

Berry's phase in the relativistic theory of spinning particles

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We show that the change of the phase of the wave function of a free relativistic spinning particle under the action of the Poincaré group is described by a certain universal connection in momentum space. This connection determines the parallel transport of the spin during the motion of the particle through external fields. In the special case of photons we reproduce the formula for Berry's phase calculated recently by Chiao and Wu.

I. INTRODUCTION

It has been shown by Berry¹ that in quantum theory the parallel transport of state vectors in the space of external parameters appearing in the Hamiltonian has striking geometrical properties when it is given by the Schrödinger equation in the adiabatic limit. Simon² has given an interpretation of Berry's result in terms of a natural Hermitian connection associated with one-parameter families of vectors (line bundles) in the Hilbert space. This connection is determined by a very simple transport law: the change $d\psi$ of a state vector is orthogonal to ψ :

$$(\psi, d\psi) = 0. \quad (1)$$

The holonomy associated with this connection determines the change of the phase $\gamma(C)$ of the state vector for every closed path C (Berry's phase). Recently, Berry³ has extended the adiabatic limit method from Schrödinger theory to Maxwell's theory. In this example the adiabatically changing external parameter is the dielectric tensor of the birefringent and gyrotropic medium.

The next step was taken by Chiao and Wu.⁴ In their proposal to measure the change of the polarization for a beam of photons traveling along a helical optical fiber, they took a somewhat different view of Berry's geometrical phase. Instead of attributing the change of the phase of the photon wave function to the adiabatic changes of the external parameters, they attributed it to the adiabatic changes of the photon wave vector. The changes of the wave vector reflect the changes of the parameters characterizing the medium. It turns out that the change of the phase can be easily calculated for a given change of the wave vector. Berry's geometrical phase is obtained when the end of the photon momentum vector follows a closed path in momentum space. The experiment proposed by Chiao and Wu has been successfully carried out by Tomita and Chiao⁵ confirming the prediction of the topological phase for photons.

Our approach closely follows that of Chiao and Wu. We will not consider explicitly the changes in the external

parameters appearing in the equations of motion, but we shall account for these changes indirectly through the changes in the particle's momentum. The final results will be, of course, the same as shown by the following chain of equations:

$$d\psi = \sum_i \frac{\partial \psi}{\partial k^i} dk^i = \sum_{i,j} \frac{\partial \psi}{\partial k^i} \frac{\partial k^i}{\partial \alpha^j} d\alpha^j = \sum_j \frac{\partial \psi}{\partial \alpha^j} d\alpha^j. \quad (2)$$

The purpose of this paper is to calculate the connection that governs the transport of the wave function in momentum space for relativistic spinning particles. In the relativistic case, the space of parameters is the momentum hyperboloid for massive particles or the light cone for massless particles and the natural transport law is obtained from the action of the Poincaré group or the conformal group. We show that the geometrical phase calculated in this manner coincides with Berry's phase.

As we have already shown some ten years ago,⁶ the generators of the Poincaré group (or the full conformal group in the case of massless particles) involve a covariant derivative \mathbf{D} in momentum space, whose curvature is universally given by the same magnetic monopole field that has been found in many other cases. This covariant derivative appears in all the equations that determine the action of the Poincaré or the conformal group on the wave functions in momentum space.

In order to apply our results, which involve only the properties of the Poincaré group, to physical situations, we have to assume that the motion of a particle in space-time (for example, the motion of a photon in an optical fiber observed in Ref. 5) can be viewed as a sequence of Lorentz boosts and rotations. This assumption, in our approach, plays the role of the adiabatic limit and enables one to draw physical conclusions from the study of the action of the Poincaré group on the momentum-space wave functions.

We shall deal in this paper with free relativistic particles described by wave functions, but everything we shall say has its direct counterpart in the theory of the quantized free fields.

II. MASSIVE PARTICLES

Let us consider a relativistic particle of mass m and spin s . The components of the wave function describing such a particle satisfy a set of differential equations (e.g., the Dirac equation, the Proca equation), which results in the selection of a single representation (or two copies of a representation, when the particles and antiparticles are distinct) characterized by m and s . Every solution $\psi_a(x)$ of such a relativistic wave equation can be expanded into plane waves,

$$\psi_a(x) = \sum_{\lambda} \int d\Gamma [u_a(\mathbf{p}, \lambda) f_1(\mathbf{p}, \lambda) e^{-ip \cdot x} + v_a(\mathbf{p}, \lambda) f_2(\mathbf{p}, \lambda) e^{ip \cdot x}], \quad (3)$$

where $d\Gamma$ is a Lorentz-invariant measure on the mass hyperboloid,

$$d\Gamma = \frac{d^3p}{2E_p (2\pi)^3}, \quad (4)$$

$u_a(\mathbf{p}, \lambda) e^{-ip \cdot x}$ and $v_a(\mathbf{p}, \lambda) e^{ip \cdot x}$ are plane-wave solutions of the wave equation with positive and negative frequencies, respectively, and λ labels different helicity states. The amplitudes $f_1(\mathbf{p}, \lambda)$ and $f_2(\mathbf{p}, \lambda)$ describe the independent degrees of freedom of the wave field $\psi_a(x)$; all the constraints imposed by the wave equation have already been fully taken into account. For neutral particles, when only one representation of the Poincaré group is present, we set $f_2(\mathbf{p}, \lambda) = f_1^*(\mathbf{p}, \lambda)$.

The ten conserved quantities, the generators of the Poincaré group, $P_{\mu} = (H, \mathbf{P})$ and $M_{\mu\nu} = (\mathbf{M}, \mathbf{N})$ can be expressed as the following bilinear combinations of the amplitudes f . For simplicity, we shall consider only one representation and drop the subscript 1 or 2:

$$H = \sum_{\lambda} \int d\Gamma f^*(\mathbf{p}, \lambda) E_p f(\mathbf{p}, \lambda), \quad (5a)$$

$$\mathbf{P} = \sum_{\lambda} \int d\Gamma f^*(\mathbf{p}, \lambda) \mathbf{p} f(\mathbf{p}, \lambda), \quad (5b)$$

$$\mathbf{M} = \sum_{\lambda} \int d\Gamma f^*(\mathbf{p}, \lambda) [-i\mathbf{p} \times \mathbf{D} + \lambda \mathbf{p}/p] f(\mathbf{p}, \lambda), \quad (5c)$$

$$\mathbf{N} = \sum_{\lambda} \int d\Gamma f^*(\mathbf{p}, \lambda) [i(\mathbf{D} E_p + E_p \mathbf{D})/2] f(\mathbf{p}, \lambda) + m \sum_{\lambda\lambda'} \int d\Gamma f^*(\mathbf{p}, \lambda) [(\mathbf{p} \times \mathcal{S}_{\lambda\lambda'})/p^2] f(\mathbf{p}, \lambda'), \quad (5d)$$

where $\mathcal{S}_{\lambda\lambda'}$ is the set of three spin matrices in the helicity basis [cf. Eq. (A6)]. The covariant derivative \mathbf{D} in momentum space, appearing in these formulas, has the form

$$\mathbf{D} = \nabla + i\lambda\boldsymbol{\alpha}(\mathbf{p}), \quad (6)$$

where ∇ is an ordinary gradient with respect to \mathbf{p} and $\boldsymbol{\alpha}(\mathbf{p})$ is the analog of the vector potential. The helicity λ , clearly, plays the role of the charge.

Equations (5a)–(5d) were derived in our earlier work⁶ for massive and massless spin-1 particles, by introducing the plane-wave expansion of the free fields into the standard formulas for the generators. It can be easily checked by the same method that the formulas for the generators

have the same form as Eqs. (5) for spin $\frac{1}{2}$. The comparison with Foldy representation made in the Appendix shows that these formulas hold universally for any spin.

Our calculations of the generators yielded the following expressions⁷ for $\boldsymbol{\alpha}(\mathbf{p})$ in the cases of spin $\frac{1}{2}$ and spin 1:

$$\text{spin } \frac{1}{2}, \quad \boldsymbol{\alpha}(\mathbf{p}) = \frac{1}{p(p+p_3)} (p_2, -p_1, 0); \quad (7a)$$

$$\text{spin } 1, \quad \boldsymbol{\alpha}(\mathbf{p}) = \frac{p_3}{p(p_1^2 + p_2^2)} (p_2, -p_1, 0). \quad (7b)$$

The “magnetic field” derived from the “vector potential” $\boldsymbol{\alpha}(\mathbf{p})$ is the same for both spins: it is the field of a magnetic monopole of unit strength

$$\nabla \times \boldsymbol{\alpha} = \mathbf{p}/p^3. \quad (8)$$

As always with magnetic monopoles, the potential is singular along a line. It is worth noting that for spin $\frac{1}{2}$ the singularity line is only a half of the p_3 axis, whereas for spin 1 the singularity line extends over the whole p_3 axis, but this may depend on the choice of the representation.

The curvature of the connection \mathbf{D} is

$$[D_i, D_j] = i\lambda \epsilon_{ijk} p_k / p^3. \quad (9)$$

We have calculated the generators (5) directly from the appropriate energy-momentum tensors, but one can understand their structure on general grounds by transforming the amplitudes $f(\mathbf{p}, \lambda)$ from the helicity basis to a basis that is fixed in p space. Then, the operators in momentum space sandwiched between the transformed amplitudes \tilde{f}^* and \tilde{f} in the formulas for P_{μ} and $M_{\mu\nu}$ become the quantum-mechanical generators of the Poincaré group given for arbitrary spin by Foldy⁸ in the form

$$H_F = E_p, \quad (10a)$$

$$\mathbf{P}_F = \mathbf{p}, \quad (10b)$$

$$\mathbf{M}_F = -i\mathbf{p} \times \nabla + \mathbf{s}, \quad (10c)$$

$$\mathbf{N}_F = i(\nabla E_p + E_p \nabla)/2 + (E_p + m)(\mathbf{p} \times \mathbf{s}). \quad (10d)$$

The components $s_{\sigma\sigma'}$ of the spin matrices in these formulas are p independent. In the Appendix we prove the equivalence of the expressions (5) and (10).

The physical significance of the covariant derivative \mathbf{D} for massive particles stems from the fact that with its help the angular momentum is split into the part $-i\mathbf{p} \times \mathbf{D}$ perpendicular to \mathbf{p} and the part $\lambda \mathbf{p}/p$ parallel to \mathbf{p} , as in formulas (5c) and (A1a). However, in the massive case we can also alternatively work in the p -independent basis using the Foldy representation (10) with ordinary derivatives. The situation is completely different in the massless case, where the use of covariant derivatives becomes a necessity, because the spin operator \mathbf{s} cannot be defined.

III. MASSLESS PARTICLES

The helicity basis has the advantages over the p -independent basis that the limit, when $m \rightarrow 0$, is well de-

fined. In this limit, the second integral in Eq. (5d) drops out while all the other terms retain their form. For massless particles all the generators of the Poincaré transformations become sums of the contributions for all different values of the helicity λ . This was to be expected, since every component $f(\mathbf{k}, \lambda)$ forms an irreducible representation of the Poincaré group. For massive particles it is only true for translations and for space rotations, but not for the Lorentz transformations, which mix different helicity states, due to the presence of the term proportional to m in the expression (5d).

The action of each of the ten generators of the Poincaré group on the helicity amplitudes $f(\mathbf{k}, \lambda)$ is given by the corresponding Poisson brackets and can be calculated with the use of the following Poisson brackets for the amplitudes

$$i\{f(\mathbf{k}, \lambda), f^*(\mathbf{k}', \lambda')\} = \delta_{\lambda\lambda'} (2\pi)^3 2\omega_k \delta(\mathbf{k} - \mathbf{k}') . \quad (11)$$

From (5) and (11), we obtain

$$i\{f, H\} = \omega_k f , \quad (12a)$$

$$i\{f, \mathbf{P}\} = \mathbf{k}f , \quad (12b)$$

$$i\{f, \mathbf{M}\} = (-i\mathbf{k} \times \mathbf{D} + \lambda k^{-1} \mathbf{k})f , \quad (12c)$$

$$i\{f, \mathbf{N}\} = i\omega_k \mathbf{D}f , \quad (12d)$$

where for the sake of clarity we have changed the notation for massless particles from (E_p, \mathbf{p}) to (ω_k, \mathbf{k}) .

Not only the action of the Poincaré group on the field amplitudes $f(\mathbf{k}, \lambda)$, but also the action of the full conformal group involves the derivatives in momentum space always in the form of the covariant derivative \mathbf{D} .

For completeness, we shall write down the expressions for the five additional generators D and $K_\mu = (K_0, \mathbf{K})$ and their Poisson-brackets relations with the field amplitudes, as derived before in Ref. 6:

$$D = \frac{1}{2} \sum_{\lambda} \int d\Gamma \mathbf{k} \cdot (f^* i \mathbf{D}f) , \quad (13a)$$

$$K_0 = \sum_{\lambda} \int d\Gamma [\omega_k (\mathbf{D}f)^* \cdot \mathbf{D}f + \omega_k^{-1} f^* f] , \quad (13b)$$

$$\begin{aligned} \mathbf{K} = \sum_{\lambda} \int d\Gamma \{ & \mathbf{k}[(\mathbf{D}f)^* \cdot \mathbf{D}f] - (\mathbf{k} \cdot \mathbf{D}f)^* \mathbf{D}f \\ & - (\mathbf{D}f)^* \mathbf{k} \cdot \mathbf{D}f + \lambda \omega_k^{-1} \mathbf{k} \times (f^* i \mathbf{D}f) \\ & - \omega_k^{-2} \mathbf{k} f^* f \} , \end{aligned} \quad (13c)$$

$$i\{f, D\} = -(i\mathbf{k} \cdot \mathbf{D} + i)f , \quad (14a)$$

$$i\{f, K_0\} = -(\omega_k \mathbf{D} \cdot \mathbf{D} - \omega_k^{-1})f , \quad (14b)$$

$$\begin{aligned} i\{f, \mathbf{K}\} = -[& k(\mathbf{D} \cdot \mathbf{D}) - 2(\mathbf{k} \cdot \mathbf{D})\mathbf{D} \\ & - 2i\lambda \omega_k^{-1} \mathbf{k} \times \mathbf{D} - \omega_k^{-2} \mathbf{k} - 2\mathbf{D}]f . \end{aligned} \quad (14c)$$

It is worth noting that the operators appearing on the right-hand side of Eqs. (12) and (14), acting on the particle wave functions $f(\mathbf{k}, \lambda)$ are Hermitian with respect to the scalar product $\int d\Gamma f_1^* f_2$ as required by the unitarity of the representation of the Poincaré and the conformal group.

IV. CONCLUSIONS

Now, we can calculate the change of the phase of the wave function using the connection $i\lambda\alpha(\mathbf{k})$ found in our study of the Poincaré group. Let us consider a particle moving with constant energy and slowly varying momentum. As we have already indicated in the Introduction, we view the changes of particle momentum as being due to a sequence of Poincaré transformations—in this case pure rotations. The calculation is the simplest for a 2π rotation around a fixed axis, i.e., for a circular closed path in momentum space, and this is also the case that has been studied experimentally. In order to calculate the change of the photon helicity amplitude during such a rotation, we shall replace the complex function $f(\mathbf{p}, \lambda)$ by a real two-dimensional vector u^α made of the real and imaginary parts of f . This will allow us to use directly the standard methods of differential geometry. The covariant derivative of the vector field u^α , determined by the connection is

$$D_i u^\alpha = \nabla_i u^\alpha + \Gamma_i^{\alpha\beta} u^\beta . \quad (15)$$

In our case, according to Eq. (6), the affine connection $\Gamma_i^{\alpha\beta}$ has the form

$$\Gamma_i^{\alpha\beta} = \lambda \alpha_i \epsilon^{\alpha\beta} , \quad (16)$$

where $\epsilon^{\alpha\beta}$ is the antisymmetric matrix $\epsilon^2_1 = 1 = -\epsilon^1_2$.

The curvature tensor $R_{ij}^{\alpha\beta}$ derived from the connection (16) is

$$R_{ij}^{\alpha\beta} = \lambda (\nabla_i \alpha_j - \nabla_j \alpha_i) \epsilon^{\alpha\beta} . \quad (17)$$

With the help of this curvature tensor we can easily calculate the rotation of the vector u^α after transporting it along a closed curve, for example, along a circle. In principle, the curvature tensor gives only the rotation angle for infinitesimal contours, but in our case, due to the factorized form of $R_{ij}^{\alpha\beta}$, we can simply add all the infinitesimal rotation angles and obtain the angle for any finite contour by the integration of the curvature tensor over the surface spanned by the contour:

$$\Delta\phi = \lambda \int d\mathbf{n} \cdot (\nabla \times \boldsymbol{\alpha}) , \quad (18)$$

which by the Stokes theorem can be written as a line integral

$$\Delta\phi = \lambda \oint d\mathbf{p} \cdot \boldsymbol{\alpha} \quad (19)$$

along a closed contour in momentum space.

This rotation angle of the real vector u^α is, of course, the same as the change $\gamma(C)$ of the phase, the Berry phase, for the complex amplitude $f(\mathbf{p}, \lambda)$ transported along the closed contour C .

Using our expression (8) for the curl of $\boldsymbol{\alpha}$, we obtain, finally from (18),

$$\gamma(C) = \pm \lambda \Omega(C) , \quad (20)$$

where $\Omega(C)$ is the solid angle subtended by the contour C as seen from the origin in the momentum space. The sign depends on the orientation of the path. This result coincides with that obtained by Chiao and Wu.

We would like to stress once again that we obtained this

result without introducing external fields in the Hamiltonian, but the formal similarity with Berry's result for a spinning particle traveling in the magnetic field is not accidental. In both cases we deal with the parallel transport of the spin of the particle resulting from the particle's motion. The motion through external fields in the adiabatic limit can be visualized as a series of Poincaré transformations. Therefore, our approach gives a unifying view of the spin propagation valid for all particles (photons, electrons, etc.), regardless of the details of the dynamics. It is also worth stressing that in our approach the role of the helicity quantum number λ , as the analog of the magnetic charge, is clearly exhibited.

We would like to end our paper with a historical note.⁹ The parallel transport law for the polarization of the electromagnetic wave in a medium with the varying index of refraction given by Berry,³ and Chiao and Wu⁴ can be traced back to the work of Levi-Civita¹⁰ on the absolute differential calculus and to the article of Bortolotti.¹¹ Bortolotti ended his article with the following conclusion: "The light vector of the ray Γ linearly polarized, propagating through a medium with a varying index of refraction $n(x,y,z)$, is transported along Γ by a parallelism with respect to a metric connection (in the sense of Weyl) in R_3 , whose components are determined by the vector $\text{grad} \log n^2$." The parallel transport of the polarization vector was later independently discovered and extensively studied by Luneburg.¹²

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APPENDIX

In this Appendix we prove the equivalence of the helicity representation and the Foldy representation of generators of the Poincaré group.

We shall start from the set of Foldy generators (10) and transform them to the helicity representation. To this end, we rewrite the expressions (10c) and (10d) in the form

$$\mathbf{M}_F = -i(\mathbf{p} \times \tilde{\mathbf{D}}) + p^{-2} \mathbf{p}(\mathbf{s} \cdot \mathbf{p}), \quad (\text{A1a})$$

$$\mathbf{N}_F = i(\tilde{\mathbf{D}} E_p + E_p \tilde{\mathbf{D}}) / 2 + m p^{-2} (\mathbf{p} \times \mathbf{s}), \quad (\text{A1b})$$

where

$$\tilde{\mathbf{D}} = \nabla - i(\mathbf{p} \times \mathbf{s}) / p^2. \quad (\text{A2})$$

The change from the fixed basis (labeled by σ) to the helicity basis (labeled by λ) is accomplished with the help of unitary matrices $W_{\sigma\lambda}(\mathbf{p})$,

$$\tilde{f}(\mathbf{p}, \sigma) = \sum_{\lambda} W_{\sigma\lambda}(\mathbf{p}) f(\mathbf{p}, \lambda), \quad (\text{A3})$$

or $\tilde{f} = W f$ in the matrix notation which we shall employ from now on. These matrices diagonalize the helicity operator $(\mathbf{s} \cdot \mathbf{p}) / p$ and, therefore, satisfy the equation

$$(\mathbf{s} \cdot \mathbf{p}) W = p W \Lambda, \quad (\text{A4})$$

where Λ is the diagonal matrix with the eigenvalues λ . The action of the covariant derivative $\tilde{\mathbf{D}}$ on the amplitude \tilde{f} can be written in the form

$$\begin{aligned} \tilde{\mathbf{D}} \tilde{f} &= [\nabla - i p^{-2} (\mathbf{p} \times \mathbf{s})] W f \\ &= W [\nabla + W^\dagger (\nabla W) - i p^{-2} (\mathbf{p} \times \mathbf{s})] f, \end{aligned} \quad (\text{A5})$$

where

$$\mathbf{s}_{\lambda\lambda'} = W_{\lambda\sigma} \mathbf{s}_{\sigma\sigma'} W_{\sigma'\lambda'}. \quad (\text{A6})$$

Now, we will show that the expression in the square brackets in (A5) is diagonal in the indices λ and, therefore, can be identified with the covariant derivative \mathbf{D} . To this end, we shall first differentiate Eq. (A4) with respect to \mathbf{p} :

$$\mathbf{s} W + (\mathbf{s} \cdot \mathbf{p}) \nabla W = p^{-1} \mathbf{p} W \Lambda + p \nabla W \Lambda. \quad (\text{A7})$$

With the help of the relations

$$\mathbf{s} = -p^{-2} \mathbf{p} \times (\mathbf{p} \times \mathbf{s}) + p^{-2} \mathbf{p}(\mathbf{s} \cdot \mathbf{p}), \quad (\text{A8})$$

$$[\mathbf{s} \cdot \mathbf{p}, \mathbf{p} \times \mathbf{s}] = -i \mathbf{p} \times (\mathbf{p} \times \mathbf{s}), \quad (\text{A9})$$

and (A4), we can rewrite (A7) in the form

$$p W [\Lambda, W^\dagger \nabla W] - i p^{-2} [\mathbf{s} \cdot \mathbf{p}, \mathbf{p} \times \mathbf{s}] W = 0 \quad (\text{A10})$$

or

$$[\Lambda, W^\dagger \nabla W - i p^{-2} W^\dagger (\mathbf{p} \times \mathbf{s}) W] = 0. \quad (\text{A11})$$

The second term in this commutator is precisely what appears in Eq. (A5) and the vanishing of the commutator means that it is diagonal in the λ indices. Therefore, we can write (A5) as

$$\sum_{\sigma'} \tilde{\mathbf{D}}_{\sigma\sigma'} \tilde{f}(\mathbf{p}, \sigma') = \sum_{\lambda} W_{\sigma\lambda} \mathbf{D} f(\mathbf{p}, \lambda), \quad (\text{A12})$$

which proves the equivalence of the two representations.

The curvature should not, of course, depend on the choice of the basis and, indeed, by a direct calculation one can confirm that

$$[\tilde{D}_i, \tilde{D}_j]_{\sigma\sigma'} = i (\mathbf{s}_{\sigma\sigma'} \cdot \mathbf{p}) \epsilon_{ijk} p_k / p^4, \quad (\text{A13})$$

in accordance with Eq. (8). Note, that $\tilde{\mathbf{D}}$ (unlike \mathbf{D}) has the transformation properties of a vector.

The form of \mathbf{D} depends on the specific choice of the $W_{\sigma\lambda}(\mathbf{p})$ matrices, which will depend on the phases of the eigenvectors of the helicity operator. For spin $\frac{1}{2}$ and spin 1 we have chosen these matrices in the form

$$W_{1/2} = \frac{1}{\sqrt{2p(p+p_3)}} \begin{pmatrix} p+p_3 & -p_1+ip_2 \\ p_1+ip_2 & p+p_3 \end{pmatrix}, \quad (\text{A14a})$$

$$W_1 = \frac{1}{[2p^2(p_1^2+p_2^2)]^{1/2}} \begin{pmatrix} p_1p_3-ipp_2 & p_1(p_1^2+p_2^2)^{1/2} & p_1p_3+ipp_2 \\ p_2p_3+ipp_1 & p_2(p_1^2+p_2^2)^{1/2} & p_2p_3-ipp_1 \\ -p_1^2-p_2^2 & p_3(p_1^2+p_2^2)^{1/2} & -p_1^2-p_2^2 \end{pmatrix}, \quad (\text{A14b})$$

which leads to the formulas (5) and (6).

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