O(N)-symmetric $\lambda\phi^4$ theory: The Gaussian-effective-potential approach

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The O(N)-symmetric generalization of $\lambda\phi^4$ theory is studied in the Gaussian-effective-potential (GEP) approach. It is shown that the GEP encompasses and transcends the leading-order $1/N$ expansion results. We find two distinct, nontrivial versions of the $(3+1)$-dimensional theory: (i) "precarious ϕ^4 theory," with negative, infinitesimal λ_B, which coincides with the $1/N$ result; and (ii) "autonomous ϕ^4 theory," with positive, infinitesimal λ_B and an infinite wave-function renormalization, which is inaccessible to perturbative or $1/N$ methods, and could well have eluded lattice-based approaches. "Autonomous ϕ^4 theory" can exhibit spontaneous symmetry breaking, and we speculate on its relevance for the Higgs mechanism.

I. INTRODUCTION

In this paper we employ the Gaussian-effective-potential (GEP) approach[1–4] to study the O(N)-symmetric $\lambda\phi^4$ theory:

$$H=\tfrac{1}{2}\dot{\boldsymbol{\phi}}^2+\tfrac{1}{2}(\nabla\boldsymbol{\phi})^2+\tfrac{1}{2}m_B{}^2\boldsymbol{\phi}^2+\lambda_B(\boldsymbol{\phi}^2)^2 \tag{1.1}$$

($\boldsymbol{\phi}^2\equiv\phi_1{}^2+\cdots+\phi_N{}^2$). This generalizes the GEP analysis of $N=1$ $\lambda\phi^4$ theory in Refs. 3 and 4, and allows us to compare the approach with the $1/N$ expansion.[5–10] We shall show explicitly that *the GEP contains the leading $1/N$ result as its formal $N\to\infty$ limit.* It is also true that the GEP contains the one-loop result as its formal $\hbar\to 0$ limit.[2,3] Thus, the GEP encompasses both of the other popular approaches to "effective" potentials: it shares their virtues, but without necessarily sharing their limitations.

The plan of the paper is as follows. Section II describes how to compute the GEP, in unrenormalized form, in theories with several scalar fields. It also shows how the $1/N$ result is recovered in the $N\to\infty$ limit. We discuss the (1 + 1)- and (2 + 1)-dimensional cases very briefly in Sec. III. The main point is that the absence of "spontaneous symmetry breaking[11–15]" (SSB) in the $1/N$ expansion is merely a consequence of the limitation of that method to rather weak coupling ($\lambda\sim 1/N$).

In 3 + 1 dimensions we find two distinct versions of the theory. (i) "Precarious ϕ^4 theory," described in Sec. IV, is the O(N) generalization of the results in Refs. 3 and 4. It is seen to be essentially identical to the theory uncovered by the $1/N$ analysis.[7–9] It has a *negative,* infinitesimal λ_B, but nevertheless appears to be stable.[16,3] (ii) "Autonomous ϕ^4 theory" discussed in Sec. V, is a generalization of the results of Ref. 17, which are in turn related to work of Consoli and collaborators.[18–20] This theory has *positive,* infinitesimal λ_B and an infinite wave-function renormalization. It may exhibit SSB, but contains massless particles if the symmetry is unbroken. We show that this theory is not accessible to perturbation theory, the loop expansion, or the $1/N$ expansion. The existence of a nontrivial, positive-λ, $(\phi^4)_{3+1}$ theory is, of course, contrary to the "triviality" dogma.[21–25] However, we shall argue that, because of the strongly coupled nature of the theory, it is quite possible that it would elude the lattice-based analyses of Refs. 21 and 22. Thus, we believe that we have found the hiding place of the long-lost nontrivial ϕ^4 theory.

Our notation follows Refs. 3 and 4 and is summarized, together with key formulas, in the Appendix.

II. THE GEP FOR O(N)-SYMMETRIC ϕ^4 THEORY

Our first point is that the generalization of the Gaussian ansatz to the case of many fields is slightly subtle.[26] One should not merely write

$$\phi^i=\phi_0^i+\hat{\phi}^i(\Omega)\equiv\phi_0^i+\int(dk)_\Omega[a_\Omega^i(\mathbf{k})e^{-ik\cdot x}+a_\Omega^{\dagger i}(\mathbf{k})e^{ik\cdot x}]\ , \tag{2.1}$$

giving each component of the field the same mass Ω. Instead, for full generality, one should put

$$\phi^i=\phi_0^i+R^i{}_j(\theta_1,\ldots,\theta_{N-1})\hat{\phi}^j(\Omega_j)\ , \tag{2.2}$$

and treat the $N-1$ angles in the rotation matrix R^i_j, as well as the N different Ω_j's, as variational parameters. The reason for this form is easy to understand in the quantum-mechanical case:[26] The point is that the most general multidimensional Gaussian wave function is not "spherical," $\exp(-\frac{1}{2}\Omega\phi^i\phi_i)$, but "ellipsoidal", $\exp(-\frac{1}{2}\phi^i\Omega_{ij}\phi^j)$.

For an O(N)-symmetric theory, where only the shift ϕ_0^i sets a direction, it is easy to see that the variational solution for the angles $\theta_1,\ldots,\theta_{N-1}$ will be such that the eigendirections of the Gaussian wave functional are radial and transverse. Moreover, because of the remaining O($N-1$) symmetry, the $N-1$ transverse quantum fields will have equal mass parameters $\Omega_{\rm trans}\equiv\omega$, while the radial quantum field will have a distinct mass parameter $\Omega_{\rm radial}\equiv\Omega$. Knowing this, we may simplify the calculation by adopting a coordinate system in which ϕ_0^i points in the $i=1$ direction, so that (with $\phi_0\equiv|\boldsymbol{\phi}_0|$)

$$\boldsymbol{\phi}^2=\phi_0^2+2\phi_0\hat{\phi}_1+\sum_1^N\hat{\phi}_i^2\ , \tag{2.3}$$

$$(\boldsymbol{\phi}^2)^2=\phi_0^4+4\phi_0^2\hat{\phi}_1^2+2\phi_0^2\sum_1^N\hat{\phi}_i^2+\left[\sum_1^N\hat{\phi}_i^2\right]^2+4\phi_0^3\hat{\phi}_1+4\phi_0\hat{\phi}_1\sum_1^N\hat{\phi}_i^2\ . \tag{2.4}$$

It is now straightforward to compute ${}_{\omega,\Omega}\langle 0|\mathscr{H}|0\rangle_{\Omega,\omega}$ to obtain[27]

$$V_G(\phi_0,\Omega,\omega)=[I_1+\tfrac{1}{2}(m_B^2-\Omega^2)I_0]+(N-1)[I_1'+\tfrac{1}{2}(m_B^2-\omega^2)I_0']+\tfrac{1}{2}m_B^2\phi_0^2+\lambda_B\phi_0^4$$
$$+\lambda_B[3I_0^2+(N^2-1)I_0'^2+2(N-1)I_0I_0'+6I_0\phi_0^2+2(N-1)I_0'\phi_0^2]\ , \tag{2.5}$$

where $I_0\equiv I_0(\Omega)$, $I_0'\equiv I_0(\omega)$, etc. Minimization of V_G with respect to Ω and ω leads to two coupled equations:

$$\Omega^2=m_B^2+4\lambda_B[(N-1)I_0'+3I_0+3\phi_0^2]\ , \tag{2.6a}$$

$$\omega^2=m_B^2+4\lambda_B[(N+1)I_0'+I_0+\phi_0^2]\ . \tag{2.6b}$$

Equations (2.5) and (2.6) specify the GEP $\overline{V}_G(\phi_0)$ in its unrenormalized form. For future use we record here some results which follow directly. The first derivative is[28]

$$\frac{d\overline{V}_G}{d\phi_0^2}=\frac{\partial V_G}{\partial\phi_0^2}=\tfrac{1}{2}\{m_B^2+4\lambda_B[(N-1)I_0'+3I_0+\phi_0^2]\}\ ,$$
$$=\tfrac{1}{2}(\Omega^2-8\lambda_B\phi_0^2)\ , \tag{2.7}$$

and, from (2.6),

$$\frac{d\Omega^2}{d\phi_0^2}=4\lambda_B[3+4(N+2)\lambda_B I'_{-1}]/A\ ,$$
$$\frac{d\omega^2}{d\phi_0^2}=4\lambda_B/A\ , \tag{2.8}$$

where

$$A=1+6\lambda_B I_{-1}+2(N+1)\lambda_B I'_{-1}+8(N+2)\lambda_B^2 I_{-1}I'_{-1}\ . \tag{2.9}$$

The second derivative is then

$$\frac{d^2\overline{V}_G}{d(\phi_0^2)^2}=\frac{2\lambda_B}{A}[1-12\lambda_B I_{-1}+4\lambda_B I'_{-1}-16(N+2)\lambda_B^2 I_{-1}I'_{-1}]\ . \tag{2.10}$$

We now want to demonstrate that the GEP contains the leading-order term of the $1/N$ expansion. We therefore consider the limit $N\to\infty$ with $(N\lambda_B)$ and (ϕ_0^2/N) held fixed, which reduces (2.5) to

$$\frac{1}{N}V_G=I_1'+\tfrac{1}{2}(m_B^2-\omega^2)I_0'+\tfrac{1}{2}m_B^2(\phi_0^2/N)+(N\lambda_B)(\phi_0^2/N)^2+(N\lambda_B)I_0'[I_0'+2(\phi_0^2/N)]\ , \tag{2.11}$$

dropping terms of order $1/N$. Note that the Ω parameter no longer appears: in this limit effects due to the single radial field are negligible compared to those of the $(N-1)$ transverse fields. In the same limit the ω equation (2.6) becomes

$$I_0'=(\omega^2-m_B^2)/(4N\lambda_B)-(\phi_0^2/N)\ . \tag{2.12}$$

Using this to eliminate I_0' in (2.11) one obtains

$$\frac{1}{N}\overline{V}_G=I_0'+\tfrac{1}{2}\omega^2(\phi_0^2/N)-\frac{1}{16(N\lambda_B)}(\omega^2-m_B^2)^2\ . \tag{2.13}$$

Recalling that $I_1'\equiv I_1(\omega)$ can be expressed as a $(\nu+1)$-dimensional Euclidean integral,[3]

$$I_1(\omega)=\frac{1}{2}\int\frac{d^{\nu+1}k_E}{(2\pi)^{\nu+1}}\ln(k_E^2+\omega^2)+\text{const}\ , \tag{2.14}$$

one recognizes Eqs. (2.12) and (2.13) as the standard leading $1/N$ result[6–10] for the effective potential, up to an irrelevant constant. Note that our parameter ω^2 is to be identified with the auxiliary field χ introduced in the $1/N$ approach.[6]

III. RESULTS IN 1 + 1 AND 2 + 1 DIMENSIONS

In this section we briefly discuss results for (1 + 1)- and (2 + 1)-dimensional theories. In these dimensions only a mass renormalization is necessary. Defining the renormalized mass to be the particle mass in the $\phi_0=0$ vacuum, we have

$$m_R^2\equiv\Omega^2|_{\phi_0=0}=\omega^2|_{\phi_0=0}=2d\overline{V}_G/d\phi_0^2|_{\phi_0=0}$$
$$=m_B^2+4(N+2)\lambda_B I_0(m_R)\ . \tag{3.1}$$

Using this equation to eliminate m_B^2, and subtracting the vacuum-energy constant

$$D_N\equiv N[I_1(m_R)-(N+2)\lambda_B I_0^2(m_R)]\ ,$$

leads to a manifestly finite GEP. Its form can be described by the following mnemonic:[29] take the unrenormalized expressions (2.5) and (2.6); write m_R^2 in place of m_B^2; and replace I_1, I_1', I_0, I_0' by their finite parts [defined by subtraction(s) at $\Omega=m_R$].

TABLE I. Critical values of $\hat{\lambda}$, which mark the onset of SSB (Ref. 11), for various N.

N	$N\hat{\lambda}_{\rm crit}(1+1)$	$N\hat{\lambda}_{\rm crit}(2+1)$
1	2.5527	3.0784
2	4.6862	5.6464
3	6.5680	7.8949
6	11.3591	13.5173
8	14.1286	16.6929
100	88.9605	89.2581

We have obtained numerical results for the GEP for various values of N and $\hat{\lambda}\equiv\lambda_B/m_R^{3-\nu}$. We shall not present these in detail, since the pictures are qualitatively similar to the $N=1$ case.[3] Basically, one sees the onset of SSB[11] as $\hat{\lambda}$ increases through some critical value $\hat{\lambda}_{\rm crit}$. Table I gives a list of values for $N\hat{\lambda}_{\rm crit}$ for various N's, which illustrates the fact that $N\hat{\lambda}_{\rm crit}\to\infty$ as $N\to\infty$. This explains why one does not see SSB[11] in the $1/N$ expansion:[6–10] The $1/N$ expansion is restricted to rather weak coupling $\hat{\lambda}=O(1/N)$ whereas SSB[11] occurs only for much larger $\hat{\lambda}$'s. This is one instance in which the GEP scores over the large-N approach.

A definite drawback of the Gaussian approach is that the Goldstone theorem[13] is not respected exactly.[19] The ω particle mass in the SSB vacuum in not exactly zero, as it should be. This is because the Gaussian ansatz does not obey the O(N) symmetry at the operator level. For example, substitution of the Gaussian ansatz for the field operators into the expression for the Noether current, J^μ does not give $\partial_\mu J^\mu=0$ when ϕ_0 is nonzero. Nevertheless, there is no reason to think that the problem is fatal. The resulting potential *does* satisfy the O(N) symmetry—it is a function of $\phi_0\equiv|\boldsymbol{\phi}_0|$ only. Moreover, the Goldstone theorem does emerge in an approximate sense.[19] Our numerical results show that in an SSB vacuum one always has $\omega^2<\frac{1}{4}\Omega^2$, and in many cases (especially for large $\hat{\lambda}$, or for large N) one finds $\omega^2\ll\Omega^2$. Thus, the ω particles, which should be the exactly massless Goldstone bosons, are indeed found to be much lighter than the Ω particle.

IV. PRECARIOUS ϕ^4 THEORY

We now turn to the (3 + 1)-dimensional case. Following Refs. 1–4 we define the renormalized mass by Eq. (3.1) and the renormalized coupling constant by

$$\lambda_R\equiv\frac{1}{4!}\frac{d^4\bar{V}_G}{d\phi_0^4}\bigg|_0=\frac{1}{2}\frac{d^2\bar{V}_G}{d(\phi_0^2)^2}\bigg|_0=\lambda_B\frac{1-8\lambda_B I_{-1}-16(N+2)\lambda_B^2 I_{-1}^2}{[1+2(N+2)\lambda_B I_{-1}](1+4\lambda_B I_{-1})}\,, \tag{4.1}$$

where $I_{-1}=I_{-1}(m_R)$ here.[30] It is straightforward, if tedious, to eliminate the bare mass using (3.1). The result is a somewhat lengthy expression whose form can be described as follows: copy Eq. (2.5) replacing m_B^2 by m_R^2 and I_1,I_1',I_0,I_0' by their finite parts $\Delta,\Delta',\Gamma,\Gamma'$ (see the Appendix); then, apart from the vacuum-energy constant D_N, the only additional terms in V_G are

$$\tfrac{1}{8}m_R^4 I_{-1}(m_R)[(x-1)^2+(N-1)(x'-1)^2]+\tfrac{1}{4}m_R^4\lambda_B[I_{-1}(m_R)]^2[3(x-1)^2+(N^2-1)(x'-1)^2+2(N-1)(x-1)(x'-1)]-m_R^2\lambda_B I_{-1}(m_R)\{(x-1)[(N-1)\Gamma'+3\Gamma+3\phi_0^2]+(N-1)(x'-1)[(N+1)\Gamma'+\Gamma+\phi_0^2]\}\,. \tag{4.2}$$

Mass renormalization of the Ω, ω equations gives

$$[1+6\lambda_B I_{-1}(m_R)](x-1)+2(N-1)\lambda_B I_{-1}(m_R)(x'-1)=(4\lambda_B/m_R^2)[(N-1)\Gamma'+3\Gamma+3\phi_0^2]\,, \tag{4.3a}$$

$$2\lambda_B I_{-1}(m_R)(x-1)+[1+2(N+1)\lambda_B I_{-1}(m_R)](x'-1)=(4\lambda_B/m_R^2)[(N+1)\Gamma'+\Gamma+\phi_0^2]\,. \tag{4.3b}$$

There are three ways in which (4.1) can give a finite λ_R, but two of them can be rapidly dismissed. First, λ_B could be finite, so that $\lambda_R=-2\lambda_B$. The Ω, ω equations (4.3) then imply $\Omega^2\simeq\omega^2\simeq m_R^2$, or more precisely,

$$(x-1)=2(\phi_0^2/m_R^2)/I_{-1}(m_R)+O(1/I_{-1}^2)\,,$$

and $(x'-1)=O(1/I_{-1}^2)$. This causes all terms in $\bar{V}_G$ to vanish except for the classical potential terms and two terms from (4.2), resulting in

$$\bar{V}_G-D=(\tfrac{1}{2}m_R^2\phi_0^2+\lambda_B\phi_0^4)+3\lambda_B\phi_0^4-6\lambda_B\phi_0^4=\tfrac{1}{2}m_R^2\phi_0^2-2\lambda_B\phi_0^4\,. \tag{4.4}$$

This potential is unbounded below as $\phi_0\to\infty$, so the theory is sick. The above analysis exactly parallels the $N=1$ case, so we may refer the reader to Ref. 3 for further discussion. Second, λ_B could be $-1/(4I_{-1})$. One can seen see, however, that this also gives a sick theory, once one realizes that here the ω, Ω equations are giving a *maximum*, not a minimum, of $\bar{V}_G(\phi_0;\Omega,\omega)$. Indeed V_G has no minimum in Ω, ω in such a theory.

The only interesting solution to (4.1) with finite λ_R is with

$$\lambda_B=\frac{-1}{2(N+2)I_{-1}(m_R)}\times\left[1+\frac{1}{2N\lambda_R I_{-1}(m_R)}+O\left[\frac{1}{I_{-1}^2}\right]\right] \tag{4.5}$$

which generalizes the $\lambda_B=-1/(6I_{-1})$, "precarious," theory of Refs. 3 and 4. Note that (4.5) is equivalent to

$$\lambda_B=\frac{-1}{2(N+2)I_{-1}(\mu)}\ , \tag{4.6}$$

where the dimensional-transmutation scale μ is related to the renormalized coupling constant by

$$\ln\left[\frac{m_R{}^2}{\mu^2}\right]=\frac{-4\pi^2}{N\lambda_R}\equiv\kappa\ . \tag{4.7}$$

With this form of λ_B the Ω, ω equations yield

$$x-x'=\frac{-4(\phi_0{}^2/m_R{}^2)}{NI_{-1}(m_R)}+O\left[\frac{1}{I_{-1}{}^2}\right]\ , \tag{4.8a}$$

$$(x-1)=\frac{4N\lambda_R}{m_R{}^2}\left[\Gamma(x)+\frac{\phi_0{}^2}{N}\right]\ . \tag{4.8b}$$

Noting that the first two of the three terms in (4.2) become proportional to $I_{-1}(m_R)(x-x')^2$ and hence of order $1/I_{-1}$, it is easy to see that the difference between x and x' can be neglected. It is then straightforward to simplify V_G to

$$V_G-D_N=N\left[\tfrac{1}{2}xm_R{}^2\frac{\phi_0{}^2}{N}-\frac{m_R{}^4}{16N\lambda_R}(x-1)^2+\Delta(x)\right]\ . \tag{4.9}$$

Equations (4.9) and (4.8b) are recognizable as the $N=1$ result [Eqs. (5.7) and (5.8) of Ref. 3] scaled by an overall N factor, and with $\lambda_R\to N\lambda_R$, $\phi_0{}^2\to\phi_0{}^2/N$. Thus, nothing changes when the large-N limit is taken, so, in this instance, the GEP result coincides with the leading-order $1/N$ result:[7–10] The terms by which the two approaches would otherwise differ are either reabsorbed by the slightly different renormalizations ($N+2$ in place of N), or simply vanish like $1/I_{-1}$.

For a detailed discussion of the interpretation of these results we can refer the reader to the literature.[7–10,3] The main points are as follows. (i) λ_R is negative,[3,8] (ii) the Ω equation (4.8b) has, in general, two solutions. However, one corresponds to a local *maximum* of $V_G(\phi_0;\Omega,\omega)$ and is not physically relevant. This was the cause of the tachyon problem found in Ref. 6 (see Refs. 7 and 8). (iii) When $\phi_0{}^2$ becomes too large, Eq. (4.8b) ceases to have a positive, real solution. However, before this happens the $\Omega=\omega=0$ end point takes over as the global minimum of $V_G(\phi_0{}^2;\Omega,\omega)$, and gives a constant potential.[10,3] (iv) The effective potential can therefore be visualized (for $N=2$ at least) as a bowl set into a table top. The bottom of the bowl is at $\phi_0=0$ and represents the O(N)-symmetric vacuum. The depth of the bowl below the table top is $N(2\kappa-1)/(128\pi^2)$, where κ is defined in (4.7), so that $\kappa>\frac{1}{2}$ is needed if the normal vacuum is to be stable.[10,3] This means that the renormalized coupling cannot be too large, and it also precludes a nontrivial massless form of the theory.[8,3] (v) If an ultraviolet cutoff were present, the table top would eventually curve downward, proportional to $-\phi_0{}^4/I_{-1}$, rendering the regularized theory unstable.[10,4,3] Reference 10 showed that the vacuum decay rate of the $\phi_0=0$ vacuum is suppressed by e^{-N} in the large-N limit. References 3 and 4 argue that, even for finite N, the vacuum decay rate should vanish as the cutoff is taken to infinity, because there is then an infinitely wide tunneling barrier. Thus, the final theory would be (just) stable.[16] The term "precarious" was introduced[4,3] to describe this situation where an instability of the regulated theory disappears once the regulator is removed.

Several other, quite different, approaches also point to a negative-λ theory (see references quoted in Ref. 3). The "precariousness" scenario suggests that such a theory can indeed have a physical meaning, and deserves to be taken seriously.

The phenomenological relevance of the negative-λ theory is quite another matter. We tend to suspect that it has none. Since it cannot sustain SSB it is difficult to imagine that it has anything to do with the Higgs mechanism. This raises the following question: "Whatever happened to SSB?" Consideration of this question led to the discovery of an exciting new possibility,[18,17] which we discuss in the next section.

V. AUTONOMOUS ϕ^4 THEORY

Recently, it has been realized that an alternative renormalization of the GEP for $(\lambda\phi^4)_{3+1}$ theory is possible.[31] In this section we shall derive the generalization of these results to the O(N) case. For convenience we refer to the new version of the theory as "autonomous ϕ^4." The name is meant to emphasize the separateness of this theory from the precarious version discussed earlier, and also (as we shall show) its lack of connection with perturbation theory or the $1/N$ expansion.

The key to autonomous ϕ^4 theory is an infinite rescaling of the classical field $\phi_0{}^2\to Z\phi_0{}^2$, which is equivalent to a wave-function renormalization, as is clear from the standard formula[32]

$$V_{\rm eff}(\phi_0)=-\sum_n[(2n)!]^{-1}(\phi_0{}^2)^n\Gamma^{(2n)}(0,\ldots,0) \tag{5.1}$$

relating the effective potential to the zero-momentum Green's functions of the symmetric vacuum (assuming no SSB).

We shall show that the following forms for the bare field, bare mass, and bare coupling constant lead to a finite, nontrivial GEP:

$$\phi_0{}^2=I_{-1}(\mu)\Phi_0{}^2\ , \tag{5.2}$$

$$m_B{}^2=-4(N+2)\lambda_B I_0(0)+m_0{}^2/[8I_{-1}(\mu)]\ , \tag{5.3}$$

$$\lambda_B=\alpha/I_{-1}(\mu)\ , \tag{5.4}$$

where α is an N-dependent number such that the *numerator* factor in Eq. (4.1) vanishes; i.e.,

$$1-8\alpha-16(N+2)\alpha^2=0 \tag{5.5}$$

(only the positive root is acceptable, see below).

The heuristic reasoning that leads to the above equations is similar to that described in Ref. 17. The uniqueness of these forms should become clear as we proceed to show how they work. Note that Eq. (5.3) means that the Ω, ω parameters vanish like $O(1/I_{-1})$ at the origin [cf.

Eq. (3.1)], which is necessary if the derivative $d\overline{V}_G/d\Phi_0^2|_0$ is to be finite in spite of the field rescaling. We also remark that (5.2)–(5.4) are sufficiently general. (i) By leaving the argument of I_{-1} in (5.4) as a free parameter μ, it is unnecessary to include an order $1/I_{-1}^2$ term explicitly. (ii) Terms vanishing faster than $1/I_{-1}$ in Eq. (5.3) would not contribute to the final result. Therefore, there is no loss of generality in choosing the argument of I_{-1} to be μ again. (iii) A further finite factor could be incorporated into (5.2)—indeed, later on it will be convenient to do so—but this is trivial and without physical consequences. (iv) We note that, as expected, the two bare parameters m_B, λ_B are being exchanged for precisely two renormalized parameters m_0, μ. The parameter μ is a dimensional transmutation[33] or "characteristic scale" parameter, analogous to the Λ parameter of QCD.

Proceeding now to substitute the renormalizations (5.2)–(5.4) into the Ω, ω equations (2.6), and making use of formulas from the Appendix, leads to

$$(1+6\alpha)\Omega^2+2(N-1)\alpha\omega^2=12\alpha\Phi_0^2+\epsilon_\Omega \ , \qquad (5.6a)$$

$$2\alpha\Omega^2+[1+2(N+1)\alpha]\omega^2=4\alpha\Phi_0^2+\epsilon_\omega \ , \qquad (5.6b)$$

where

$$\epsilon_\Omega=\frac{1}{8\pi^2 I_{-1}(\mu)}\left\{\pi^2 m_0^2+2\alpha\left[(N-1)\omega^2\left[\ln\frac{\omega^2}{\mu^2}-1\right]+3\Omega^2\left[\ln\frac{\Omega^2}{\mu^2}-1\right]\right]\right\} ,$$
$$\epsilon_\omega=\frac{1}{8\pi^2 I_{-1}(\mu)}\left\{\pi^2 m_0^2+2\alpha\left[(N+1)\omega^2\left[\ln\frac{\omega^2}{\mu^2}-1\right]+\Omega^2\left[\ln\frac{\Omega^2}{\mu^2}-1\right]\right]\right\} . \qquad (5.7)$$

Taking linear combinations of (5.6) yields

$$\Omega^2=8\alpha\Phi_0^2+2\{[1+2(N+1)\alpha]\epsilon_\Omega-2(N-1)\alpha\epsilon_\omega\}/\beta \ , \qquad (5.8a)$$

$$\omega^2=8\alpha\Phi_0^2/\beta+O(1/I_{-1}) \ , \qquad (5.8b)$$

where

$$\beta\equiv 3+4(N+2)\alpha=(1+4\alpha)/(4\alpha) \ , \qquad (5.9)$$

and where crucial use has been made of the fact that α obeys (5.5). Notice that Ω^2 and ω^2 emerge as proportional to Φ_0^2, up to terms vanishing like $1/I_{-1}$. Since $\Omega^2,\omega^2,\Phi_0^2$ are intrinsically positive, only the positive solution for α in (5.5) is acceptable.

It is now simplest to calculate $\overline{V}_G$ from Eq. (2.7) which becomes

$$\frac{d\overline{V}_G}{d\Phi_0^2}=\tfrac{1}{2}I_{-1}(\mu)(\Omega^2-8\alpha\Phi_0^2) \ . \qquad (5.10)$$

Using (5.8a), one observes that the divergent terms cancel leaving a finite term which simplifies to

$$\frac{d\overline{V}_G}{d\Phi_0^2}=\tfrac{1}{2}m_0^2\alpha+\frac{\alpha}{4\pi^2\beta}\left[(N-1)\omega^2\left[\ln\frac{\omega^2}{\mu^2}-1\right]+\beta\Omega^2\left[\ln\frac{\Omega^2}{\mu^2}-1\right]\right] . \qquad (5.11)$$

Dropping terms $O(1/I_{-1})$ one may now use just the leading terms of (5.8) and thus obtain the result explicitly as a function of Φ_0^2. Integration then yields

$$\overline{V}_G-D'=\tfrac{1}{2}m_0^2\alpha\Phi_0^2+\frac{\alpha^2\Phi_0^4}{\pi^2\beta}\left\{2(\beta-2)\left[\ln\left[\frac{8\alpha\Phi_0^2}{\mu^2}\right]-\frac{3}{2}\right]-(\beta-4)\ln\beta\right\} , \qquad (5.12)$$

where D' is the vacuum energy constant. We can tidy up this expression by defining

$$\tilde{\Phi}_0^2=\alpha\Phi_0^2 \ ,$$
$$\gamma^2=\tfrac{1}{8}\mu^2\exp\{1+\tfrac{1}{2}[(\beta-4)/(\beta-2)]\ln\beta\} \ , \qquad (5.13)$$

so that[34]

$$\overline{V}_G-D'=\tfrac{1}{2}m_0^2\tilde{\Phi}_0^2+\frac{2(\beta-2)}{\pi^2\beta}\tilde{\Phi}_0^4\left[\ln\left[\frac{\tilde{\Phi}_0^2}{\gamma^2}\right]-\frac{1}{2}\right] . \qquad (5.14)$$

Note that, in terms of N,

$$\beta=2+(N+3)^{1/2} \ . \qquad (5.15)$$

The shape of the potential depends on the ratio

$$\rho\equiv\frac{\beta}{(\beta-2)}\frac{m_0^2}{\gamma^2} \ , \qquad (5.16)$$

and is shown in Fig. 1. The figure shows a strong family resemblance to the lower-dimensional results (cf. Ref. 3). We may also note that the "Goldstone ratio"

$$\omega^2/\Omega^2=1/\beta \ , \qquad (5.17)$$

which is always less than $\frac{1}{4}$ and tends to zero as $N\to\infty$, shows the same behavior as in lower dimensions. These similarities to the better-understood superrenormalizable cases give us increased confidence that our results reflect some real physics.

Superficially, Eq. (5.14) shows a startling resemblance to the Coleman-Weinberg one-loop result.[33] However, this is illusory.[35] If one discards the I_0^2 terms of the GEP in (2.5), thereby obtaining[3] the unrenormalized one-loop result, and tries to renormalize it in the manner used above, one cannot obtain (5.14), or indeed anything sensible. When one examines where the $\hbar$'s should go one finds that the vanishing of the numerator of Eq. (4.1) [i.e., the condition (5.5)] requires a cancellation among dif-

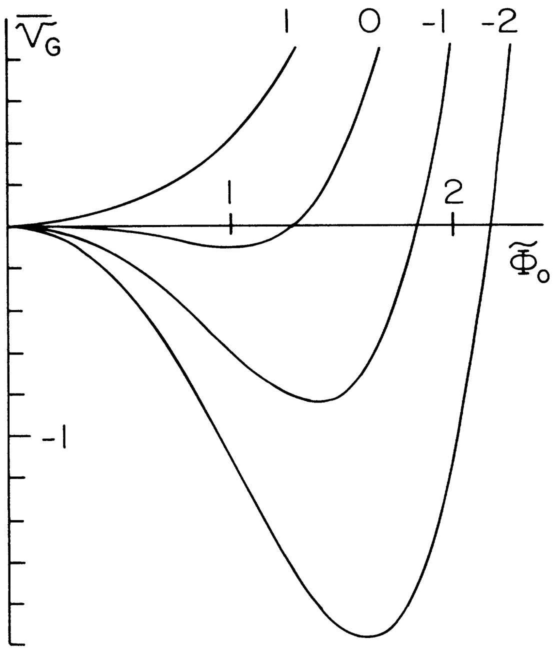

FIG. 1. The GEP of autonomous ϕ^4 theory for various values of the parameter ρ. See Eqs. (5.14)–(5.16). The curves represent a radial slice through the O(N)-symmetric potential, with $\overline{\mathcal{V}}_G \equiv (\overline{V}_G - D)/[\gamma^4(\beta-2)/\beta]$ and $\tilde{\Phi}_0$ in units of γ.

ferent orders in $\hbar$ thus making any distinction between classical and quantum terms meaningless. It leads to $\lambda_B = \alpha/[\hbar I_{-1}(\mu)]$, and hence to $1/\hbar$ factors in the GEP, which clearly shows that autonomous ϕ^4 lies outside the reach of the loop expansion.

Similarly, autonomous ϕ^4 is inaccessible to perturbation theory. Some insight into this fact can be gained by introducing a renormalized coupling constant

$$\lambda_R(M) \equiv \frac{\lambda_B}{1-\alpha^{-1}\lambda_B I_{-1}(M)} , \tag{5.18}$$

where M is some arbitrary scale (e.g., one could use $M = m_0$, as long as $m_0 \neq 0$). This is designed to be (i) finite, and (ii) of the perturbative form $\lambda_B[1+O(\lambda_B)]$ in the formal limit $\lambda_B \to 0$ (at fixed cutoff). Evaluating (5.18) gives

$$\lambda_R(M) = \frac{8\pi^2\alpha}{\ln(M^2/\mu^2)} \tag{5.19}$$

which one may use to eliminate the parameter μ^2 in favor of λ_R. This reveals that $\overline{V}_G$ contains $1/\lambda_R$ terms, and so cannot be expanded as a perturbative series in λ_R. (One could absorb the $1/\lambda_R$ by a further field rescaling, but this would give a $\ln\lambda_R$ which still resists power-series expansion.) It is interesting to note that (5.19) suggests that the theory is asymptotically free. Further study is needed to decide that question, however.

We can also show that autonomous ϕ^4 theory is not accessible to the $1/N$ expansion: it is too strongly coupled. Equations (5.4) and (5.5) imply that $N\lambda_B$ does not remain finite as $N \to \infty$, but diverges as $\sqrt{N}$ does. This recalls our observation in Sec. III that in lower dimensions the $1/N$ expansion does not see SSB because SSB occurs at values of λ out of its reach. Another point is the following. Autonomous ϕ^4 theory hinges on the cancellation between the two terms of (5.10), which could alternatively be expressed as

$$d\overline{V}_G/d\Phi_0{}^2 = \tfrac{1}{2} I_{-1}(\mu)[\omega^2 - 8\lambda_B(I_0' - I_0)] , \tag{5.20}$$

where again there is a cancellation between the two terms. In the $1/N$ expansion the second term would be discarded as order $1/N$, and so the cancellation would not be obtained. In our analysis λ_B is $O(1/\sqrt{N})$ and the $\sqrt{N}$ factor is gained back because $\Omega^2 \simeq \sqrt{N}\,\omega^2$, making the two terms of the same order in N.

Nothing resembling autonomous ϕ^4 theory has shown up in Monte Carlo studies,[21] which are all consistent with the "triviality" picture. However, this too may be understandable. In the $\Phi_0 = 0$ vacuum, Eq. (5.8) gives $\Omega^2 = \omega^2 = O(1/I_{-1})$, so that the symmetric phase of autonomous ϕ^4 theory is a massless theory. The singularity of the fourth derivative of $\overline{V}_G$ at the origin is another symptom of the infrared complications due to masslessness. By contrast, the Monte Carlo studies assume that one is looking for a regular, massive, unbroken-symmetry phase, and generally start out by fixing the renormalized mass to some finite, *nonzero* value.

A related argument is that autonomous ϕ^4 may have eluded Monte Carlo studies because it is a *strongly coupled* theory. This might seem a surprising statement given that λ_B vanishes as $1/I_{-1}$ does. However, one must take into account the wave-function renormalization as well, and if we count $\boldsymbol{\phi}^2$ as $O(I_{-1})$ then the interaction term $\lambda_B(\boldsymbol{\phi}^2)^2$ is *large*, $O(I_{-1})$: the kinetic term $\frac{1}{2}(\partial_\mu\boldsymbol{\phi})(\partial^\mu\boldsymbol{\phi})$ is of the same order, but the mass term (if we ignore the quadratic divergent terms[36]) is only $\frac{1}{2}[m_0{}^2/(8I_{-1})]\boldsymbol{\phi}^2 = O(1)$. As remarked by Khuri,[24] this situation is difficult to handle in Monte Carlo simulations.

The points raised above (and also the need for an infinite wave-function renormalization) may perhaps explain why autonomous ϕ^4 theory escapes the rigorous, lattice-based analyses,[22] which claim a proof of triviality given certain assumptions. We conjecture that autonomous ϕ^4 theory can be shown to lie outside this set of assumptions, though our woefully inadequate understanding of Ref. 22 prevents us from making any definite statement.

VI. CONCLUSIONS

We have shown how, even at the unrenormalized level, the GEP evidently contains the leading-order $1/N$ result. The GEP displays a much richer structure, though, which enables it to transcend the limitations of the large-N approach. In 1 + 1 and 2 + 1 dimensions the GEP results smoothly approach the $1/N$ results in the appropriate limit, but the GEP also shows that SSB[11] can occur for couplings $\hat{\lambda} \gg 1/N$ out of range of $1/N$ methods.

In 3 + 1 dimensions we find two quite distinct $\lambda\phi^4$ theories. One is precisely the theory found in the $1/N$ expansion.[7–10] Two points are added by the GEP analysis: (i) a strong indication that the earlier results are not mere artifacts of the $N\to\infty$ limit, but correspond to real physics at finite N, and (ii) some illumination of the puzzle of the theory's stability, even though λ_B is negative, through the notion of "precariousness."[3,4] In this version of the theory $\overline{V}_G$ has a weak-coupling expansion, which is related to perturbation theory.[3]

The other $(3+1)$-dimensional $\lambda\phi^4$ theory[17,18] is new and surprising—and extremely elusive. We have shown that it is inaccessible to perturbation theory, the loop expansion, or the $1/N$ expansion. We have also argued that it would probably have eluded the Monte Carlo studies[21] and the rigorous analyses,[22] principally because the new theory is strongly coupled, and in its symmetric phase the quanta are massless.

Of course the sceptical reader may prefer to attribute our findings to some artifact of the Gaussian approximation. We think there is a strong point against that view. If the situation had been reversed, and we had found only triviality in a theory which was known to be interacting, it would be very reasonable to believe that our free-field ansatz was simply inadequate. However, it is very difficult to understand, if $(\lambda\phi^4)_{3+1}$ genuinely were a free theory, why a variational approach based on a free-field ansatz would fail to confirm that fact. If the theory wants to be free, our Gaussian ansatz gives it every opportunity to say so.

The most exciting aspect of autonomous ϕ^4 theory (if one accepts its reality), is that it can display SSB. This clearly raises the question of its relevance for the Higgs mechanism in the standard model. The usual picture has SSB "preexisting" in the scalar sector which gives the gauge bosons masses once these are coupled. This picture has been questioned because of the triviality scenario.[37] However, if autonomous ϕ^4 is relevant, the usual picture would be right for the wrong reasons. In particular, the quantitative description of the Higgs phenomenon in classical or semiclassical terms would be completely wrong, but qualitatively the outcome would be similar. We hope to examine these questions in the near future by studying scalar electrodynamics in the GEP approach.[38]

Note added. A recent article by Y. Brihaye and M. Consoli [Nuovo Cimento A **94**, 1 (1986)] overlaps with our results somewhat. In particular, it contains results corresponding to the autonomous ϕ^4 theory, though for the special case $m_0=0$ only.

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APPENDIX: NOTATION AND FORMULAS

This appendix summarizes some notations and formulas from Refs. 3 and 4. In $\nu+1$ dimensions

$$(dk)_\Omega \equiv \frac{d^\nu k}{(2\pi)^\nu 2\omega_{\underline{k}}(\Omega)}\,,\quad \omega_{\underline{k}}{}^2=\mathbf{k}^2+\Omega^2\,, \tag{A1}$$

and the integrals

$$I_N(\Omega)\equiv\int (dk)_\Omega[\omega_{\underline{k}}{}^2(\Omega)]^N \tag{A2}$$

have the property

$$dI_N/d\Omega=(2N-1)\Omega I_{N-1}\,. \tag{A3}$$

The presence of an ultraviolet cutoff in these integrals is implicit. In 3 + 1 dimensions the integrals I_1,I_0,I_{-1} are, respectively, quartically, quadratically, and logarithmically divergent. They obey the algebra

$$\begin{aligned} I_1(\Omega)-I_1(m)&=\tfrac{1}{2}(\Omega^2-m^2)I_0(m)\\ &\quad-\tfrac{1}{8}(\Omega^2-m^2)^2I_{-1}(m)+\Delta(x)\,,\\ I_0(\Omega)-I_0(m)&=-\tfrac{1}{2}(\Omega^2-m^2)I_{-1}(m)+\Gamma(x)\,,\\ I_{-1}(\Omega)-I_{-1}(m)&=-\ln x/(8\pi^2)\,, \end{aligned} \tag{A4}$$

where

$$\begin{aligned} \Delta(x)&=m^4[2x^2\ln x-2(x-1)-3(x-1)^2]/128\pi^2\,,\\ \Gamma(x)&=m^2[x\ln x-(x-1)]/16\pi^2\,,\\ x&=\Omega^2/m^2\,. \end{aligned} \tag{A5}$$

Useful special cases are

$$\begin{aligned} I_1(0)-I_1(m)&=-\tfrac{1}{2}m^2I_0(0)+\tfrac{1}{8}m^4I_{-1}(m)\\ &\quad+\frac{3}{128\pi^2}m^4\,,\\ I_0(0)-I_0(m)&=\tfrac{1}{2}m^2\left[I_{-1}(m)+\frac{1}{8\pi^2}\right]. \end{aligned} \tag{A6}$$

(Note that I_{-1} is infrared singular when its mass argument tends to zero.)

We shall systematically use a prime to indicate when the mass argument is ω rather than Ω; e.g., I_0', Δ', Γ', x', etc. A distinction is maintained between V_G, which is a function of ϕ_0, Ω ω separately, and the GEP itself $\overline{V}_G$, which is a function of ϕ_0 alone; Ω and ω having been fixed so as to minimize V_G.

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[11]To avoid cumbersome circumlocutions we use the term "spontaneous symmetry breaking (SSB)" in any case where the GEP develops a minimum at $\phi_0 \neq 0$ which lies deeper than its value at $\phi_0 = 0$. Strictly speaking, though, this terminology is inappropriate in $1+1$ (or lower) dimensions. The point is that, in low dimensions, there can be mixing among the $O(N-1)$ degenerate "protovacua" with $|\boldsymbol{\phi}_0| = v$, so that $\langle \boldsymbol{\phi} \rangle$ remains zero and the $O(N)$ symmetry is preserved. This is very familiar in the $0+1$ (quantum-mechanics) case. Something similar is believed to happen in $1+1$ dimensions, thereby evading Coleman's theorem (Ref. 12), which forbids massless particles in $1+1$ dimensions, and hence, by Goldstone's theorem (Ref. 13) rules out true SSB. [As far as we understand the literature (Refs. 14 and 15), the result of the intervacuum mixing in the would-be SSB phase is a "Kosterlitz-Thouless phase," which "almost" displays the masslessness of a Goldstone phase in that the correlation functions fall off as an inverse power of the separation, instead of falling off with the exponential behavior characteristic of a massive intermediate particle.] In $2+1$ and higher dimensions the intervacuum mixing is expected to be suppressed by infinite-volume factors, so that the appearance of a $|\boldsymbol{\phi}_0| = v$ vacuum can legitimately be interpreted as SSB from $O(N)$ to $O(N-1)$ with the consequnt appearance of $N-1$ massless Goldstone bosons.

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[23]For heretical points of view see Refs. 24 and 25. As yet, however, we have been unable to make any connection between these heresies and our own.

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[27]The $N=2$ result has been given previously in Ref. 19.

[28]Unlike our previous practice, we work with the first (and second) derivatives with respect to ϕ_0^2 rather than the second (and fourth) derivatives with respect to ϕ_0.

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[30]This differs from the expression in Ref. 4, which was based on the more primitive Gaussian ansatz (2.1). Since, as we shall show, the Ω, ω distinction turns out to be negligible in precarious ϕ^4 theory, the other $O(N)$ results mentioned in Ref. 4 remain valid.

[31]This version of the theory was first discussed in Ref. 18. The question of whether it was truly renormalizable—i.e., completely cutoff insensitive—remained confused (cf. comments in Refs. 3, 19, and 20) until the analysis of Ref. 17, whose approach we follow here.

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