\sin\pi v_{13}\sin\pi v_{24}}=\frac{\cos2\pi\alpha-\cos2\pi\beta}{\cos2\pi\alpha-\cos2\pi\gamma} \tag{29}$$

and

$$S_n=\cos2n\pi\alpha(\cos2n\pi\gamma-\cos2n\pi\beta)\ . \tag{30}$$

We then find that V_λ can be written in the form

$$V_\lambda=\ln\eta-\lambda\ln(1-\eta)+4\sum_{n=1}^{\infty}\frac{w^n}{1-w^n}\frac{S_n-\lambda T_n}{n}\ . \tag{31}$$

T_n is defined in analogy to S_n. Note that the integrand now has the form

$$e^{-s/2V_\lambda}=\eta^{1-\alpha(s)}(1-\eta)^{1-\alpha(t)}[1+O(w)]\ , \tag{32}$$

and bears some resemblance to the Veneziano Born term.

The integrand now can be seen to have a surface of critical points of the second kind at

$$w=0,\quad \left.\frac{\partial V_\lambda}{\partial\eta}\right|_{w=0}=0\ . \tag{33}$$

The solution to the latter equation is

$$\eta_0=\frac{1}{1-\lambda}\ . \tag{34}$$

Note that $\eta_0>1$, whereas η is restricted to be positive and *less* than 1 in the original region of integration. At η_0 we have (where $\phi_i=\alpha,\beta,\gamma$)

$$\left.\frac{\partial V_\lambda}{\partial\phi_i}\right|_{\eta_0}=\left.\frac{\partial V_\lambda}{\partial\eta}\right|_{\eta_0}\frac{\partial\eta}{\partial\phi_i}=0\ . \tag{35}$$

We now find it convenient to change variables to (η,β,γ), because the critical point will then be the sheet $\eta=\eta_0$, $w=0$. The integral over η then becomes nothing other than the Veneziano Born-term amplitude, and its asymptotic form is the same. We must also deal with the w integration, which is expedited by recalling that the region of convergence of our integral is a cone around the imaginary s axis. We expand $V_\lambda(\eta,w)$ about $w=0$, keeping only the linear term, and expand in η about η_0. We find

$$L=64i(2\pi i)^3g^4K\delta^{10}\left[\sum_i k_i\right]e^{-sV_\lambda(\eta_0,0)/2}\times\int_0^{1/2}d\beta\int_\beta^{1/2}\frac{d\gamma}{J_0}\int_0^{\eta_m}d\eta\int_0^1\frac{dw}{w}[e^{-s(\eta-\eta_0)^2V_\lambda''(\eta_0,0)/4}(e^{-swV_w/2}-e^{+swV_w/2})]\ , \tag{36}$$

where

$$V_w=\left.\frac{\partial V_\lambda}{\partial w}\right|_{\substack{w=0\\ \eta=\eta_0}}=(S_1-\lambda T_1)_{\eta=\eta_0}\ . \tag{37}$$

Note

$$V_w=\left[\frac{1-\lambda}{\lambda}\right](\cos2\pi\beta-\cos2\pi\gamma)^2>0\ . \tag{38}$$

J is the Jacobian $\partial\eta/\partial\alpha$, which we have approximated by its value J_0 at $\eta=\eta_0$.

Letting $s\rightarrow i\sigma$, the w integral becomes

$$\int_0^1\frac{dw}{w}\left[-2i\sin\left[\frac{\sigma wV_w}{2}\right]\right]\underset{\sigma\rightarrow\infty}{\longrightarrow}(-i\pi)\ . \tag{39}$$

The η integral is a simple Gaussian as $s\rightarrow\infty$, yielding

$$L=64\pi(2\pi i)^3g^4K\delta^{10}\left[\sum_i k_i\right]e^{-sV_\lambda(\eta_0,0)/2}\frac{\pi}{sV_\lambda''}\int_0^{1/2}d\beta\int_\beta^{1/2}d\gamma(1/J_0)\ . \tag{40}$$

The remaining integrations in Eq. (40) cannot be carried out in closed form. However, it is easy to verify that the integral is finite, and does not vanish. In fact, it is complex, as the integrand possesses a single square-root branch point within the region of integration.

Equation (40) represents our approximation for the asymptotic form of the amplitude. The dominant behavior for fixed λ is given by the exponential, with

$$V_\lambda(\eta_0,0)=-\ln[(-\lambda)^\lambda(1-\lambda)^{1-\lambda}]\ . \tag{41}$$

We have thus found the asymptotic behavior of the (s,t) diagram at the one-loop level. The (s,u) and (t,u) diagrams are quite similar, with only minor modifications. For the (s,u) diagram, we need only replace $t\rightleftarrows u$. The resulting change is that

$$V_\lambda^{su}=V_1-\lambda' V_2\ , \tag{42}$$

where

$$\lambda'=\tfrac{1}{2}(1+\cos\theta_{\text{c.m.}})=-u/s\ . \tag{43}$$

The asymptotic behavior is given by (40) and (41), with $\lambda \to \lambda'$.

For the (t,u) diagram, we find

$$V_\lambda^{tu} = -(\lambda' V_1 + \lambda V_2) \; . \quad (44)$$

In this case, the critical point is at $\eta_0 = \lambda'$ and the asymptotic behavior is given by (40), with

$$V^{tu}(\eta_0, 0) = -(\lambda \ln\lambda + \lambda' \ln\lambda') \; . \quad (45)$$

We now turn to the question of the allowed behavior of the amplitude, according to the asymptotic theorems. Assuming Mandelstam analyticity, Cerulus and Martin[4] have found the lower limit

$$|F(s, \cos\theta_{\text{c.m.}})| > \exp[-s^{1/2} \ln s C(\theta_{\text{c.m.}})] \; . \quad (46)$$

We see that the type-I superstring amplitude falls more rapidly than this at high energies, at least at the one-loop level. The Cerulus-Martin bound, of course, applies to the full amplitude and not necessarily to low-order terms in perturbation theory. Our result, therefore, serves only to point out that superstring perturbation theory is a poor approximation to the full theory in at least some regions of the kinematic variables. Chiu and Tan[5] have argued that, if there are indefinitely rising linear trajectories, the Cerulus-Martin bound can be exceeded. The superstring theory has such trajectories and is consistent with the Chiu-Tan result.

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