Classical adiabatic holonomy and its canonical structure

Ennio Gozzi
Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

William D. Thacker
Max-Planck-Institut für Physik und Astrophysik, Werner-Heisenberg-Institut für Physik, P.O. Box 40 12 12, Munich, Federal Republic of Germany

(Received 15 December 1986)

In this paper we introduce some mathematical tools to further study the classical adiabatic holonomy effect known as Hannay's angle. In particular, we prove with purely classical methods that the area (or angle) two-form associated with this effect can be seen as a modification of the *symplectic structure* of the slow-variable dynamics. We also show that, as in the quantum case, degeneracies cause singularities in this two-form. We conclude with some considerations concerning the triviality or nontriviality of the phase-space bundle associated with this phenomenon.

I. INTRODUCTION

The remarkable quantum adiabatic effect known as Berry's phase[1] has recently attracted considerable interest. This phenomenon occurs in certain systems which depend on a number of external parameters and consists of an additional phase acquired by the wave function during adiabatic transport around a closed loop in parameter space. Berry's phase has found applications in molecular[2] and solid-state[3] physics, in the understanding of anomalies in quantum field theory,[4] and its classical limit has even been detected in an optics experiment[5] performed recently.

Our understanding of Berry's phase and its implications has been deepened by a beautiful mathematical interpretation of this phenomenon given by Simon[6] and studied in some detail by Kiritsis.[7] It is possible to introduce a bundle of eigenstates of the parametrized Hamiltonian and a natural connection on it. Berry's phase is the bundle holonomy associated with this connection. The curvature of this connection, integrated over a closed two-dimensional surface in parameter space, gives the first Chern class which characterizes the topological twisting of the bundle. Using this formalism, Simon was able to relate the quantization of this characteristic class to the appearance of integers in the analysis of the quantum Hall effect by Thouless, Kohmoto, Nightingale, and Nijs.[8]

It has recently been found[9,10] that Berry's phase possesses a classical counterpart now known as Hannay's angle. This consists, for integrable systems, of an extra shift picked up by the angle variables of the system as the parameters undergo a closed adiabatic excursion. We have found this classical phenomenon worthy of study[11] and for a further introduction to the subject we suggest Refs. 9–11. In view of the great insight brought on by the differential geometric interpretation of Berry's phase, we thought it interesting to develop a similar mathematical framework for Hannay's angle. This is the purpose of this paper. Naturally the mathematical tools we will provide in this paper are only preliminary and we do pretend to exhaust the whole mathematical structure hidden behind the Hannay-angle phenomenon.

The paper is organized as follows. In Sec. II we review briefly the interpretation of Berry's phase as the holonomy of the connection in a Hilbert line bundle and set up a similar framework for the classical case, introducing an analogous structure which we call a phase-space bundle and deriving the parallel-transport equation in this new bundle. In Sec. III we promote the adiabatic external parameters to slowly changing dynamical variables and obtain a system with coupled fast and slow degrees of freedom. To obtain effective equations of motion for the slow variables, we have to average over the fast degrees of freedom. Then we find that the connection associated with Hannay's angle shows up as an effective gauge potential, as is familiar in the case of the quantum adiabatic holonomy.[12] What is new is that the curvature of this connection can then be interpreted as an additional term in the *symplectic form* on the slow-variable phase space. This additional term modifies some Poisson brackets and the canonical structure of phase space. Based on this phase-space analysis of the classical adiabatic holonomy, we put forward in Sec. IV some new ideas about the possible interplay between Hannay's angle and the quantum implementability of symmetries. Some mathematical details are confined to two appendices.

II. THE PHASE-SPACE BUNDLE

Let us begin by reviewing the geometric setting in which Berry's phase arises. In this case there is a family of Hilbert spaces, each one spanned by the eigenstates of a parameter-dependent Hamiltonian $H(B)$. This family forms a Hilbert bundle over parameter space. Anandan and Stodolsky[13] have shown how to describe Berry's phase as the holonomy in this bundle, for the case that the Hilbert space is finite dimensional, by considering all of the eigenstates of H together. Since in the adiabatic limit an eigenstate at a given energy level stays at the same level as the parameters change, one can also consider the Berry phase acquired by a single nondegenerate eigenstate.[6] We

 ©1987 The American Physical Society

find it more convenient for our purposes to follow this latter approach. When we restrict our attention to a single energy level $|n(B)\rangle$, we obtain as a subspace of the Hilbert bundle a line bundle in which the fiber at each parameter space point B consists of the ray in Hilbert space above $|n(B)\rangle$.

When the initial state $|n(B_0)\rangle$ is adiabatically transported in this bundle around a closed loop C in parameter space it acquires a dynamical phase $\exp[i\int_C E_n(B(t))dt]$ and the additional Berry phase $e^{i\gamma(C)}$. Berry's phase has a purely geometric nature and cannot be written in terms of an integral of the instantaneous energy $E_n(B(t))$. It is convenient to eliminate the dynamical phase by rescaling the Hamiltonian so that the nth level has zero energy. We can then introduce a local section in the Hilbert line bundle by choosing a particular phase for the solution $|n(B)\rangle$ of $H(B)|n(B)\rangle = E_n(B)|n(B)\rangle$, at each point of some neighborhood in parameter space.

Let us consider the evolution of the state $|\psi(t)\rangle$, initially given by $|\psi(0)\rangle = |n(B_0)\rangle$, as the parameters $B(t)$ change adiabatically. According to the quantum adiabatic theorem,[14] the adiabatically transported state $|\psi(t)\rangle$ can be written in terms of the local section $|n(B)\rangle$ as

$$|\psi(t)\rangle = e^{i\gamma(t)}|n(B(t))\rangle . \tag{1}$$

Simon[6] showed that the adiabatic transport of $|\psi(t)\rangle$ is equivalent to a parallel transport in the Hermitian line bundle given by the condition

$$\langle\psi(t+\delta t)|\psi(t)\rangle = 1 + O(\delta t^2) . \tag{2}$$

Physically this means that after the dynamical phase has been eliminated, the evolution of the state $|\psi\rangle$ is so slow that the projection of $|\psi\rangle$ at time t onto $|\psi\rangle$ at the infinitesimally later time $t+\delta t$ is just 1 to first order in δt.

Equation (2) implies that

$$\left\langle \frac{d\psi(t)}{dt} \middle| \psi(t) \right\rangle = 0 ,$$

which with the help of (1) yields

$$i\dot{\gamma}(t) + \langle n|\nabla_B|n\rangle \cdot \frac{d\mathbf{B}}{dt} = 0 . \tag{3}$$

This is the parallel-transport equation for the phase γ with the "gauge field"

$$\mathbf{A} = i\langle n(B)|\nabla_B|n(B)\rangle .$$

Integrating Eq. (3) from $t=0$ to $t=T$, we find the total phase

$$\gamma(T) = i\int_0^T \langle n(B(t))|\nabla_B|n(B(t))\rangle \cdot \frac{d\mathbf{B}}{dt} dt .$$

This is not invariant under parameter-dependent changes in the choice of phase for $|n(B)\rangle$ unless $B(T) = B(0) = B_0$, in which case the phase accumulated by $|\psi\rangle$ is given by Berry's expression[1]

$$\gamma_n(C) = i\oint \langle n(B)|\nabla_B|n(B)\rangle \cdot d\mathbf{B} \tag{4a}$$

which can be rewritten as the surface integral

$$\begin{aligned}\gamma_n(C) &= i\int_S\int \langle d_B n(B)| \wedge |d_B n(B)\rangle \\ &= -\int_S\int V_n ,\end{aligned} \tag{4b}$$

where $\partial S = C$ and

$$V_n = \mathrm{Im}[\langle d_B n(B)| \wedge |d_B n(B)\rangle] \tag{5}$$

is called the phase two-form.

In considering Hannay's angle we must restrict our attention to parameter-dependent Hamiltonians $H(B)$ which are integrable so that the system can be expressed in action-angle variables. The classical analog to the Hilbert bundle of the quantum case is a "phase-space bundle" in which the base space is again the parameter manifold and the fiber at the base space point B is the phase space for the system, filled by the invariant tori of $H(B)$. Local coordinates for points in the phase-space bundle are given by $(B;(\theta,I))$, where B stands for the M parameters characterizing a point in the base space, while (θ,I) represent the N angles and N actions characterizing a point in the fiber above B. We are interested in the evolution of a system whose trajectory starts on a given torus, fixed by the N actions $I=(I_1,\ldots,I_N)$, in the fiber above an initial parameter space point B_0, as the parameters change adiabatically. Since the actions are adiabatic invariants, we need not consider the entire phase-space bundle but can restrict our attention to the bundle of phase-space tori with the fixed set of actions I. This bundle of tori, with one torus at each point in parameter space, is the classical analog of the Hilbert line bundle discussed above.

In order to develop a notion of parallel transport in this torus bundle, we take the classical limit of Eq. (2). We use the semiclassical expression[10]

$$\psi(q;B) = \sum_\alpha a_{(\alpha)}(q,I;B)\exp\left[\frac{i}{\hbar}S^{(\alpha)}(q,I;B)\right] , \tag{6}$$

where $a_{(\alpha)}{}^2 = \det(\partial\theta_i^{(\alpha)}/\partial q_j)(1/2\pi)^N$ and α labels different branches of the multivalued generating function $S^{(\alpha)}(q,I;B)$. When (6) is substituted into the left-hand side of (2) only products of terms from the same branches of α survive, as explained by Berry,[10] giving

$$\langle\psi(t+\delta t)|\psi(t)\rangle = \int d^N q \sum_\alpha a_{(\alpha)}(q(t+\delta t),I;B(t+\delta t))a_{(\alpha)}(q(t),I;B(t)) \times \exp\left[\frac{i}{\hbar}[S^{(\alpha)}(q(t+\delta t),I;B(t+\delta t)) - S^{(\alpha)}(q(t),I;B(t))]\right] .$$

Expanding to first order in δt we get

$$\langle \psi(t+\delta t) \mid \psi(t) \rangle = \int dq \sum_{(\alpha)} \left\{ a_{(\alpha)}{}^2 + \delta t \left[a_{(\alpha)} \frac{da_{(\alpha)}}{dt} + a_{(\alpha)}{}^2 \left[-\frac{i}{\hbar} \right] \frac{dS^{(\alpha)}}{dt} \right] + O(\delta t^2) \right\}$$

$$= 1 + \delta t \left[\int dq \frac{1}{2} \frac{d}{dt} (a_{(\alpha)}{}^2) + \int \frac{d^N\theta}{(2\pi)^N} \left[-\frac{i}{\hbar} \right] \sum_{(\alpha)} \frac{dS^{(\alpha)}}{dt} \right] + O(\delta t^2)$$

$$= 1 - \frac{i}{\hbar} \delta t \int \frac{d^N\theta}{(2\pi)^N} \sum_{(\alpha)} \frac{dS^{(\alpha)}}{dt} .$$

Because of the Jacobian $a_{(\alpha)}{}^2$, the integral over q has become an integral over angles. Equation (2) then implies that the coefficient of δt vanishes

$$0 = \int \frac{d^N\theta}{(2\pi)^N} \sum_{(\alpha)} \frac{dS^{(\alpha)}(q(t);I;B(t))}{dt}$$

$$= \left\langle \sum_{(\alpha)} \left[\frac{\partial S^{(\alpha)}}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial S^{(\alpha)}}{\partial B_l} \frac{dB_l}{dt} \right] \right\rangle$$

$$= \left\langle p_i \dot{q}_i + \left[\frac{\partial S}{\partial B_l} - p_i \frac{\partial q_i}{\partial B_l} \right] \frac{dB_l}{dt} \right\rangle ,$$

where we have introduced the single-valued function[10]

$$\mathscr{S}(\theta, I; B) = S^{(\alpha)}(q(\theta, I; B), I; B) \quad (0 \le \theta \le 2\pi) . \tag{7}$$

Thus we have obtained the parallel-transport equation

$$\langle p_i \dot{q}_i \rangle + \left\langle \frac{\partial \mathscr{S}}{\partial B_l} - p_i \frac{\partial q_i}{\partial B_l} \right\rangle \frac{dB_l}{dt} = 0 . \tag{8}$$

Let us recall that in the defintion of ψ the dynamical phase has been eliminated by the energy normalization. Therefore, from Eq. (8) we see that the quantity which gets parallel transported is the total surplus area in phase space swept by the system, left over after the "dynamical area" has been subtracted off. In order to obtain the total "surplus" $\Delta\mathscr{A}$ acquired by the system during adiabatic transport around a closed circuit, we integrate (8) from $t=0$ to T with $B(0)=B(T)$:

$$\Delta\mathscr{A} = \int_0^T dt \langle p_i \dot{q}_i \rangle = \int_0^T dt \left\langle p_i \frac{\partial q_i}{\partial B_l} \right\rangle \frac{dB_l}{dt}$$

$$= \oint d\mathbf{B} \cdot \langle p_i \nabla_B q_i \rangle$$

$$= \oint A , \tag{9}$$

where we have dropped the term $\langle \nabla_B \mathscr{S} \rangle \cdot d\mathbf{B}/dt$ since it does not contribute to the line integral around a closed loop, and introduced the connection

$$A = \langle p_i d_B q_i \rangle . \tag{10}$$

To obtain the familiar expression[10] for the jth Hannay angle we differentiate with respect to the jth action

$$\Delta\theta_j = -\frac{\partial}{\partial I_j} \Delta\mathscr{A} = -\frac{\partial}{\partial I_j} \oint \langle p_i d_B q_i \rangle . \tag{11}$$

$\Delta\mathscr{A}$ and $\Delta\theta_j$ can be reexpressed as integrals over the surface S whose boundary is C:

$$\Delta\mathscr{A} = \int_S \int \langle d_B p \wedge d_B q \rangle = \int_S \int W , \tag{12}$$

$$\Delta\theta_j = -\frac{\partial}{\partial I_j} \int_S \int \langle d_B p \wedge d_B q \rangle = -\frac{\partial}{\partial I_j} \int_S \int W , \tag{13}$$

where we have introduced the area two-form

$$W = \langle d_B p \wedge d_B q \rangle \tag{14}$$

which is the curvature of the connection A.

Further properties of the phase-space bundle, including its triviality or nontriviality in the presence of Hannay's angle and the relationship between classical degeneracies and singularities of W, will be discussed in Appendix A.

III. THE SYMPLECTIC STRUCTURE CONNECTED WITH THE CLASSICAL ADIABATIC HOLONOMY

In the previous section the parameters play a passive role and their adiabatic evolution is prescribed from the outside. However, in many systems of physical interest the adiabatic parameters are themselves dynamical variables which evolve very slowly in comparison with the subsystem which depends on them. It is therefore of interest to consider systems with coupled, fast, and adiabatic degrees of freedom. In the quantum case one can eliminate the fast degrees of freedom either by integrating them out of the path integral[15] or in a Born-Oppenheimer treatment.[2,12] Then one finds that the quantum adiabatic holonomy induces an effective gauge field acting on the slow variables. In this section we develop a similar approach to the classical adiabatic holonomy.

Let us consider a system with N fast degrees of freedom, described by coordinates and momenta (q,p), coupled to M slow (adiabatic) degrees of freedom, with coordinates and momenta (Q,P), in such a way that the fast variables depend only on the slow coordinates Q and not on their momenta P. The coordinates Q then take the place of the parameters B of the last section. A generic Hamiltonian for such a system is

$$H(q,p;Q,P) = H_1(q,p;Q) + H_2(Q,P) , \tag{15}$$

where H_1 is the Hamiltonian ordinarily appearing in discussions of Hannay's angle and H_2 depends only on the slow variables.

We assume that the system is instantaneously integrable in its fast degrees of freedom, which can be written in action-angle variables (θ,I). The transformation $(q,p) \to (\theta,I)$ is implemented by the many-valued generating function $S^{(\alpha)}(q,I;Q)$ satisfying

$$P_i^{(\alpha)} = \frac{\partial S^{(\alpha)}}{\partial q_i}, \quad \theta_i = \frac{\partial S^{(\alpha)}}{\partial I_i} .$$

Since Q depends on t this generating function is explicitly time dependent and hence the Hamiltonian H_1, expressed in action-angle variables, becomes

$$\overline{H}_1(\theta,I,Q(t))=\mathscr{H}(I;Q)+\frac{\partial S^{(\alpha)}(q,I;Q(t))}{\partial t}$$
$$=\mathscr{H}(I;Q)+\dot{Q}_l\frac{\partial S^{(\alpha)}}{\partial Q_l}$$
$$=\mathscr{H}(I;Q)+\dot{Q}_l\left[\frac{\partial \mathscr{S}}{\partial Q_l}-P_i\frac{\partial q_i}{\partial Q_l}\right], \quad (16)$$

where $\mathscr{H}(I;Q)=H_1[q(\theta,I,Q),p(\theta,I;Q)]$ and again we have introduced the single-valued function

$$\mathscr{S}(\theta,I;Q)=S^{(\alpha)}(q(\theta,I;Q),P(\theta,I;Q);Q) \quad (0\leq\theta\leq 2\pi),$$

following Berry.[10]

Now, in order to find approximate equations of motion for the slow variables, we average over the fast degrees of freedom. The influence of the fast variables on the slow ones is then given to a "good approximation"[16] by the averaged Hamiltonian

$$\langle \overline{H}_1\rangle=\int \frac{d^N\theta}{(2\pi)^N}\overline{H}_1(\theta,I;Q)$$
$$=\mathscr{H}(I;Q)+\dot{Q}_l\left\langle\frac{\partial \mathscr{S}}{\partial Q_l}-p_i\frac{\partial q_i}{\partial Q_l}\right\rangle, \quad (17)$$

where the angular brackets denote the average over all angles. (We are ignoring here the possibility of parametric resonance in which a natural frequency of the fast variables is commensurate with a natural frequency of the slow variables.) Thus after averaging over the angle variables, we obtain the Hamiltonian

$$H_{\rm av}(I;P,Q)=\mathscr{H}_1(I;Q)+H_2(P,Q)+\dot{Q}_l\left\langle\frac{\partial \mathscr{S}}{\partial Q_l}-p_i\frac{\partial q_i}{\partial Q_l}\right\rangle. \quad (18)$$

We insert the averaged Hamiltonian into the Hamiltonian variational principle

$$\delta S_{\rm eff}=\delta\int_0^T dt[P_l\dot{Q}_l-H_{\rm av}(I;P,Q)]=0$$

to obtain equations of motion for Q and P. The term $\sum_{l=1}^M \dot{Q}_l\langle\partial\mathscr{S}/\partial Q_l\rangle$ in $H_{\rm av}$, being a total time derivative, will not affect the equations of motion and hence we drop it from the action principle. Grouping together terms proportional to $\dot{Q}$, we obtain the effective action

$$S_{\rm eff}=\int_0^T dt\left[\left[P_l+\left\langle p_i\frac{\partial q_i}{\partial Q_l}\right\rangle\right]\dot{Q}_l-\overline{H}\right], \quad (19)$$

where have defined the effective Hamiltonian $\overline{H}(I;P,Q)=\mathscr{H}_1(I;Q)+H_2(P,Q)$.

We see that the averaged fast motion induces an effective "gauge field" which acts on the slow variables through the minimal coupling replacement

$$P_l\to P_l+\left\langle p_i\frac{\partial q_i}{\partial Q_l}\right\rangle.$$

This is the same connection that was associated with Hannay's angle in the previous section. Varying the effective action independently with respect to Q and P, while keeping the end points fixed, we obtain

$$\delta S_{\rm eff}=\int_0^T dt\left[\left[\delta P_l+\delta\left\langle p_i\frac{\partial q_i}{\partial Q_l}\right\rangle\right]\dot{Q}_l+\left[P_l+\left\langle p_i\frac{\partial q_i}{\partial Q_l}\right\rangle\right]\delta\dot{Q}_l-\frac{\partial\overline{H}}{\partial Q_l}\delta Q_l-\frac{\partial\overline{H}}{\partial P_l}\delta P_l\right]=0$$

from which, after integrating by parts, discarding the boundary term, and noting that $\langle p_i(\partial q_i/\partial Q_l)\rangle$ depends only on Q and not on P, we get

$$\delta S_{\rm eff}=0=\int_0^T dt\left\{\delta P_l\left[\dot{Q}_l-\frac{\partial\overline{H}}{\partial P_l}\right]+\delta Q_l\left[\left[\frac{\partial}{\partial Q_l}\left\langle p_i\frac{\partial q_i}{\partial Q_m}\right\rangle-\frac{\partial}{\partial Q_m}\left\langle p_i\frac{\partial q_i}{\partial Q_l}\right\rangle\right]\dot{Q}_m-\frac{\partial\overline{H}}{\partial Q_l}-\dot{P}_l\right]\right\}.$$

Setting the coefficients of δP_l and δQ_l equal to zero we obtain the equations of motion

$$\dot{Q}_l=\frac{\partial\overline{H}}{\partial P_l}, \quad (20a)$$

$$\dot{P}_l=-\frac{\partial\overline{H}}{\partial Q_l}+\left[\frac{\partial}{\partial Q_l}\left\langle p_i\frac{\partial q_i}{\partial Q_m}\right\rangle-\frac{\partial}{\partial Q_m}\left\langle p_i\frac{\partial q_i}{\partial Q_l}\right\rangle\right]\dot{Q}_m. \quad (20b)$$

The effective gauge potential

$$A_l=\left\langle p_i\frac{\partial q_i}{\partial Q_l}\right\rangle, \quad (21)$$

appearing in $S_{\rm eff}$, has a curvature (or field strength)

$$\tilde{\omega}_{lm}=\frac{\partial}{\partial Q_l}\left\langle p_i\frac{\partial q_i}{\partial Q_m}\right\rangle-\frac{\partial}{\partial Q_m}\left\langle p_i\frac{\partial q_i}{\partial Q_l}\right\rangle$$
$$=\left\langle\frac{\partial p_i}{\partial Q_l}\frac{\partial q_i}{\partial Q_m}-\frac{\partial p_i}{\partial Q_m}\frac{\partial q_i}{\partial Q_l}\right\rangle \quad (22)$$

which exerts a velocity-dependent force on the slow degrees of freedom.

Now we substitute for $\dot{Q}$ from (20a) into (20b) to obtain the following equation for $\dot{P}$:

$$\dot{P}_l=-\frac{\partial\overline{H}}{\partial Q_l}+\tilde{\omega}_{lm}\frac{\partial\overline{H}}{\partial P_m}. \quad (23)$$

We can put Eqs. (20a) and (23) into canonical form

$\dot{Q}=\{\overline{H},Q\}$ and $\dot{P}=\{\overline{H},P\}$ by introducing the modified Poisson brackets on the slow-variable phase space:

$$\{f(Q,P),g(Q,P)\}=\left[\frac{\partial f}{\partial P_l}\frac{\partial g}{\partial Q_l}-\frac{\partial f}{\partial Q_l}\frac{\partial g}{\partial P_l}\right]-\widetilde{\omega}_{lm}\frac{\partial f}{\partial P_l}\frac{\partial g}{\partial P_m} . \quad (24)$$

It is shown in Appendix B that this modified Poisson brackets correspond to the symplectic form

$$\omega=dP_l\wedge dQ_l+\tfrac{1}{2}\widetilde{\omega}_{lm}dQ_l\wedge dQ_m . \quad (25)$$

Because of the coupling of the slow variables to the fast degrees of freedom, the symplectic form on the slow-variable phase space has an additional piece

$$\widetilde{\omega}=\frac{1}{2}\left\langle\frac{\partial p_i}{\partial Q_l}\frac{\partial q_i}{\partial Q_m}-\frac{\partial p_i}{\partial Q_m}\frac{\partial q_i}{\partial Q_l}\right\rangle dQ_l\wedge dQ_m =\left\langle\frac{\partial p_i}{\partial Q_l}\frac{\partial q_i}{\partial Q_m}\right\rangle dQ_l\wedge dQ_m \quad (26)$$

which is just precisely the area two-form W found in the previous section. *Thus we see that the classical adiabatic holonomy reveals itself as an additional term in the symplectic form on the slow-variable phase space.*

As an example, let us consider the classical Grassmannian model[11] for a spin-$\frac{1}{2}$ magnetic dipole coupled to an external magnetic field with an additional "kinetic energy" term for the external field. This model is a classical version of the "spinning solenoid" discussed by Stone.[12] The Hamiltonian for this system is

$$H=-\frac{i}{2}\epsilon_{klm}Q_k\psi_l\psi_m+\tfrac{1}{2}P_k{}^2 , \quad (27)$$

where all indices are summed from 1 to 3, ψ_l are three Grassmann variables, $\mathbf{Q}$ is the external magnetic field, and $\mathbf{P}$ its conjugate momentum. After transforming the ψ's to action-angle variables and averaging over the angles we obtain the effective Hamiltonian

$$\overline{H}=(-I_2+I_3)Q+\tfrac{1}{2}P^2 , \quad (28)$$

where

$$Q=(Q_1{}^2+Q_2{}^2+Q_3{}^2)^{1/2} ,$$
$$P=(P_1{}^2+P_2{}^2+P_3{}^2)^{1/2} ,$$

$I_2=\frac{1}{2}\widetilde{\psi}_2^*\widetilde{\psi}_2$ is the action of the normal mode $\widetilde{\psi}_2(t)=\widetilde{\psi}_2(0)e^{iQt}$, $I_3=\frac{1}{2}\widetilde{\psi}_3^*\widetilde{\psi}_3$ is the action of the normal mode $\widetilde{\psi}_3=\widetilde{\psi}_2^*$, and $I=I_3-I_2=\widetilde{\psi}_3^*\widetilde{\psi}_3$. The area two-form $\widetilde{\omega}$, which modifies the symplectic form in the (Q,P) phase space was found in Ref. 11; its components are

$$\widetilde{\omega}_{lm}=-I\frac{Q_k}{Q^3}\epsilon_{klm} . \quad (29)$$

Notice that this two-form has a singularity at $Q=0$. This is the point in Q space where the frequencies of the originally fast variables ψ vanish.

Substituting Eqs. (28) and (29) into (20a) and (23), we obtain the equations of motion

$$\dot{Q}_k=P_k , \quad (30a)$$

$$\dot{P}_k=-I\frac{Q_k}{Q}+I\epsilon_{klm}\frac{Q_l}{Q^3}P_m . \quad (30b)$$

Combining (30a) with (30b) we get

$$\ddot{Q}_k=-I\frac{Q_k}{Q}+I\epsilon_{klm}\frac{Q_l\dot{Q}_m}{Q^3} \quad (31)$$

which corresponds to motion of a unit charge with unit mass in the field of a "magnetic monopole" of strength I and with an additional constant force of strength I directed towards the origin in Q space.

Essentially the same result can be obtained using Berry's bosonic model[17] for a spin coupled to a magnetic field $\mathbf{Q}$. In Berry's model the action is given by

$$I=\mathbf{S}\cdot\frac{\mathbf{Q}}{Q} ,$$

where $\mathbf{S}$ is the classical spin vector. This action will then play exactly the same role in (31) as the monopole strength and the strength of the constant central force. The difference between Berry's bosonic model and the Grassmannian model is that while Berry's action is a c number, the Grassmannian action, although bosonic, is constructed from anticommuting variables.

Finally, we would like to remark that if, following Stone,[12] we constrain the magnetic field to lie on the unit sphere, the effective Hamiltonian becomes $H=I+\frac{1}{2}P^2$, the constant central force in (31) disappears, and the system becomes equivalent to a charge constrained to a sphere moving in the field of a magnetic monopole at the center of the sphere.[18] This is precisely the result that Stone finds at the quantum level by integrating the fermionic degrees of freedom out of the path integral.

IV. CONCLUSION

The reader might ask what is the purpose of the formal analysis presented in the previous sections. The use we have in mind for it is related to a semiclassical interpretation of anomalies and we will briefly sketch here the main ideas. These ideas will be further expanded in a forthcoming paper.[19]

We know[4] that systems with anomalies (non-Abelian chiral anomalies, or global anomalies), must develop Berry's phase when they are treated in a Hamiltonian formalism. To explain this, let us review the nice argument given by Niemi and Semenoff:[4] Consider a single Weyl fermion ψ in a complex representation of a non-Abelian gauge group G and let us work in the gauge for which $A^0=0$. The object we are interested in is

$$Z(A)=\int d\psi d\psi^\dagger\exp\left[i\int dt\,\psi^\dagger[i\partial_t+H(t)]\psi\right] =\prod_n\lambda_n , \quad (32)$$

where d^3x is understood and the path integral has periodic boundary conditions on A and antiperiodic on ψ. The operator H is the usual Dirac operator

$$H(t) = -i(\partial_j + A_j)\gamma^j . \tag{33}$$

The background field A belongs to the infinite-dimensional manifold $\mathcal{A}^3$ of all static gauge fields. A t-dependent A field describes an orbit in $\mathcal{A}^3$. In the path integral (32) the trajectories in ψ space, on which we sum, have a different operator H associated to them at each instant of time. This family of H's describes a closed loop in A^3. The calculation of $Z(A)$ can be done once the eigenvalues λ_n of

$$i[\partial_t + H(t)]\psi_n = \lambda_n \psi_n \tag{34}$$

are known. This (for more detail see Ref. 4) boils down to solving the Schrödinger-type equation

$$H\langle x \mid r;t\rangle = E_r \langle x \mid r;t\rangle , \tag{35}$$

where r is an index for the zero modes of (34). As H (through its dependence on A) is a time-dependent Hamiltonian, we have to find an approximation to solve (35). We can use, for example, a Born-Oppenheimer approximation and consider A as an adiabatic parameter upon which the solution of (35) depends. The solution will have the form

$$\langle x \mid r;t\rangle = \exp\left[i \int dt' E_r(t') + i\gamma(t) \right] ,$$

where γ is the Berry phase associated to the loop in $\mathcal{A}^3$. Niemi and Semenoff proved that γ has to be different from zero if the system has anomalies. Conversely if the Hamiltonian H has a $\gamma \neq 0$, then when we couple Weyl fermions to it, the system will develop anomalies. Thus Berry's phase is an indicator that the operator H might "get in trouble" when coupled with Weyl fermions.

Now we want to go back to the Hannay angles. Let us recall[10,11] that if a system has a Hannay angle $\Delta\theta$ different from zero, then the corresponding Berry phase is also different from zero. In fact, in an expansion in powers of $\hbar$ we have

$$\frac{\partial \gamma}{\partial n} \simeq \Delta\theta + O(\hbar)$$

and so if $\Delta\theta \neq 0$ then $\partial\gamma/\partial n \neq 0$ which implies that $\gamma \neq 0$. Because of this fact we can limit ourselves to study the operator H of (33) at the purely classical level and calculate the corresponding Hannay angles. If they are different from zero then we know that also γ will be different from zero, and this indicates that the system will develop anomalies (once coupled to Weyl fermions).

The reader may ask how we can treat H classically. This has been done in the literature.[20] The idea is to write down a Lagrangian L for coupled x_i and ξ_i, where x_i are the position variables and ξ_i the Grassmann variables. This Lagrangian has a corresponding Hamiltonian and, once it is quantized, the algebra of the ξ_i goes into the algebra of Dirac γ matrices reproducing the operator H of (33). Using this classical Lagrangian and considering A as an adiabatic external parameter, the calculation of the corresponding Hannay angles goes along the same line as in Ref. 11. This calculation will be reported in a forthcoming paper.[19]

What is nice about this analysis is that, *even at the classical level,* we have an indicator $\Delta\theta$ of possible anomalies. This means that the system even "feels" classically if, once quantized, it can develop anomalies. Of course this *does not mean* that classically certain symmetries are not preserved: it means something more subtle is happening at the classical level that will show up at the quantum level as the nonconservation of certain currents. We would like to give an indication of the "something more subtle" that is happening at the classical level.

We are usually taught that a symmetry can always be implemented at the classical level with canonical transformations. This is true locally but it might not be the case globally on the whole of phase space.[2] Those symmetry generators which cannot be implemented globally in a canonical way are called nonglobally Hamiltonian.

We believe that the presence of generators that are not globally Hamiltonian at the classical level may be a *necessary condition* to have anomalies at the quantum level. For some systems it might even be a sufficient condition and we are currently exploring this possibility in several models. Our belief that it might be, for some models, a necessary and sufficient condition, has to do with the following simple (and maybe naive) idea: if a generator of a symmetry is not globally Hamiltonian, then it means that, "at some point" the transformation it generates is not canonical any more. If it is no longer canonical, then the volume of phase space is not preserved by the symmetry transformation around that point (designating a breakdown of the Liouville theorem). This means that the symmetry transformation can shrink any volume of phase space down to an infinitesimally small volume. This, of course, will clash at the quantum level with the Planck-Heisenberg principle that tells us that the minimum amount of phase space that can be occupied by a system is $\Delta p \Delta q = h$. So quantum mechanics will reject such a transformation as a symmetry transformation, because it violates the principles of quantization. The reader may ask how quantum mechanics can detect something that is happening only at some points. The answer is that quantum mechanics naturally feels if the symmetry is doing "something wrong" even at "some points only," because it has a "global detector": the wave function (an extended object). Classical mechanics, on the contrary, probes phase space only locally. To detect global effects, we have to go beyond the Lagrange equations of motion and build some "global detector." This is what the Hannay's angle is—a global detector at the classical level—and it is for this reason that it gives us information on the possible appearance of anomalies. Of course these are just speculations for the moment, but work is in progress to confirm them.

Note added. While we were completing this work it came to our attention that S. Iida and H. Kuratsuji had proposed Report No. PHANTOM 865 [Kyoto University (unpublished)] that Berry's phase is associated with a modification of the symplectic structure of phase space, using a mixture of classical and quantum mechanics. In contrast, we have established this result at the purely classical level.

ACKNOWLEDGMENTS

It is a pleasure to thank M. V. Berry, E. Kiritsis, J. Lauer, J. Preskill, E. Seiler, J. Sidenius, B. Simon, and L. Stodolsky for helpful correspondence and conversations. The work of E. Gozzi was supported by the Danish Research Council. W.T. acknowledges the financial support of the Fulbright Foundation.

APPENDIX A

In this appendix we will explore some facts about the phase-space bundle introduced in Sec. II. We will basically be concerned with the topics listed below.

(1) We prove that classical degeneracies, caused by changing the parameters B, produce singularities in the angle two-form W analogously to what happens at the quantum level.[1]

(2) We analyze how the angle one-form A changes under B-dependent canonical transformations in the fast variables (p,q). Using these transformations we will prove that the Hannay's angle *can* (in principle) *be obtained as a dynamical angle* of a canonically equivalent Hamiltonian only if the relevant bundle is *trivial.*

1. Classical degeneracies and singularities of the angle two-form

Berry,[1] in his first paper on this subject, found a way to write the phase two-form $V_n(B)$ in a form different from the one introduced in Sec. II of this paper. His expression is

$$V_n(B)=\frac{\langle n\,|\,dH\,|\,m\,\rangle\wedge\langle m\,|\,dH\,|\,n\,\rangle}{(E_n-E_m)^2}\ , \tag{A1}$$

where d denotes throughout the exterior derivative in parameter space and we sum over m for m different from n. For details on how to derive this expression we refer the reader to the paper of Berry. From (A1) we see that quantum degeneracies [occurring at some point B^* of the parameter space in which $E_n(B^*)=E_m(B^*)$] cause singularities in the phase two-form V. Moreover it has been proved[1,6] that a circuit C, which passes close to this singularity, picks up a contribution different from zero for the Berry phase. So quantum degeneracies act as sources for the phase two-form. Of course they might not be the only sources.

It would be interesting to see if, in an analogous way, classical degeneracies cause singularities in the angle two-form W_j of Hannay. The angle two-form is given in terms of the area two-form W, introduced in Sec. II by $W_j=\partial W/\partial I_j$. This two-form can be written in many ways.[9,10] We find it convenient here to use the expression[9]

$$W_j(I,B)=d\langle d\theta_j\rangle=\left\langle\frac{\partial}{\partial I_j}(d\theta_l\wedge dI_l)\right\rangle \tag{A2}$$

sum over l only. Let us recall that the Hamiltonian H is a function of the n action variables I_j,

$$H=H(I_1,\ldots,I_N)\ ,$$

and that it is an invertible function[16] so that

$$I_1=F_1(H,I_2,\ldots,I_N)$$

and in general

$$I_j=F_j(I_1,\ldots,H,\ldots,I_N)\ . \tag{A3}$$

From (A3) we have

$$dI_l=\frac{\partial F_l}{\partial H}dH+\frac{\partial F_l(I_1,\ldots,H,\ldots,I_N)}{\partial I_k}dI_k\ ,$$

that is,

$$dI_l=\frac{dH}{\omega_l}+\frac{\partial F_l}{\partial I_k}dI_k\ ,$$

where ω_l is the frequency associated to the angle θ_l. We can insert this into (A2) and obtain

$$\langle d\theta_l\wedge dI_l\rangle=\frac{\langle d\theta_l\wedge dH\rangle}{\omega_l}+\left\langle d\theta_l\wedge\frac{\partial F_l}{\partial I_k}dI_k\right\rangle\ . \tag{A4}$$

From here we see that if, for some value of B, let us say B^*, one frequency ω_l goes to zero, then the angle two-form W_j becomes singular. We should remember that a frequency that goes to zero is a signal of a degeneracy (see Ref. 16). In fact, let us imagine a system with only two degrees of freedom $(\theta_1,\theta_2;I_1,I_2)$ which has a degeneracy $\omega_1=\omega_2$. Then it is possible to go to new canonical variables $(\theta_1',\theta_2';I_1',I_2')$ given by

$$\theta_1'=\theta_1-\theta_2,\quad I_1'=I_1\ ,$$

$$\theta_2'=\theta_2,\quad I_2'=I_1+I_2\ ,$$

and in these new variables $\omega_1'=0$ due to the fact that $\omega_1=\omega_2$ and that $\omega_1'=\omega_1-\omega_2$. Thus we can conclude that classical degeneracies cause singularities in the angle two-form, in complete analogy to the quantum case.

The next question to answer is whether these singularities can act as sources for Hannay's angle. Let us consider a circuit C in B space which passes close to B^*. The angle two-form in (A4) is then dominated by the term with the frequency that goes to zero. Let us suppose this frequency is ω_2. Then we have in (A4)

$$d\langle d\theta_j\rangle=\left\langle\frac{\partial}{\partial I_j}\left[d\theta_2\wedge\frac{dH}{\omega_2}\right]\right\rangle\ . \tag{A5}$$

As only two frequencies are involved in the degeneracy, we can choose, as interpolating Hamiltonian in (A5), a Hamiltonian with only two degrees of freedom. Of course H has to have the property of being integrable and having a degeneracy for some value of the parameters. There is a whole family of Hamiltonians that can do the job without losing generality. One possibility is

$$H=B_1P_1{}^2+B_2P_2{}^2-B_3Q_1Q_2\ .$$

For this Hamiltonian the calculation of $\Delta\theta$ can be done along the same lines as in the model studied in Ref. 11 and it is not difficult to see that the $\Delta\theta$ is different from zero. So going back to the full-fledged system with n degrees of freedom from which we started we see that the singularity (i.e., the classical degeneracy) acts as a source for the angle two-form. Of course there may be other sources for W_j.

2. Hannay's angle and the nontriviality of the phase-space bundle

The issue we want to address in this part of the appendix is whether Hannay's angle $\Delta\theta$, given by

$$\theta_j(T)=\int_0^T \omega_j(t)dt+\Delta\theta_j \ ,$$

where $\omega=\partial H/\partial I$ is a real "geometrical" part of the angle variable of the system, or whether it can be obtained dynamically by using a set of variables different from (q,p). It might in fact happen that, as we are allowed to do canonical transformations on our system, we find a set of variables (q',p') and a new Hamiltonian H' (canonically equivalent to the previous one) such that the new frequency ω' associated with it is enough to describe the full angle swept. This means that

$$\theta_j(T)=\int_0^T \omega_j'(t)dt, \quad \omega_j'=\frac{\partial H'}{\partial I_j} \ .$$

This would not indicate that the phenomenon and the physics connected with the Berry phase (or the Hannay angle) could be taken away. It would mean that this phenomenon has become a dynamical effect absorbed into the new frequency ω'.

In these new variables (q',p') the Hannay angle would be zero

$$\Delta\theta_j=\int\int W_j'=0$$

or equivalently, the new angle two-form W_j would have to be zero. The *first question* to ask is if there is any fundamental reason which forbids this possibility and the *second* is which transformation of variables would do the job. We will answer only the first question and prove that, if the phase-space bundle is trivial then there is nothing to forbid the existence of a set of coordinates (q',p') such that its associated two-form W' is zero. The second question can only be answered partially: we will only prove that the set of *canonical* transformations, allowed on (q,p), deform A in a way that can in principle make the two-form W' zero. We will also set up the formal differential equation for the generating function of the canonical transformation that achieves this.

Let us first see which transformations can change W'. Certainly they are not the gauge transformations examined by Berry:[1]

$$\theta'=\theta+\alpha(B)$$

(a B-dependent change in the origin of θ). Under such transformations $A=\langle d\theta\rangle$ behaves like a gauge field

$$A'=A+d\alpha$$

and W is left invariant. (Here we are omitting indices; A is not to be confused with the connection discussed in Secs. II and III.) On the other hand, a general B-dependent canonical transformation on (θ,I) causes A to transform differently from a connection, and so it modifies W. Let us perform, for example, the transformation

$$I'=I+f(\theta,I,B)\ , \qquad \theta'=\theta+g(\theta,B)\ . \tag{A6}$$

(I' and θ' are no longer action-angle variables). The f and g ($C^{(\infty)}$ functions) have to satisfy the constraint

$$\frac{\partial f}{\partial I}+\frac{\partial g}{\partial \theta}+\frac{\partial f}{\partial I}\frac{\partial g}{\partial \theta}=0 \tag{A7}$$

(for the transformation to be canonical). This is a partial differential equation in two variables with two unknowns. So, most probably, the solutions are underdetermined. This means that we will have a whole family of solutions f and g for this equation. The new angle one-form $A'=\langle d\theta'\rangle$ is

$$A'=A+\langle dg\rangle \ . \tag{A8}$$

Note that this transformation is not a gauge transformation such as

$$A'=A+d\langle g\rangle \tag{A9}$$

because the d cannot be pulled out of the angular brackets (see Hannay, Ref. 9). In fact the average is B-dependent so that $d\langle g\rangle\neq\langle dg\rangle$.

In the case of the simple canonical transformation examined by Berry

$$\theta'=\theta+\alpha(B)$$

the function g is α and, as it is not dependent on θ, it can be pulled out of the angular brackets. Therefore $d\langle\alpha\rangle=d\alpha\langle\ \rangle=k\,d\alpha$.

So we see from (A8) that a B-dependent canonical transformation on (q,p) deforms A in a very general way. This is exactly what we need, in fact it is possible to prove[23] that any continuous deformation of the connection $A\rightarrow A'$ will only change the two-form W by an exact differential $dQ(A,A')$:

$$W'=W+dQ \ .$$

The one-form Q is obtained once the deformation is given. If we want now to deform A so that $W'=0$ then Q has to satisfy the relation

$$W=-dQ \ . \tag{A10}$$

This means that the W has to be an exact form and this is always possible for a trivial bundle. So, once W is given, it is always possible to find Q. Having Q, it is easy to find which transformations g of (A8) on (q,p) will deform A so that W' is zero: Combining (A8) and (A10) we get the following equation for g:

$$d\langle dg\rangle=dQ \ . \tag{A11}$$

Thus (A11) combined with the constraint (A7) are the equations that determine the canonical transformation we were looking for. We do not see, in principle, any obstruction which prevents these two equations from having solutions. Of course we have not proved an existence theorem which states that, for sure, these two equations have a solution. Resorting to a familiar case with a trivial bundle (the Aharonov-Bohm effect) we see that there the phase is already dynamical. In fact the bundle connection there is the same as the gauge potential of the current producing the effect. This gauge potential is already in the Hamiltonian, so that the phase generated by it is dynamical.

For *nontrivial bundles,* on the contrary, we cannot find a globally defined Q that makes $W'=0$. This implies that we will also not be able to find globally well-defined solutions g to Eq. (A11). *Physically this means that, in the case that the bundle is nontrivial, the Berry phase (or Hannay angle) can never be obtained as a dynamical phase (or angle) through a globally well-defined unitary (or canonical) transformation.* This is what happens in most cases and that makes the effect a truly "geometrical" one.

APPENDIX B

In this appendix we show that the symplectic form (25) corresponds to the Poisson brackets (24):

$$\{f(Q,P),g(Q,P)\}=\left[\frac{\partial f}{\partial P_l}\frac{\partial g}{\partial Q_l}-\frac{\partial f}{\partial Q_l}\frac{\partial g}{\partial P_l}\right]-\tilde{\omega}_{lm}\frac{\partial f}{\partial P_l}\frac{\partial g}{\partial P_m}\ . \tag{B1}$$

The term in large parentheses corresponds, as is well known, to the symplectic form $dP_l\wedge dQ_l$. The symplectic form which we seek then has the form

$$\omega=dP_l\wedge dQ_l+\tfrac{1}{2}\omega_{Q_l,Q_m}dQ_l\wedge dQ_m+\tfrac{1}{2}\omega_{P_l,P_m}dP_l\wedge dP_m\ . \tag{B2}$$

The Poisson brackets are defined[22] abstractly by

$$\{f,g\}=X_f(g)\ , \tag{B3}$$

where

$$X_f=(X_f)_{P_l}\frac{\partial}{\partial P_l}+(X_f)_{Q_l}\frac{\partial}{\partial Q_l} \tag{B4}$$

is the Hamiltonian vector field corresponding to the observable $f(Q,P)$ and is given in terms of the symplectic form by

$$i_{X_f}\omega=-df \tag{B5}$$

where the left-hand side denotes the contraction of the form ω with the evector field X_f. In (Q,P) coordinates, we have

$$[(X_f)_{P_m}+(X_f)_{Q_l}\omega_{Q_l,Q_m}]dQ_m+[(X_f)_{P_l}\omega_{P_l,P_m}-(X_f)_{Q_m}]dP_m=-\frac{\partial f}{\partial Q_m}dQ_m-\frac{\partial f}{\partial P_m}dP_m\ . \tag{B6}$$

Combining (B3) with (B4) and comparing with (B1), we find

$$(X_f)_{P_l}=-\frac{\partial f}{\partial Q_l}+\tilde{\omega}_{lm}\frac{\partial f}{\partial P_m}\ ,\qquad (X_f)_{Q_l}=\frac{\partial f}{\partial P_l}\ . \tag{B7}$$

Then, substituting (B7) into (B6) and comparing coefficients of dQ and dP we obtain $\omega_{P_l,P_m}=0$ and $\omega_{Q_l,Q_m}=-\tilde{\omega}_{ml}=\tilde{\omega}_{lm}$. Our final result is therefore

$$\omega=dP_l\wedge dQ_l+\tfrac{1}{2}\tilde{\omega}_{lm}dQ_l\wedge dQ_m\ . \tag{B8}$$

[1]M. V. Berry, Proc. R. Soc. London **A392**, 457 (1984).
[2]C. A. Mead and B. G. Trular, J. Chem. Phys. **70**, 2284 (1984); J. Moody, A. Shapere, and F. Wilczek, Phys. Rev. Lett. **56**, 893 (1986); R. Jackiw, *ibid.* **56**, 2779 (1986).
[3]J. E. Avron, R. Seiler, and B. Simon, Phys. Rev. Lett. **51**, 51 (1983); J. E. Avron and R. Seiler, *ibid.* **54**, 259 (1985).
[4]P. Nelson and L. Alvarez-Gaume, Commun. Math. Phys. **99**, 103 (1985); H. Sonoda, Phys. Lett. **156B**, 220 (1985); Nucl. Phys. **B266**, 410 (1986); A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **55**, 927 (1985); A. J. Niemi, G. W. Semenoff, and Y. S. Wu, Nucl. Phys. **B276**, 173 (1986).
[5]A. Tomita and R. Y. Chiao, Phys. Rev. Lett. **57**, 937 (1986).
[6]B. Simon, Phys. Rev. Lett. **51**, 2167 (1983).
[7]E. Kiritsis, Caltech Report No. 68-1338, 1986 (unpublished).
[8]D. Thouless, M. Kohmoto, M. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[9]J. H. Hannay, J. Phys. A **18**, 221 (1985).
[10]M. V. Berry, J. Phys. A **18**, 15 (1985).
[11]E. Gozzi and W. D. Thacker, preceding paper, Phys. Rev. D **35**, 2388 (1987).
[12]M. Stone, Phys. Rev. D **33**, 1191 (1986).
[13]J. Anandan and L. Stodolsky (unpublished).
[14]A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1962), Vol. 2.
[15]H. Kuratsuji and S. Iida, Prog. Theor. Phys. **74**, 439 (1985); G. W. Semenoff, report, 1986 (unpublished). See, also, Stone (Ref. 12).
[16]V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1978).
[17]M. V. Berry, in *Fundamental Aspects of Quantum Theory,* edited by V. Gorini and A. Frigerio (Plenum, New York, in press).
[18]A. P. Balachandran, G. Marmo, B. S. Skagerstam, and A. Stern, *Gauge Symmetries and Fibre Bundles* (Springer, New York, 1983).
[19]E. Gozzi and W. D. Thacker, MPI report (unpublished).
[20]R. Casalbuoni, Nuovo Cimento **33**, 389 (1976); **33**, 115 (1976); Phys. Lett. **62B**, 49 (1976); P. Di Vecchia and F. Ravndal, *ibid.* **73A**, 371 (1979).
[21]B. Zumino, in *Relativity, Groups and Topology II,* edited by B. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).
[22]N. Woodhouse, *Geometric Quantization* (Clarendon, Oxford, 1980).
[23]T. Eguchi, P. B. Gilkey, and A. J. Hanson, Phys. Rep. **66**, 213 (1980).