PHYSICAL REVIEW D VOLUME 35, NUMBER 8 15 APRIL 1987

New algorithm for asymptotic expansions of the heat kernel

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(Received 12 January 1987)

A new algorithm generating asymptotic expansions for general minimal differential operators of any order is derived. At each space-time point we introduce a tangent space forming the normal-coordinate system and a fiber frame obtained by a parallel transportation from the base point. The differential operators can be reexpressed in this local representation of vector bundle. With these operators we consider the heat kernels and derive an algorithm for the asymptotic expansions. We apply this method to most general fourth-order minimal differential operators in a curved space-time and find the first two terms of the expansions including the divergencelike terms which had been neglected in many calculations. Some interesting cases of general higher-order operators are also considered.

I. INTRODUCTION

Recently there has been much progress on fourth-order gravity theories.[1–6] In spite of the relatively long history of fourth-order gravity, it was only recently that these models proved to be renormalizable[3] and asymptotically free.[5,6] Many calculations on the one-loop β functions in these models, which determine whether or not these theories are asymptotically free, relied partly on the Schwinger-DeWitt proper-time algorithm.[7,8] A similar theory also exists in mathematics, usually called the "heat-kernel method."[9,10] The heat-kernel method is used to calculate the local index of various elliptic operators. The original DeWitt algorithm is restricted to second-order operators. Therefore to find β functions in fourth-order gravity, this algorithm must be generalized to higher-order operators.

This was done by many authors.[6,10–12] When a certain operator can be written as a product of second-order operators, we may obtain the functional determinant of this operator by multiplying that of each operator. Many works[6] on fourth-order gravity relied on such a property of functional determinants and used the results about second-order operators to find the one-loop ultraviolet divergences. In more recent work on the generalized Schwinger-DeWitt technique, Barvinsky and Vilkovisky,[11] using similar properties of functional determinants and the results on second-order operators, made a reduction scheme with which asymptotic expansion for a most general fourth-order operator was obtained. For another approach, there is an algorithm based on the pseudodifferential operators, which is studied mainly by mathematicians.[9,10]

In the algorithms where the multiplicative property of functional determinants is used, information about divergencelike terms[13] can be lost in the course of the multiplication of operators. For finding ultraviolet divergences, it may be safe to discard such terms. However, when we consider a theory on a manifold with a boundary, we should keep the divergencelike terms. Even for a theory on a manifold without a boundary, such terms can be important. For example, in quantum gravity, terms of the form $\nabla^2 R$ may be responsible for the conformal anomaly.[14] The Schwinger-DeWitt proper-time algorithm can be used to calculate a chiral anomaly,[15] where such divergencelike terms are necessary. On the other hand, in mathematics, various indices associated with pseudodifferential operators can be expressed as a total trace of an asymptotic coefficient acted by some operator. In such a calculation, we cannot neglect the divergencelike terms.

Recently two of us (Lee and Pac) formulated an algorithm[12] of finding asymptotic expansion for general minimal operators[16] with any order. This algorithm does not rely on the multiplicative property of functional determinants which has been used in many previous works. In Ref. 12, using this algorithm, we calculated an asymptotic series for a restricted class of fourth-order operators. Our results were different from the early works in the divergencelike terms. With our method we can evaluate an asymptotic series, in a direct manner, including the divergencelike terms for higher-order operators. Although our previous work presented a new way to calculate asymptotic series, it still requires some tedious combinatoric calculations.

In this paper we improve our previous algorithm by incorporating the normal-coordinate system accompanied by the parallel transportation on the fiber frame and present complete first two coefficient functions in the asymptotic series for a most general fourth-order minimal operator including divergencelike terms. In this paper we shall call such a formalism simply by "normal-coordinate method." The advantage of incorporating this normal-coordinate system of vector bundles is that an operator expressed in this representation can be expanded by tangent vectors covariantly. In our opinion, this formalism can be also used to obtain covariant momentum expansions in gauge theory and quantum gravity. Using the results on the asymptotic expansions, we will test the multiplicative property of functional determinant. We also calculate asymptotic series for a $2d$th-order operator obtained by multiplying second-order operators of the form ∇^2+C.

In the next section a general formalism on the algorithm for the asymptotic expansions is given. In Sec. III we apply this technique to various operators. The final section contains the discussions.

II. GENERAL FORMALISM

We begin this section by reviewing our previous works. Before doing this let us be precise about the notation. Throughout the paper we work in Riemannian space-time manifold and use a metric tensor $g_{\mu\gamma}(x)$ with metric signature $(+,\ldots,+)$. A curvature tensor is defined by

$$R_{\alpha\mu\gamma}{}^{\tau}=\partial_\alpha\Gamma_{\mu\gamma}{}^{\tau}-\partial_\mu\Gamma_{\alpha\gamma}{}^{\tau}+\Gamma_{\mu\gamma}{}^{\delta}\Gamma_{\alpha\delta}{}^{\tau}-\Gamma_{\alpha\gamma}{}^{\delta}\Gamma_{\mu\delta}{}^{\tau}\ , \quad (1)$$

where $\Gamma_{\mu\gamma}{}^{\tau}$ denotes the torsion-free Riemannian connection determined by $g_{\mu\gamma}(x)$. Ricci tensor and scalar curvature are defined by

$$R_{\mu\gamma}=R_{\alpha\mu\gamma}{}^{\alpha} \text{ and } R=R_\alpha{}^\alpha\ . \quad (2)$$

Denoting Yang-Mills fields by $A_\mu\equiv A_\mu{}^aT^a$ with group generator T^a, the covariant derivative can be written as

$$\nabla_\mu=\partial_\mu-iA_\mu+\Gamma_\mu\ , \quad (3)$$

where Γ_μ denotes the Riemannian connection operator acting on space-time tensors. Hereafter we shall represent the covariant derivative by a semicolon. For example, we use $R_{;\mu}{}^{\mu}$ for $\nabla_\mu\nabla^\mu R$.

In the proper-time method, the one-loop effective action is given by the logarithm of the determinant of some operator M:

$$\ln[\det(M)]=-\int_\xi^\infty\frac{d\tau}{\tau}\mathrm{Tr}[\exp(-\tau M)]\ , \quad (4)$$

where ξ denotes the proper-time cutoff. Here M can be any second-order partial differential operator with suitable background fields. Generally we may write $M=M(\nabla,\phi)$, where ϕ denotes the additional background fields. It is assumed that, when all the background fields vanish, M reduces to[16]

$$M_0\equiv(-\partial^2)^d\ . \quad (5)$$

A. The normal-coordinate method

The normal-coordinate method is a way of representing vector bundles by using the normal coordinates to describe the manifold and the fiber frame obtained by the parallel transportation from the base point to describe the fibers. Now we introduce this formalism briefly. Consider a heat kernel

$$\langle y\tau\,|\,x\rangle\equiv\langle y\,|\,\exp(-\tau M)\,|\,x\rangle\ . \quad (6)$$

Here any Green's function can be taken instead of a heat kernel. We can extend the heat kernel to tangent bundle $T\Sigma$ of space-time manifold Σ. To do this, we first consider a product space $\Sigma\times\Sigma$. Then a point in $\Sigma\times\Sigma$ is represented by (x,x'). $\Sigma\times\Sigma$ is locally diffeomorphic to $T\Sigma$. Each point in $T\Sigma$ is described by (x,X), where X is a tangent vector at $x\in\Sigma$. Consider a local diffeomorphism defined by

$$X_\alpha\equiv\sigma(x,x')_{;\alpha}\ , \quad (7)$$

where $\sigma(x,x')$ denotes a biscalar,[8] half the square of the geodesic distance from x to x'. In differential geometry, this diffeomorphism is called an "exponential map": $x'=\exp(-X)x$, and the tangent vectors form a normal-coordinate system. $\sigma(x,x')$ is also defined by

$$\begin{aligned}&\sigma(x,x')=\sigma(x',x)\ ,\\&\sigma_{;\alpha}\sigma^{;\alpha}=2\sigma\ ,\quad \sigma_{;\mu\gamma}(x,x)=g_{\mu\gamma}\ .\end{aligned} \quad (8)$$

The Riemannian measure on Σ is given by

$$dx\equiv g^{1/2}(x)d^nx\ , \quad (9)$$

where $g(x)=\det(g_{\mu\gamma})$ and n is the space-time dimension. The measures on $\Sigma\times\Sigma$ and $T\Sigma$ induced from Σ are given by

$$g^{1/2}(x)g^{1/2}(x')d^nx\,d^nx'=\Delta^{-1}(x,x')d^nx\,d^nX\ , \quad (10)$$

where

$$\Delta(x,x')=(g_xg_{x'})^{-1/2}\det(-\sigma_{;\mu\gamma'}) \quad (11)$$

with

$$\sigma_{;\mu\gamma'}=\nabla_\mu\nabla_{\gamma'}\sigma(x,x')\ . \quad (12)$$

For a given point x in Σ, we define the measure on the tangent space at x by

$$dX=dx'=g^{-1/2}(x')\Delta^{-1}(x,x')d^nX\ . \quad (13)$$

Then the measure on $T\Sigma$ can be written as $dx\,dX$.

Next we introduce the parallel transportation operator T defined by

$$\begin{aligned}&Tf(x,X)=I(x,x')f(x',X')\ ,\\&T^{-1}f(x,X)=I(x,\bar{x})f(\bar{x},\bar{X})\ ,\end{aligned} \quad (14)$$

where

$$\begin{aligned}&\sigma(x,x')_{;\mu}=X_\mu\ ,\quad \sigma(x',x)_{;\mu'}=X'_{\mu'}\ ,\\&\sigma(x,\bar{x})_{;\mu}=-X_\mu\ ,\quad \sigma(x,\bar{x})_{;\bar{\mu}}=\bar{X}_{\bar{\mu}}\ ,\end{aligned} \quad (15)$$

and $I(x,x')$ denotes the parallel transportation matrix from x to x' (Ref. 8) satisfying

$$\begin{aligned}&\sigma^\mu\vec{\nabla}_\mu I(x,x')=0=I(x,x')\overleftarrow{\nabla}_{\mu'}\sigma^{\mu'}\ ,\\&I(x,x')\,|_{x=x'}=\mathbb{1}\ ,\\&I(x,y)I(y,x')=I(x,x')\ ,\end{aligned} \quad (16)$$

for y lying on the geodesic from x to x'. Bitensor $g_{\mu\gamma'}(x,x')$ satisfies the similar equation except for

$$g_{\mu\gamma'}(x,x')\,|_{x=x'}=g_{\mu\gamma}(x)\ . \quad (17)$$

Now we return to Eq. (6). Regarding M as an operator on $T\Sigma$ by restricting it to the first coordinate of (x,x'), we can write

$$\langle y\tau | x\rangle = \int dY \langle yY \,|\, \exp(-\tau M) \,|\, x0\rangle$$
$$= \int dY\, I(y,y')\langle y'Y' \,|\, \exp(-\tau \overline{M}) \,|\, x0\rangle \ , \qquad (18)$$

where we used Eq. (13) and the relation $T^{-1}|x0\rangle = |x0\rangle$, and

$$\overline{M} \equiv T^{-1}MT$$
$$= M(\overline{\nabla},\overline{\phi}) \ , \qquad (19)$$

where $\overline{\nabla} = T^{-1}\nabla T$ and $\overline{\phi} = T^{-1}\phi T$.

Using Eq. (13) we can easily prove that

$$\overline{\nabla}_\mu = g_\mu{}^{\bar\gamma}(x,\bar x)\left[I(x,\bar x) I(\bar x,x)_{;\bar\gamma} - \sigma_{;\alpha\bar\gamma}\left[\frac{\partial}{\partial X_\alpha}\right]_x \right] . \qquad (20)$$

ϕ can have tensor indices, e.g., ϕ_μ. As an example consider ϕ_μ:

$$\overline{\phi}_\mu = g_\mu{}^{\bar\gamma}(x,\bar x) I(x,\bar x)\phi_{\bar\gamma}(\bar x) I(\bar x,x) \ . \qquad (21)$$

Since $\overline{\nabla}_\mu$ is not a differential operator for the first coordinate in (x,X), we can integrate Eq. (17) and find

$$\langle y\tau | x\rangle = I(\bar x,x)\langle X \,|\, \exp(-\tau\overline{M}) \,|\, 0\rangle_x \ , \qquad (22)$$

where $y=\bar x$. In the coincidence limit $y=x$, we obtain

$$\langle x\tau | x\rangle = \langle 0 \,|\, \exp(-\tau\overline{M}) \,|\, 0\rangle_x \ . \qquad (23)$$

Equations (22) and (23) describe the heat kernel by Green's functions defined on the tangent spaces. In Eqs. (22) and (23), the coordinate x is considered as a constant parameter. The merit of this representation of Green's functions is that these functions are covariant at x regardless of tangent vector X and x can be regarded as a constant. Since the diffeomorphism defined in Eq. (7) is a local one, Eqs. (22) and (23) may not hold exactly in curved space-time. However, the asymptotic series depends on the local property of the operator.[9] So we can safely use Eqs. (22) and (23) to find asymptotic series.

For the later calculation of asymptotic series, let us now expand $\overline{M}$ by X_μ. Since $I(x,\bar x)$ and $\sigma(x,\bar x)$ have well-behaved series expansions at $x=\bar x$ by X_μ, we can write

$$\overline{M} = \sum a_{\alpha\cdots\gamma\mu\cdots\delta}(x) X^{\alpha\cdots\gamma}\left[\frac{\partial}{\partial X_\mu}\right]\cdots\left[\frac{\partial}{\partial X_\delta}\right] , \qquad (24)$$

where the $a_{\alpha\cdots\delta}$'s are covariant tensor matrices and

$$X_{\alpha_1\cdots\alpha_n} \equiv X_{\alpha_1}\cdots X_{\alpha_n} \ . \qquad (25)$$

To expand $\overline{M}$ in the form (24), consider an operator

$$D \equiv X_\mu\left[\frac{\partial}{\partial X_\mu}\right] = \sigma^{;\bar\mu}\partial_{\bar\mu} \ . \qquad (26)$$

Since $[D,X_\mu]=X_\mu$ and $[D,\sigma^{;\bar\mu}]=\sigma^{;\bar\mu}$, we can say that operator D counts powers in X_μ and/or $\sigma_{;\bar\mu}$. By successive application of D we can expand various quantities appearing in $\overline{M}$.

In the case of flat space-time, since $g^\mu{}_{\bar\gamma} = g^\mu{}_\gamma = -\sigma^{;\mu}{}_{\bar\gamma}$, we may write

$$\overline{\nabla}^\mu(\bar x) = I(x,\bar x) I(\bar x,x)^{;\bar\mu} + \left[\frac{\partial}{\partial X_\mu}\right]_x \ . \qquad (27)$$

By a simple calculation we have

$$(D+1) I(x,\bar x) I(\bar x,x)_{;\bar\mu} = X^\alpha I(x,\bar x)\Omega_{\bar\alpha\bar\mu} I(\bar x,x) \ , \qquad (28)$$

where

$$\Omega_{\bar\mu\bar\gamma} \equiv [\nabla_{\bar\mu},\nabla_{\bar\gamma}] \ . \qquad (29)$$

Using Eq. (28), we can find

$$\overline{\nabla}^\mu = \left[\frac{\partial}{\partial X_\mu}\right]_x - \sum \frac{n-1}{n!} X_{\alpha_1\cdots\alpha_n}\Omega^{\mu\alpha_1;\cdots\alpha_n} \ . \qquad (30)$$

In curved space-time, although the calculations are somewhat complicated due to space-time curvature, in a similar way, we obtain

$$g(x\bar x)_\mu{}^{\bar\alpha} I(x\bar x) I(\bar x x)_{;\bar\alpha} = \tfrac{1}{2}X^\alpha \mathbb{R}_{\alpha\mu} + \tfrac{1}{3}X^{\alpha\delta}\mathbb{R}_{\alpha\mu;\delta} + \tfrac{1}{8}X^{\gamma\alpha\delta}\mathbb{R}_{\gamma\mu;\alpha\delta} - \tfrac{1}{16}X^{\gamma\alpha\epsilon}\sigma^{;\delta}{}_{\mu\gamma\alpha}\mathbb{R}_{\epsilon\delta} + O(X^4) \ , \qquad (31a)$$

$$-g(x\bar x)_\mu{}^{\bar\alpha}\sigma_{;\bar\alpha\gamma} = g_{\mu\gamma} - \sum_{n=2}^{\infty}\frac{1}{n!}X^{\alpha_1\cdots\alpha_n}\sigma_{;\gamma\bar\mu\bar\alpha_1\cdots\bar\alpha_n} \ , \qquad (31b)$$

$$\overline{\phi}_\mu = \sum_{n=0}^{\infty}\frac{1}{n!}X^{\alpha_1\cdots\alpha_n}\phi_{\mu;\alpha_1\cdots\alpha_n} \ , \qquad (31c)$$

where

$$\mathbb{R}_{\mu\gamma} = [\nabla_\mu,\nabla_\gamma] \ . \qquad (32)$$

In Eq. (31) $\mathbb{R}_{\mu\gamma}$ is a curvature operator acting on any tensor following it and all the tensors with the name σ have been assumed to be evaluated at $x=\bar x$. Since $-X^\alpha g_\alpha{}^{\bar\delta}\sigma_{\bar\delta\mu} = X_\mu$, each tensor with the name σ in Eq. (31.b) vanishes if symmetrized with respect to all barred indices, which can be used to find these tensors in terms of curvature tensor $R_{\mu\gamma\alpha\delta}$. Inserting these results into Eq. (31), we finally obtain

$$-g(x\bar x)_\mu{}^{\bar\alpha}\sigma_{;\bar\alpha\gamma} = g_{\mu\gamma} - \tfrac{1}{6}X^{\tau\alpha}R_{\gamma\alpha\mu\tau} - \tfrac{1}{12}X^{\tau\alpha\delta}R_{\gamma\alpha\mu\tau;\delta} + X^{\tau\alpha\delta\xi}\left(-\tfrac{1}{40}R_{\mu\alpha\gamma\tau;\delta\xi} + \tfrac{7}{360}R_{\mu\alpha\epsilon\tau}R^\epsilon{}_{\xi\gamma\delta}\right) + O(X^5) \ . \qquad (33)$$

Similar calculation gives, when $x=\bar{x}$,

$$\sigma^{;\alpha}{}_{\mu(\gamma\delta)}=-\tfrac{2}{3}R^{\alpha}{}_{(\gamma\delta)\mu} \; . \tag{34}$$

This completes our description on the normal-coordinate method.

B. Asymptotic expansions

In this subsection we derive an algorithm for asymptotic expansions using the representation of the heat kernel given in Eqs. (22) and (23). First consider the case of flat space-time and vanishing background fields. In this case, $M=M_0$ and we easily find

$$\overline{M}_0=\left[-\left[\frac{\partial}{\partial X_\mu}\right]_x\left[\frac{\partial}{\partial X^\mu}\right]_x\right]^d , \tag{35}$$

$$\langle X\mid \exp(-\tau\overline{M}_0)\mid 0\rangle_x=t^{-n}\Phi(\tfrac{1}{2}z^2) , \tag{36}$$

where

$$t=\tau^{1/2d} , \quad z_\mu=t^{-1}X_\mu ,$$
$$\Phi(\tfrac{1}{2}z^2)=\int\frac{d^np}{(2\pi)^n}\exp(ip^\mu z_\mu-p^{2d}) . \tag{37}$$

Next we consider $\langle X\mid \exp(-\tau\overline{M})\mid 0\rangle_x$, which can be written as

$$\begin{aligned}\langle X\mid \exp(-\tau\overline{M})\mid 0\rangle_x&=\langle X\mid[\exp(-\tau\overline{M})\exp(\tau\overline{M}_0)]\exp(-\tau\overline{M}_0)\mid 0\rangle\\&=[\exp(-\tau\overline{M})\exp(\tau\overline{M}_0)]\langle X\mid \exp(-\tau\overline{M}_0)\mid 0\rangle .\end{aligned} \tag{38}$$

In Eq. (38) the two operators within the brackets may be meaningless, when considered one by one, but not as a whole. From Eqs. (36) and (38), we have

$$\langle X\mid \exp(-\widetilde{M})\mid 0\rangle_x=t^{-n}[\exp(-\widetilde{M})\exp(\widetilde{M}_0)]\Phi , \tag{39}$$

where $\widetilde{M}=\tau\overline{M}$ and $\widetilde{M}_0=\tau\overline{M}_0$. Using Eq. (39), we calculate the asymptotic series. From Eqs. (24) and (39), we can see that it is not possible, except for $d=1$, to write

$$\langle X\mid \exp(-\widetilde{M})\mid 0\rangle_x=t^{-n}\Phi\sum t^k h_k(x,X)$$

with some functions h_k's which are regular at $X=0$. This is possible only when $d=1$. In the next two paragraphs, we show that for arbitrary d, $\langle X\mid \exp(-\widetilde{M})\mid 0\rangle_x$ has an expansion by z_μ and t rather than X_μ and τ.

From now on let us take as independent variables x, t, and z_μ. Note that

$$[\tilde{\partial}_\mu,\tilde{\partial}_\gamma]=0=[z_\mu,z_\gamma] ,$$
$$[\tilde{\partial}_\mu,z_\gamma]=g_{\mu\gamma}(x) , \tag{40}$$

where $\tilde{\partial}^\mu=(\partial/\partial z_\mu)_x$. In general, $\widetilde{M}$ can be expanded by t, i.e.,

$$\widetilde{M}=\widetilde{M}_0+t\widetilde{M}_1+t^2\widetilde{M}_2+t^3\widetilde{M}_3+t^4\widetilde{M}_4+\cdots . \tag{41}$$

From Eq. (35), it is obvious that

$$\widetilde{M}_0=(-\tilde{\partial}^2)^d \text{ with } \tilde{\partial}^2=\tilde{\partial}^\mu\tilde{\partial}_\mu . \tag{42}$$

For the operator inside the brackets in Eq. (39), we define operator m by

$$\exp(m)\equiv\exp(-\widetilde{M})\exp(\widetilde{M}_0) . \tag{43}$$

Since $\widetilde{M}-\widetilde{M}_0=O(t)$, m can be written as

$$m=\sum_{k=1}^{\infty}t^k m_k=tm_1+t^2m_2+t^3m_3+t^4m_4+\cdots . \tag{44}$$

Notice that $m=O(t)$. Hence to find asymptotic series, we can use

$$\langle X\mid \exp(-\widetilde{M})\mid 0\rangle_x=t^{-n}\sum_{k=0}^{\infty}\frac{1}{k!}(m)^k\Phi . \tag{45}$$

Generally $\widetilde{M}_k$ in Eq. (41) and m_k in Eq. (44) are regular at $z=0$ and have the form

$$\sum a^{\alpha\cdots\delta\mu\cdots\gamma}(x)z_{\alpha\cdots\delta}\tilde{\partial}_{\mu\cdots\gamma} , \tag{46}$$

where

$$z_{\alpha\cdots\delta}=z_\alpha\cdots z_\delta , \quad \tilde{\partial}_{\mu\cdots\gamma}=\tilde{\partial}_\mu\cdots\tilde{\partial}_\gamma . \tag{47}$$

Comparing Eq. (46) with Eq. (24) and noting that m is given by the sum of repeated commutators involving $\widetilde{M}$ and $\widetilde{M}_0$, we find

$$\#(z)-\#(\tilde{\partial})+2d=k \text{ and } \#(\tilde{\partial})\le 2d \text{ for } \widetilde{M} , \tag{48a}$$

$$\#(z)-\#(\tilde{\partial})+j2d=k \text{ for } m , \tag{48b}$$

where $\#(z)$ and $\#(\tilde{\partial})$ denote the degrees of the monomials of (47) in z and $\tilde{\partial}$, respectively, and j is a non-negative integer. On the other hand, $\tilde{\partial}^{\alpha\cdots\delta}$, when operated on Φ, yields a regular function around $z=0$.

From these results we conclude that Eq. (45) gives a well-defined series expansion by the variables z_μ and t:

$$\langle X\mid \exp(-\widetilde{M})\mid 0\rangle_x=t^{-n}\sum_{k=0}^{\infty}t^k h_k(x,z) , \tag{49}$$

where h_k's are C^∞ at $z=0$. In particular, we are interested in the coincidence limit of the heat kernel. From Eqs. (45) and (49), we conclude

$$\begin{aligned}\langle x\tau\mid x\rangle&\simeq t^{-n}\sum_{k=0}^{\infty}\frac{1}{k!}(m)^k\Phi\\&=t^{-n}\sum_{k=0}^{\infty}a_k(x)t^k ,\end{aligned} \tag{50}$$

where the symbol $\simeq$ means "equal to in the coincidence limit." Here we would like to emphasize that the expressions of the form (50) used in this paper always mean the asymptotic expansions. It may be impossible to write the exact form of $\langle x\tau | x \rangle$ in this way, which also depends on the global properties of M and the underlying manifold.

In the remaining part of this section, we explain how to calculate the series given in Eqs. (50). First we expand $\widetilde{M}$ by t as in Eq. (41) and write it in the form (46), using the results obtained in the last subsection, e.g., Eqs. (19)–(21), (24), and (31). For later use, we list some useful formulas. From Eq. (31), we find

$$t\overline{\nabla}_\mu = \widetilde{\partial}_\mu + t^2\left(-\tfrac{1}{6}z^{\tau\delta}\widetilde{\partial}^\alpha R_{\alpha\delta\mu\tau} + \tfrac{1}{2}z^\tau \mathbb{R}_{\tau\mu}\right) + t^3\left(-\tfrac{1}{12}z^{\tau\delta\gamma}\widetilde{\partial}^\alpha R_{\alpha\delta\mu\tau;\gamma} + \tfrac{1}{3}z^{\tau\delta}\mathbb{R}_{\tau\mu;\delta}\right)$$
$$+ t^4\left[z^{\tau\delta\gamma\xi}\widetilde{\partial}^\alpha\left(-\tfrac{1}{40}R_{\alpha\delta\mu\tau;\gamma\xi} + \tfrac{7}{360}R_{\mu\delta\lambda\tau}R^\lambda{}_{\xi\alpha\gamma}\right) + z^{\tau\delta\gamma}\left(\tfrac{1}{8}\mathbb{R}_{\tau\mu;\delta\gamma} + \tfrac{1}{24}R_{\mu\tau\delta}{}^\lambda \mathbb{R}_{\gamma\lambda}\right)\right] + \cdots , \tag{51a}$$

$$\overline{\phi}_\mu = \phi_\mu + tz^\gamma\phi_{\mu;\gamma} + \tfrac{1}{2}t^2 z^{\gamma\delta}\phi_{\mu;\gamma\delta} + \tfrac{1}{6}t^3 z^{\gamma\delta\alpha}\phi_{\mu;\gamma\delta\alpha} + \cdots . \tag{51b}$$

Inserting Eqs. (51) into Eq. (19), we obtain an expansion of $\widetilde{M}$ by t. In Eq. (51) do not confuse $\widetilde{\partial}_\mu$ with ∂_μ.

Next we calculate m defined in Eq. (43) using the Campbell-Baker-Hausdorf formula. Denoting $\epsilon \equiv -\widetilde{M} + \widetilde{M}_0$, we have

$$m = \ln[\exp(-\widetilde{M}_0 + \epsilon)\exp(\widetilde{M}_0)]$$
$$= \int_0^1 ds\, \epsilon_s + \tfrac{1}{2}\int_0^1 ds_1 \int_0^{s_1} ds_2 [\epsilon_{s_2}, \epsilon_{s_1}] + \tfrac{1}{6}\int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 ([[\epsilon_{s_1},\epsilon_{s_2}],\epsilon_{s_3}] + [\epsilon_{s_1},[\epsilon_{s_2},\epsilon_{s_3}]]) + \cdots \tag{52}$$
$$= (1 + \tfrac{1}{2}L + \tfrac{1}{6}L^2 + \tfrac{1}{24}L^3)\epsilon + \tfrac{1}{12}[\epsilon, L\epsilon] + \tfrac{1}{24}[\epsilon, L^2\epsilon] + \cdots , \tag{53}$$

where $\epsilon_s \equiv \exp(sL)\epsilon$, and $L \equiv [-\widetilde{M}_0,]$. From Eq. (48a) we easily find

$$L^k \widetilde{M}_j = 0 \quad \text{for } k \geq j . \tag{54}$$

Inserting Eq. (41) into Eq. (53) and using Eq. (54), we find, for the coefficient functions of the asymptotic series defined in Eq. (50),

$$a_0(x) \simeq \Phi , \quad a_2(x) \simeq \left[\tfrac{1}{2}\widetilde{M}_1{}^2 - \left[1 + \frac{L}{2}\right]\widetilde{M}_2\right]\Phi ,$$

$$a_4(x) \simeq \left[\frac{1}{4!}\widetilde{M}_1{}^4 - \frac{1}{3!}(\widetilde{M}_1{}^2\widetilde{M}_2 + \widetilde{M}_1\widetilde{M}_2\widetilde{M}_1 + \widetilde{M}_2\widetilde{M}_1{}^2) - \frac{1}{4!}L(3\widetilde{M}_1{}^2\widetilde{M}_2 + 2\widetilde{M}_1\widetilde{M}_2\widetilde{M}_1 + \widetilde{M}_2\widetilde{M}_1{}^2)\right.$$
$$+ (\tfrac{1}{2}\widetilde{M}_2 + \tfrac{1}{6}L\widetilde{M}_2)(\widetilde{M}_2) + (\tfrac{1}{8}L\widetilde{M}_2 + \tfrac{1}{3}\widetilde{M}_2)(L\widetilde{M}_2) + \tfrac{1}{24}(12 + 8L + 3L^2)\widetilde{M}_1\widetilde{M}_3$$
$$\left. + \tfrac{1}{24}(12 + 4L + L^2)\widetilde{M}_3\widetilde{M}_1 - (1 + \tfrac{1}{2}L + \tfrac{1}{6}L^2 + \tfrac{1}{24}L^3)\widetilde{M}_4\right]\Phi , \tag{55}$$

and so on. The a_k's for odd number k vanish, which will be clear in the next paragraph. In many cases, $\widetilde{M}_1 = 0$ and then Eq. (55) is greatly simplified.

Each term in Eq. (55) can be reduced into the standard form (46). Equation (48a) is also valid here. Then every term with $\#(z) > 0$ vanishes in the coincidence limit. So for the calculations of a_k using Eq. (55), it is sufficient to consider the terms of the form $a^{\mu\cdots\gamma}\widetilde{\partial}_{\mu\cdots\gamma}\Phi$. For this let us introduce the fully symmetric tensors defined by the recursive relation

$$g_{\mu_1\cdots\mu_k} = \sum_{j=2}^{k} g_{\mu_1\mu_j} g_{\mu_2\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_k} . \tag{56}$$

Then we can find

$$\widetilde{\partial}_{\mu_1\cdots\mu_{2j}}\Phi \simeq \frac{1}{C_j} g_{\mu_1\cdots\mu_{2j}} (\widetilde{\partial}^2)^j \Phi \tag{57}$$

and

$$(\widetilde{\partial}^2)^j\Phi \simeq (4\pi)^{-n/2}\frac{(-1)^j}{d}\Gamma\left[\frac{n+2j}{2d}\right] \Big/ \Gamma\left[\frac{n}{2}\right] , \tag{58}$$

where C_j satisfies the recurrence relation

$$C_j = n[C_{j-1} + j(j-1)C_{j-2}] \quad \text{and} \quad C_0 = 1 . \tag{59}$$

In the special case of $n = 4$, $C_j = (j+1)!2^j$. In conclusion, by using Eq. (51) to find the expansion (41), and applying Eqs. (55), (57), and (58), we can calculate the asymptotic series $t^{-n}\sum a_k t^k$.

Even with these tools, we still need some labor to calculate a_k. So now we briefly introduce techniques allowing us to contract tensors in advance. In calculating a_k, we frequently meet expressions of the form

$$(\tilde{\partial}^2)^i z_{\alpha_1\cdots\alpha_p}\tilde{\partial}_{\gamma_1\cdots\gamma_q}(\tilde{\partial}^2)^j\Phi \ . \qquad (60)$$

To find the coincidence limit of Eq. (60), normally, we reduce it into the standard form (46) by commuting z_α's with $\tilde{\partial}_\gamma$'s, and then use Eqs. (57) and (58). Since Eq. (60) is symmetric for the two sets of indices $(\alpha_1\cdots\alpha_p)$ and $(\gamma_1\cdots\gamma_q)$, it will be the sum of tensors of the form

$$g_{2r}^{\alpha_1\cdots\alpha_p;\gamma_1\cdots\gamma_q}\equiv g^{\alpha_1\alpha_2}\cdots g^{\alpha_{2r-1}\alpha_{2r}}\times g^{\alpha_{2r+1}\cdots\gamma_q}+\cdots \ , \qquad (61)$$

where $p+q=$ even and $p\geq 2r$. So it is convenient to introduce an operator S_{2r}^{pq} such that

$$(\tilde{\partial}^2)^i z^{\alpha_1\cdots\alpha_p}\tilde{\partial}^{\gamma_1\cdots\gamma_q}(\tilde{\partial}^2)^j\Phi\simeq\sum_{r=0}^{[p/2]}(\tilde{\partial}^2)^i S_{2r}^{pq}(\tilde{\partial}^2)^j\Phi g_{2r}^{\alpha_1\cdots\alpha_p;\gamma_1\cdots\gamma_q} \ . \qquad (62)$$

Hence we may write

$$z^{\alpha_1\cdots\alpha_p}\tilde{\partial}^{\gamma_1\cdots\gamma_q}\sim\sum_{r=0}^{[p/2]}S_{2r}^{pq}g_{2r}^{\alpha_1\cdots\alpha_p;\gamma_1\cdots\gamma_q} \ , \qquad (63)$$

where the symbol $\sim$ means "equal to in the sense of Eq. (62)." By a simple combinatoric calculation we find

$$(\tilde{\partial}^2)^i S_{2r}^{pq}\sim\frac{i!}{(i-p+r)!}\frac{2^{p-r}}{C_{(p+q)/2-r}}(\tilde{\partial}^2)^{i+(q-p)/2} \quad \text{for } i\geq p-r \ . \qquad (64)$$

Note that $S_{2r}^{pq}\sim S_{2r-2u}^{(p-u)(q-u)}$. Hence, denoting $S^{pq}\equiv S_0^{pq}$, we can write

$$S_{2r}^{pq}\sim S^{(p-r)(q-r)} \ . \qquad (65)$$

In this case, Eq. (64) reduces to

$$(\tilde{\partial}^2)^i S^{pq}\sim\frac{i!}{(i-p)!}\frac{2^p}{C_{(p+q)/2}}(\tilde{\partial}^2)^{i+(q-p)/2} \quad \text{for } i\geq p \ . \qquad (66)$$

Given p and q, only a subset of g_{2r}'s $(r=0,\ldots,[\min(p,q)/2])$ are linearly independent. For example, $g_4=g_2-g_0$ for $p=4$ and $q=2$, and $g_2=g_0$ for $p=3$ and $q=1$.

III. APPLICATIONS

In this section we calculate a_2 and a_4 for various higher-derivative operators defined in four-dimensional space-time. First we consider a most general fourth-order minimal operator

$$M=(\nabla^2)^2+B^{\mu\gamma\delta}\nabla_\mu\nabla_\gamma\nabla_\delta+C^{\mu\gamma}\nabla_\mu\nabla_\gamma+D^\mu\nabla_\mu+E \ , \qquad (67)$$

where the tensors $B^{\mu\gamma\delta}$ and $C^{\mu\gamma}$ are fully symmetric in their indices. Even if torsion is present, we can use Eq. (67) again, since torsion can be regarded as a tensor field.

First we should find $\tilde{M}$ and expand it by t. Before doing this, let us consider a second-order operator ∇^2. Using Eq. (51), we find

$$(\tilde{\nabla}^2)_2=-\tfrac{1}{3}z^{\mu\gamma}\tilde{\partial}^{\alpha\delta}R_{\alpha\gamma\delta\mu}-\tfrac{2}{3}z^\mu\tilde{\partial}^\alpha R_{\alpha\mu}+z^\mu\tilde{\partial}^\alpha\Omega_{\mu\alpha} \ , \qquad (68a)$$

$$(\tilde{\nabla}^2)_3=-\tfrac{1}{6}z^{\mu\gamma\sigma}\tilde{\partial}^{\alpha\delta}R_{\alpha\gamma\delta\mu;\sigma}+z^{\mu\gamma}\tilde{\partial}^\alpha(-\tfrac{5}{12}R_{\alpha\mu;\gamma}-\tfrac{1}{12}R_{\alpha\gamma\sigma\mu}{}^{;\sigma}+\tfrac{2}{3}\Omega_{\mu\alpha;\gamma})+\tfrac{1}{3}z^\mu\Omega_{\mu\alpha}{}^{;\alpha} \ , \qquad (68b)$$

where $\tilde{\nabla}_\mu=t\bar{\nabla}_\mu$. Here $\mathbb{R}_{\mu\gamma}$'s have been replaced by $\Omega_{\mu\gamma}$'s if they sit on the right-most side of each term. Note that $\mathbb{R}_{\mu\gamma}$ operates on any tensor index following it, whereas $\Omega_{\mu\gamma}$ does not. For instance, $[\mathbb{R}_{\mu\gamma},z_\alpha]=R_{\mu\gamma\delta\alpha}z^\delta$. Since we are interested in a_2 and a_4, from now on, we shall contract tensors with $O(t^4)$ in advance, as described in the last section. In this way, we obtain

$$(\tilde{\nabla}^2)_4\sim-\tfrac{1}{5}S'^{31}[R_{;\alpha}{}^{\alpha}+\tfrac{1}{9}(R^{\alpha\delta}R_{\alpha\delta}+\tfrac{3}{2}R^{\alpha\mu\gamma\delta}R_{\alpha\mu\gamma\delta})]+\tfrac{1}{4}S'^{20}\Omega_{\mu\gamma}\Omega^{\mu\gamma} \ , \qquad (68c)$$

where $S'^{31}=S^{31}+S^{20}$ and $S'^{20}=S^{20}+S_2^{20}$.

Now we turn to $\tilde{M}$. In a similar way, we find

$$\tilde{M}_1=\tilde{\partial}^{\mu\gamma\delta}B_{\mu\gamma\delta} \ , \quad \tilde{M}_2=\{\tilde{\partial}^2,(\tilde{\nabla}^2)_2\}+z^\mu\tilde{\partial}^{\gamma\delta\alpha}B_{\gamma\delta\alpha;\mu}+\tilde{\partial}^{\gamma\delta}C_{\gamma\delta} \ ,$$
$$\tilde{M}_3=\{\tilde{\partial}^2,(\tilde{\nabla}^2)_3\}+\tfrac{1}{2}z^{\mu\sigma}\tilde{\partial}^{\gamma\delta\alpha}(B_{\gamma\delta\alpha;\mu\sigma}-B_{\gamma\delta\epsilon}R_{\alpha\sigma}{}^{\epsilon}{}_{\mu})+z^\mu\tilde{\partial}^{\gamma\delta}(-2B_{\sigma\epsilon\delta}R_{\gamma}{}^{\sigma\epsilon}{}_{\mu}+\tfrac{3}{2}B_{\gamma\delta\sigma}\Omega_\mu{}^\sigma+C_{\gamma\delta;\mu})+\tilde{\partial}^\gamma D_\gamma \ . \qquad (69)$$

$\tilde{M}_4$ can be also found by contracting indices in advance:

$$\tilde{M}_4\sim\{\tilde{\partial}^2,(\tilde{\nabla}^2)_4\}+(\tilde{\nabla}^2)_2(\tilde{\nabla}^2)_2+S^{33}B_{\mu\gamma\delta}{}^{;\mu\gamma\delta}+\tfrac{3}{2}(S^{33}+S^{22})B_\alpha{}^{\alpha}{}_{\dot{\delta}}{}^{;\dot{\delta}}{}_{\mu}{}^{\mu}$$
$$-S^{22}B^\alpha{}_{\mu\gamma;\alpha}R^{\mu\gamma}+\tfrac{1}{2}(S^{22}+S^{11})(3B^\alpha{}_{\alpha\mu;\gamma}\Omega^{\gamma\mu}+2B^\alpha{}_{\alpha\mu}\Omega_\gamma{}^{\mu;\gamma})$$
$$-\tfrac{1}{4}(S^{11}+3S^{22})B^{\mu\gamma\delta}R_{\mu\gamma;\delta}+S^{22}C_{\mu\gamma}{}^{;\mu\gamma}+\tfrac{1}{2}(S^{22}+S^{11})C_\alpha{}^{\alpha;\delta}{}_\delta-\tfrac{1}{3}S^{11}C_{\alpha\delta}R^{\alpha\delta}+S^{11}D_\alpha{}^{;\alpha}+E \ , \qquad (70)$$

where the tensor with dotted indices is symmetrized in these indices. Hereafter we shall use this notation to represent the symmetrization.

Inserting Eqs. (69) and (70) into Eq. (55) and calculating the combinatoric factors using Eqs. (58) and (66), we can find a_2 and a_4. a_2 is given by

$$a_2(x)=\frac{\pi^{1/2}}{64\pi^2}\left(\tfrac{1}{3}R+\tfrac{1}{4}C-\tfrac{1}{8}g^{\mu\gamma\delta\alpha}B_{\mu\gamma\delta;\alpha}-\tfrac{1}{256}g^{\mu\gamma\delta\alpha\sigma\epsilon}B_{\mu\gamma\delta}B_{\alpha\sigma\epsilon}\right), \tag{71}$$

where $C\equiv C_\alpha{}^\alpha$. Next let us consider a_4. Because a_4 contains many terms, it may be necessary to write down the results systematically. Our result on a_4 is

$$a_4(x)=\frac{1}{32\pi^2}(h_0+h_1+h_2+h_3), \tag{72}$$

where

$$\begin{aligned}
h_0={}&-\tfrac{1}{90}R_{\alpha\delta}R^{\alpha\delta}+\tfrac{1}{90}R_{\alpha\mu\gamma\delta}R^{\alpha\mu\gamma\delta}+\tfrac{1}{36}R^2+\tfrac{1}{6}\Omega_{\alpha\delta}\Omega^{\alpha\delta}-\tfrac{1}{6}C_{\alpha\delta}R^{\alpha\delta}+\tfrac{1}{12}CR\\
&+\tfrac{1}{24}C_{\alpha\delta}C^{\alpha\delta}+\tfrac{1}{48}C^2-E+\tfrac{1}{15}R_{;\alpha}{}^{\alpha}-\tfrac{5}{18}C_{\alpha\delta}{}^{;\alpha\delta}+\tfrac{1}{9}C_\alpha{}^{\alpha;\delta}{}_\delta+\tfrac{1}{2}D_\alpha{}^{;\alpha},\\
h_1={}&\tfrac{1}{4}B^{\alpha\mu\gamma}{}_{;\alpha}R_{;\mu\gamma}-\tfrac{1}{8}B_\alpha{}^{;\alpha}R-\tfrac{1}{4}b_\alpha\Omega_\delta{}^{\alpha;\delta}-\tfrac{1}{8}C^{\alpha\delta}B_{\alpha\delta\sigma}{}^{;\sigma}-\tfrac{1}{8}b_\alpha C^{\alpha\delta}{}_{;\delta}-\tfrac{1}{16}Cb_\alpha{}^{;\alpha}\\
&+\tfrac{1}{16}\{b_\alpha,D^\alpha\}+\tfrac{1}{6}B_{\alpha\delta\sigma}{}^{;\alpha\delta\sigma}-\tfrac{1}{6}b_\alpha{}^{;\dot\alpha}{}_{\dot\delta}{}^{\delta}+\tfrac{1}{9}(b^\alpha R_{\alpha\delta}+3b^\alpha\Omega_{\delta\alpha})^{;\delta}+\tfrac{5}{72}[b_\alpha,\Omega_\delta{}^{\alpha;\delta}]\\
&+\tfrac{1}{24}([B_{\alpha\delta\sigma},C^{\alpha\delta;\sigma}]-[B_{\alpha\delta\sigma}{}^{;\sigma},C^{\alpha\delta}]+[C_{\alpha\delta},b^{\alpha;\delta}]-[C_{\alpha\delta}{}^{;\delta},b^\alpha])+\tfrac{1}{48}[b_\alpha,C^{;\alpha}],\\
h_2={}&\tfrac{1}{480}B_{000;\alpha}B_{000}{}^{;\alpha}-\tfrac{3}{80}B_{\alpha00}B_{\delta00}{}^{;\dot\alpha\dot\delta}+\tfrac{7}{80}b^\gamma B_{\alpha\delta\gamma}{}^{;\alpha\delta}+\tfrac{1}{40}b_{\gamma;\alpha\delta}B^{\alpha\delta\gamma}\\
&-\tfrac{1}{576}B_{000}B_{000}R-\tfrac{1}{32}B_{\alpha00}\Omega^{\alpha\delta}B_{\delta00}+\tfrac{1}{16}B_{\alpha00}B_{\delta00}\Omega^{\alpha\delta}+\tfrac{1}{32}R^{\alpha\delta}\{B_{\alpha\delta\sigma},b^\sigma\}\\
&-\tfrac{7}{8!}[\{(B_{000})^2,C_{00}\}+B_{000}C_{00}B_{000}]-\tfrac{1}{480}(B_{000}B_{000})_{;\alpha}{}^{\alpha}+\tfrac{1}{32}(B_{\alpha00}B_{00\delta}{}^{;\delta}-B_{\delta00}B_{00\alpha;}{}^{\delta})^{;\alpha}\\
&+\tfrac{1}{40}(B_{\alpha00}{}^{;\dot\alpha}B_{\delta00})^{;\dot\delta}+\tfrac{1}{40}(B^{\alpha\delta\sigma}b_\sigma)_{;\alpha\delta}+\tfrac{7}{80}[b_{\gamma;\alpha}B^{\alpha\delta\gamma}{}_{;\delta}],\\
h_3={}&-\frac{1}{16\times 5!}(\{B_{\alpha00},B_{000}\}B_{000}{}^{;\alpha}+B_{\alpha00}B_{000}{}^{;\alpha}B_{000})+\frac{1}{32\times 8!}(B_{000})^4\\
&+\frac{1}{32\times 6!}\{2(B_{000})^3{}_{;0}+[(B_{000})^2,B_{000;0}]\},
\end{aligned}$$

with $b_\alpha\equiv B_{\alpha\delta}{}^{\delta}$ and the notation $T_{0\cdots0}\equiv g^{\alpha\cdots\delta}T_{\alpha\cdots\delta}$.

Our results summarized in Eq. (72) include the divergence and the commutatorlike terms which vanish after taking a trace and integrating over a space-time manifold without boundary. By careful consideration of Eq. (72), we can see that our results are in complete agreement with the previous one,[11] up to these terms. When $B_{\alpha\delta\sigma}=0$, Eq. (72) is greatly simplified

$$a_4(x)=\frac{1}{32\pi^2}h_0(x)\quad\text{for } B_{\alpha\delta\sigma}=0\,. \tag{73}$$

As noted in our previous paper,[12] the divergencelike terms in Eq. (73) disagree with the results obtained in Ref. (6), where the multiplicative property of functional determinant had been assumed.

Next we consider $2d$th-order operators of the form

$$M=(-1)^d(\nabla^2+C_1)\cdots(\nabla^2+C_d)\,. \tag{74}$$

For these operators, $\widetilde{M}_1=0$ and therefore the calculations are relatively simple. Here, we omit the detailed calculations and report only the results, which can be summarized as

$$\begin{aligned}
a_2(x)&=\frac{1}{d}\Gamma\left[1+\frac{1}{d}\right]\sum a_{j2}(x)\,,\\
a_4(x)&=\frac{1}{d}\sum_j a_{j4}(x)+\frac{1}{16\pi^2 d}\left[\tfrac{1}{2}\sum_{j>i}[C_j,C_i]-\tfrac{1}{12}\sum C_{j;\alpha}{}^{\alpha}[3(j-1)(j-2)+(d-1)(d-2)-6(d-2)(j-1)]\right],
\end{aligned} \tag{75}$$

where a_{j2} and a_{j4} are the corresponding coefficient functions for the operator ∇^2+C_j

$$a_{j2}(x)=\frac{1}{16\pi^2}C_j\ ,$$

$$a_{j4}(x)=\frac{1}{16\pi^2}\Big(-\tfrac{1}{180}R_{\alpha\delta}R^{\alpha\delta}+\tfrac{1}{180}R_{\alpha\mu\gamma\delta}R^{\alpha\mu\gamma\delta}+\tfrac{1}{72}R^2+\tfrac{1}{12}\Omega_{\alpha\delta}\Omega^{\alpha\delta}+\tfrac{1}{6}C_jR+\tfrac{1}{2}C_j^{\,2}+\tfrac{1}{30}R_{;\alpha}{}^{\alpha}+\tfrac{1}{6}C_{j;\alpha}{}^{\alpha}\Big)\ . \qquad (76)$$

In Eq. (75), the additional factor $1/d$ in a_4 disappears when we express the ultraviolet divergences by a momentum cutoff instead of proper-time cutoff ξ. Hence we again find that a_4 is an additive quantity up to the divergencelike terms.

IV. DISCUSSIONS

In the last section we used the algorithm obtained in Sec. II to calculate the first two terms in the asymptotic series for some higher-order operators. In our formulas the divergencelike terms are correctly included, which may have important physical consequences in higher derivative quantum field theories.

We postpone such considerations in the near future, and here we would like to discuss the additive property of a_4, which can be tested using Eqs. (75) and (72), for a restricted class of operators. From Eq. (75) we can see that a_4 of the operator (74) is the sum of those of the individual second-order operators up to the divergencelike terms. Using Eq. (72), we can prove that this is also true when the fourth-order operators are the product of general second-order minimal operators. We expect that this property of a_4 will be satisfied generally. However, when we calculate the divergencelike terms, we cannot use the additive property of a_4 at all.

ACKNOWLEDGMENTS

This work was supported in part by the Ministry of Education, Republic of Korea, and by the Korea Science and Engineering Foundation.

[1]R. Utiyama and B. S. DeWitt, J. Math. Phys. **3**, 608 (1962); S. Deser, in *Proceedings of the Conference on Gauge Theories and Modern Field Theories,* edited by R. Arnowitt and P. Nath (MIT, Cambridge, 1975); S. Weinberg, in *Proceedings of the XVII International Conference on High Energy Physics,* edited by J. R. Smith (Chilton, Didcot, 1974).
[2]M. Kaku, P. K. Townsend, and P. van Nieuwenhuizen, Phys. Rev. D **17**, 3179 (1978); P. K. Townsend and P. van Nieuwenhuizen, *ibid.* **19**, 3166 (1979); S. Ferrara and B. Zumino, Nucl. Phys. **B134**, 301 (1978).
[3]K. S. Stelle, Phys. Rev. D **16**, 953 (1977).
[4]J. Julve and M. Tonin, Nuovo Cimento B **46**, 137 (1978).
[5]E. Tomboulis, Phys. Lett. **97B**, 77 (1980).
[6]E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B201**, 469 (1982); Phys. Lett. **104B**, 377 (1980); Nucl. Phys. **B203**, 157 (1982); Phys. Lett. **110B**, 117 (1982).
[7]J. Schwinger, Phys. Rev. **82**, 664 (1951).
[8]B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965); Phys. Rep **19C**, 295 (1975).
[9]P. B. Gilkey, *Invariance Theory, Heat Equation, and the Atiyah-Singer Index Theorem* (Publish or Perish, Wilmington, Delaware, 1984), and references therein.
[10]P. B. Gilkey, J. Diff. Geom. **10**, 601 (1975).
[11]A. O. Barvinsky and G. A. Vilkovisky, Phys. Rep. **119**, 1 (1985).
[12]Hae Won Lee and Pong Youl Pac, Phys. Rev. D **33**, 1012 (1986).
[13]In a field theory with boundary, the boundary terms in the asymptotic expansions can arise from the boundary itself as well as interior points (see Ref. 9). Here we are interested in the latter case, which may be classified into two categories: divergencelike terms and topological terms. However, in this paper, we use the terminology "divergencelike terms" in a broad sense by including the possibility of the terms with the form of commutator.
[14]R. Critchley, Phys. Rev. D **18**, 1849 (1978).
[15]Choonkyu Lee, Hyunsoo Min, and P. Y. Pac, Nucl. Phys. **B202**, 336 (1982).
[16]By the "minimal operator," we mean any differential operator having the highest-order part of the form $(\nabla^2)^d$.