Inflation in spherically symmetric inhomogeneous models

Jaime A. Stein-Schabes
Theoretical Astrophysics Group, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510
(Received 24 November 1986)

Exact analytical solutions of Einstein's equations are found for a spherically symmetric inhomogeneous metric in the presence of a massless scalar field with a flat potential. The process of isotropization and homogenization is studied in detail. It is found that the time dependence of the metric becomes de Sitter for large times. Two cases are studied. The first deals with a homogeneous scalar field, while the second deals with a spherically symmetric inhomogeneous scalar field. In the former case the metric is of the Robertson-Walker form, while the latter is intrinsically inhomogeneous.

I. INTRODUCTION

One of the best explanations so far as to why the observable Universe looks so remarkably flat, homogeneous, and isotropic is provided by the inflationary models.[1] In these models the Universe undergoes a phase transition characterized by the evolution of a Higgs field ϕ, that is, initially displaced from the minimum of its potential $V(\phi)$, towards the minimum. The Higgs field has initially a zero vacuum expectation value (VEV) and it evolves, by moving inside the potential, to a state where it acquires a nonzero VEV. Inflation will take place if the potential $V(\phi)$ has a "flat" region and the ϕ field evolves slowly, spending a considerable amount of time in this part of the potential. At the same time the Universe is expanding in an exponential way, driven by the vacuum field energy, until the ϕ field reaches a steep region where it starts moving fast. Eventually, it gets to the bottom of the potential and after a few oscillations it stops[1-3] (for a comprehensible and up-to-date review on inflation see Turner[4]). It is the presence of this flat region in the potential that makes the Universe enter an inflationary period. I shall not dwell any longer on the fine subtleties of the inflationary models, as I should only need the existence of the flat region in the potential. In terms of more physical ideas, I will only consider the first stage of inflation where the Universe enters a phase of very rapid expansion, and shall assume that the reheating process follows in the standard fashion from there on. This assumption may seem a bit too strong; however, I will argue that once the Universe enters the exponential expansion phase, rapidly becomes homogeneous and isotropic on scales of the order the horizon size, and by the time the scalar field reaches the end of the flat part of the potential and starts to roll down, it is essentially isotropic. Of course, there is the tacit assumption that there will be sufficient inflation, i.e., the scalar field will take sufficient time to cross the flat region of the potential to allow the Universe to expand the necessary number of e-folds, otherwise, inflation could not do the job. The flat part of the potential is naturally associated with a vacuum energy that dominates the dynamics for a period of time, so I shall identify this vacuum energy with an effective cosmological constant Λ. It has been shown by Jensen and Stein-Schabes[5] that inhomogeneous cosmologies that have a cosmological constant, an energy-momentum tensor satisfying the strong and weak energy conditions and a nonpositive three-curvature will become isotropic and homogeneous for large times on the scale of the observable Universe, essentially becoming the de Sitter model for late times. This is a generalization of a similar result proven within the context of homogeneous cosmologies by Wald.[6] Even though these results indicate that dynamically the Universe goes from an inhomogeneous and anisotropic phase to one which is isotropic and homogeneous on scales of the observable Universe, it gives no details on the evolution of the scalar field or any other content of the Universe. In order to learn more about the evolutionary details of the isotropization process we have to study specific examples. Most inflationary models have been constructed assuming the background space-time metric is homogeneous, either one of the Bianchi models or a Kantowski-Sachs model.[7,8] For these models it has been shown that once inflation starts, the process of isotropization is remarkably efficient. Furthermore, once inflation has successfully ended, the Universe remains isotropic for a very long time. Very little analytical work has been done to solve the problem in the case of inhomogeneous space-times, but some numerical results have been obtained[9] for the case of an inhomogeneous scalar field; however, the background metric is still homogeneous and isotropic.

In this paper I will explicitly solve Einstein's equations obtaining analytical expressions for both the metric components and the scalar field in the case where the metric describes an inhomogeneous spherically symmetric space-time, the so-called Tolman-Bondi[10] metric in the presence of a massless scalar field ϕ with a potential $V(\phi)$ that has a flat region.

One of the most general exact solutions to Einstein's equations with a cosmological constant and dust was found by Barrow and Stein-Schabes.[11] This model describes a quasispherical space-time that does not have Killing vectors, the so-called Szekeres metrics.[12] It has been shown that asymptotically these solutions possess the same event horizon structure as the de Sitter metric, in accord with the no-hair theorem.[13] However, in this case there is no scalar field present. To include a completely inhomogeneous scalar field and solve the field equation is extremely difficult. However, there is a subclass of metrics for which the problem can be solved completely. These metrics represent spherically symmetric space-times that are both inhomogeneous and anisotropic. A full description of the model will be given in the next section, where the field equations for the gravitational field coupled to a real scalar field will be presented and solved. I shall study the dynamical evolution towards the de Sitter phase.

II. THE MODEL

The Lagrangian will be that of gravity minimally coupled to a scalar field $\phi(r,t)$ with a potential $V(\phi)$,

$$S=\int \sqrt{-g}\,[R-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)]d^4x\ , \tag{2.1}$$

where $g=\det(g_{\mu\nu})$ and R is the Ricci scalar. Units are taken so that $16\pi G=c=1$.

By varying the action with respect to the dynamical fields the following equations are obtained:

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=T_{\mu\nu} \tag{2.2}$$

with

$$T_{\mu\nu}=\tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi-[\tfrac{1}{4}(\partial_\alpha\phi\partial^\alpha\phi)+\tfrac{1}{2}V(\phi)]g_{\mu\nu}\ , \tag{2.3}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,\partial^\mu\phi)=-\frac{dV(\phi)}{d\phi}\ . \tag{2.4}$$

The metric is given by the Tolman-Bondi line element,

$$ds^2=-dt^2+X^2(r,t)dr^2+Y^2(r,t)(d\theta^2+\sin^2\theta\,d\varphi^2) \tag{2.5}$$

with r, θ, and φ the normal spherical coordinates (in the comoving frame). The field equations can be written as (ϕ will be rescaled to 2 ϕ)

$$2\frac{\dot X}{Y}\frac{\dot Y}{X}+\left[\frac{\dot Y}{Y}\right]^2+\frac{1}{Y^2}-\frac{1}{X^2}\left[2\frac{Y''}{Y}+\left[\frac{Y'}{Y}\right]^2-2\frac{X'}{X}\frac{Y'}{Y}\right]=(\dot\phi)^2+\frac{(\phi')^2}{X^2}+\tfrac{1}{2}V(\phi)\ , \tag{2.6}$$

$$2\frac{\ddot Y}{Y}+\left[\frac{\dot Y}{Y}\right]^2+\frac{1}{Y^2}-\frac{1}{X^2}\left[\frac{Y'}{Y}\right]^2=-(\dot\phi)^2-\frac{(\phi')^2}{X^2}+\tfrac{1}{2}V(\phi)\ , \tag{2.7}$$

$$\frac{\ddot X}{X}+\frac{\ddot Y}{Y}+\frac{\dot X}{X}\frac{\dot Y}{Y}-\frac{1}{X^2}\left[\frac{Y''}{Y}-\frac{X'}{X}\frac{Y'}{Y}\right]=-(\dot\phi)^2+\frac{(\phi')^2}{X^2}+\tfrac{1}{2}V(\phi)\ , \tag{2.8}$$

$$\frac{\dot Y'}{Y}-\frac{\dot X}{X}\frac{Y'}{Y}=-\dot\phi\phi'\ , \tag{2.9}$$

$$\ddot\phi+\left[\frac{\dot X}{X}+2\frac{\dot Y}{Y}\right]\dot\phi-\frac{1}{X^2}\left[\phi''+\left[3\frac{X'}{X}+2\frac{Y'}{Y}\right]\phi'\right]=-\frac{dV(\phi)}{d\phi}\ , \tag{2.10}$$

where an overdot $=\partial_t$ and a prime $=\partial_r$. As is usually the case the system of partial differential equations is overdetermined, so some of the equations will be regarded as dynamical while the rest as constraint equations. These equations will be solved in two different cases. First, assuming that the scalar field is homogeneous, i.e., $\phi=\phi(t)$, and second, taking it to be of the general form $\phi=\phi(r,t)$. We shall also assume that for the epoch that I am interested in, the potential can be well approximated by a constant value $V(\phi)\simeq 2\Lambda$. This can be justified by noticing that in general the quantity multiplying the $\dot\phi$ term in Eq. (2.10) acts as a friction term and it has been shown elsewhere[5,6] that this term is always larger than in the homogeneous and isotropic model. What this means is that the field will always move slower in an anisotropic and inhomogeneous model, spending more time than usual on the flat part of the potential.

III. THE HOMOGENEOUS CASE: $\phi=\phi(t)$

In this case it is easy to see that Eqs. (2.9) and (2.10) immediately integrate to give, respectively,

$$X(r,t)=f(r)Y'(r,t)\ , \tag{3.1}$$

$$\dot\phi(t)=\frac{l(r)}{XY^2}=\frac{l(r)}{f(r)}\frac{1}{Y^2Y'} \tag{3.2}$$

with $l(r)$ and $f(r)$ arbitrary integration functions. Equation (3.2) is a very interesting equation as it forces the spatial dependence of $Y(r,t)$ to be such that $\dot\phi$ is only a function of t. Taking now the following combination of Eq. (2.6)−(2.7)−2×(2.8) we get

$$\frac{\ddot X}{X}+2\frac{\ddot Y}{Y}=-2(\dot\phi)^2\Lambda\ . \tag{3.3}$$

Using Eqs. (3.1) and (3.3) we can decouple the equations and get one for $Y(r,t)$:

$$\frac{\ddot Y'}{Y'}+2\frac{\ddot Y}{Y}=\frac{l^2(r)}{f^2(r)}\frac{1}{Y^4(Y')^2}+\Lambda\ . \tag{3.4}$$

To solve this partial differential equation we use separation of variables. We look for solutions of the form

$$Y(r,t)=Y_r(r)Y_t(t) \tag{3.5}$$

which implies from (3.1) that $X(r,t)=f(r)Y_r'X_t$. This decomposition allows a separation into two ordinary instead of partial differential equations of the form

$$3\frac{\ddot{Y}_t}{Y_t}-\frac{m_1}{Y_t^{\,6}}=\Lambda\ , \tag{3.6}$$

$$Y_r'Y_r^2=\left[\frac{-2f^2(r)}{m_1 l^2(r)}\right]^{1/2} . \tag{3.7}$$

Clearly this demands that the separation constant $m_1<0$. Equation (3.6) has a general first integral of the form

$$\left[\frac{\dot{Y}_t}{Y_t}\right]^2=\frac{\Lambda}{3}-\frac{k}{Y_t^{\,2}}-\frac{m_1}{6Y_t^{\,6}} \tag{3.8}$$

with k an integration constant.

This is exactly the evolution equation obeyed by the geometric mean scale factor with a cosmological constant, a curvature, and anisotropy terms characterized by k and m_1, respectively.[14] The general solution to this equation is given in terms of some complicated elliptical functions. Since this is a well-studied equation and its asymptotic behavior well known we shall not say more about its general solution. These have a stable asymptotic limit of the form $Y_t=Y_0e^{(\Lambda/3)^{1/2}t}$ which is the same time evolution as in the de Sitter model. Then by Eqs. (3.1) and (3.5) the time evolution for $X(r,t)$ is the same as that for Y_t, namely,

$$X(r,t)=f(r)Y_r'Y_t\to X_r(r)e^{(\Lambda/3)^{1/2}t}\ , \tag{3.9}$$

where $f(r)$ is an arbitrary integration function. By substituting Eq. (3.5) into (3.2) and using (3.1) we get, for ϕ,

$$\dot{\phi}(t)=\left[\frac{-m_1}{2}\right]^{1/2}\frac{1}{Y_t^{\,3}} \tag{3.10}$$

which is perfectly consistent with the functional dependence of ϕ. For the spatial part of Y_r Eq. (3.7) cannot be integrated until we specify the form of the arbitrary function. The fact that the Universe is expanding exponentially fast, becoming more and more like de Sitter, makes the scalar field ϕ go rapidly to a constant value ϕ_0.

The ansatz [Eq. (3.5)] used to find the solution for $X(r,t)$ and $Y(r,t)$ together with the assumed homogeneity of the scalar field determined the dynamics of the model to be identical to the homogeneous and isotropic model for any potential $V(\phi)$. From Eqs. (3.1) and (3.5) we get that $X_t=Y_t$ and since the scalar field evolution is determined by (2.10) (without the spatial derivatives), it becomes the same equation as for the homogeneous scalar field. Furthermore, if we run the constraint equations we immediately find that $f^2(r)=(1-kY_r^{\,2})^{-1}$. This is a direct consequence of the separability of the metric components into space and time parts, which immediately forces the Universe to expand isotropically with an expansion rate that approaches the de Sitter one. Even though the integration method is general there is an inbuilt bias in it that only picks up several solutions. This suggests that there may be a nonseparable solution to the field equations (this is known to occur in nonlinear partial differential equations, a general solution may have several distinct branches), and this is indeed the case. The set of equations (2.7) and (2.8) accept a nonseparable type of solution (this was done in Ref. 10 for the case of dust). If we use (3.1) in (2.7) and (2.8) and we introduce a new variable $U(r,t)=Y[1-f^{-2}+(\dot{Y})^2]$ these can be rewritten as

$$\frac{\dot{U}}{Y^2\dot{Y}}=-\dot{\phi}^2+\tfrac{1}{2}V(\phi)\ , \tag{3.11}$$

$$\frac{1}{2YY'}\left[\frac{\dot{U}}{\dot{Y}}\right]'=-\dot{\phi}^2+\tfrac{1}{2}V(\phi)\ . \tag{3.12}$$

By equating these two equations we get one very simple differential equation

$$\left[\frac{\dot{U}}{Y^2\dot{Y}}\right]'=0 \tag{3.13}$$

which after some manipulation becomes

$$\left[\frac{\dot{Y}}{Y}\right]^2=\frac{\Lambda}{3}-\frac{k(r)}{Y^2}+\frac{c_1(r)}{Y^3}\ , \tag{3.14}$$

where $k(r)\equiv 1-f^{-2}$ this equation has the same form as the Friedmann equation for a Robertson-Walker model in the presence of a cosmological constant, and a fluid whose density scales like the inverse of the volume. Of course this is the scalar field energy density. The point to notice is that this equation is still a partial differential equation and no functional form has been set for the solution. This equation has been solved in Ref. 11. In general the solution is not expressible in terms of elementary functions. For the flat case $k(r)=0$ the solution has the following form:

$$Y(r,t)=\left[\frac{3c_1(r)}{\Lambda}\right]^{1/3}\times\sinh^{2/3}\left[\left[\frac{3\Lambda}{4}\right][t-t_0(r)]\right] \tag{3.15}$$

with $t_0(r)$ an arbitrary integration function. Only in the large-time limit does the solution become separable into its space and time parts. However, in order to have a consistent solution for $\dot{\phi}(t)$ we are forced to take $t_0(r)=0$, bringing the solution back to the separable form. The homogeneity of the field is so restrictive that it determines the functional dependence of the metric components. The physical reason is that both the scalar field and the cosmological constant produce "isotropic forces" so there is nothing to keep the model from becoming isotropic as it evolves and it would seem that any initial anisotropy and inhomogeneity present in the model must be put in by hand. It is then impossible to have an anisotropic and inhomogeneous universe if the scalar field is to be homogeneous. The only way out is to allow the field to contain some inhomogeneities. In that case we could envisage the case just studied as an intermediate stage between complete inhomogeneity and homogeneity. The extension to the inhomogeneous case is done in the next section.

We can now write the full solution for the metric,

$$ds^2=-dt^2+X_t^2X_r^2dr^2 + Y_t^2Y_r^2(d\theta^2+\sin^2\theta\, d\varphi^2) \tag{3.16}$$

with the help of Eq. (3.1) and using the freedom to redefine the coordinate system, we introduce a new radial variable $\rho=Y_r$ then Eq. (3.16) becomes, in the large-time limit,

$$ds^2=-dt^2+e^{(\Lambda/3)^{1/2}t}\left[\frac{d\rho^2}{1-k\rho^2}+\rho^2(d\theta^2+\sin^2\theta\, d\varphi^2)\right] \tag{3.17}$$

which is exactly the de Sitter solution in its open, flat, or closed version. What we can conclude in this case is that the model was never inhomogeneous but rather looked like one due to a bad choice of coordinates, so even if we assumed that the metric was more general than the standard model, the equations only accepted the homogeneous and isotropic solution. The flatness of the model is then explained as in the standard Friedmann-Robertson-Walker (FRW) model. By calculating the three-curvature scalar ${}^{(3)}R$,

$$\begin{aligned}{}^{(3)}R &= e^{-2(\Lambda/3)^{1/2}t}\left[\frac{f'\rho+f^3-f}{f^2\rho}\right] \\ &= e^{-2(\Lambda/3)^{1/2}t}\left[\frac{2k\rho}{1-k\rho^2}\right]\end{aligned} \tag{3.18}$$

we can see that it becomes zero exponentially fast; however, the individual components of the three-Ricci tensor do not vanish in the comoving frame (unless, of course, $k=0$). However, when a measurement is done in the observers frame then all components of the curvature tensor vanish asymptotically fast in the same way the curvature scalar does. For a very interesting account of these models when there is only dust, see Ref. 15.

IV. THE INHOMOGENEOUS CASE: $\phi=\phi(r,t)$

In this general case we look for solutions of the form

$$X(r,t)=X_r(r)X_t(t)\ , \tag{4.1}$$

$$Y(r,t)=Y_r(r)Y_t(t)\ . \tag{4.2}$$

This decomposition of the scale factors forces the scalar field to be of the following form [this can be immediately seen from Eqs. (3.3), (4.1), and (4.2)]:

$$\phi(r,t)=\phi_r(r)+\phi_t(t)\ . \tag{4.3}$$

Equation (2.9) is again immediately separable and integrable:

$$\frac{Y_r'}{Y_r}=-\phi_r'\ , \tag{4.4}$$

$$\frac{\dot Y_t}{Y_t}-\frac{\dot X_t}{X_t}=\dot\phi_t\ . \tag{4.5}$$

To close the system of equations we use Eq. (2.10), which again splits naturally into the following equations:

$$\ddot\phi_t+\left[\frac{\dot X_t}{X_t}+2\frac{\dot Y_t}{Y_t}\right]\dot\phi_t=\frac{m_2}{X_t^2}\ , \tag{4.6}$$

$$\phi_r''+\left[3\frac{X_r'}{X_r}+2\frac{Y_r'}{Y_r}\right]\phi_r'=m_2X_r^2 \tag{4.7}$$

with m_2 the separation constant. We require one more equation for the time-dependent part of the solution. We can use Eq. (3.3) which has no spatial part, and for the spatial part we use the r-dependent part of Eq. (2.8) (of course the time-dependent part of this equation becomes one more of the constraint equations):

$$\frac{\ddot X_t}{X_t}+2\frac{\ddot Y_t}{Y_t}=-2(\dot\phi_t)^2+\Lambda\ , \tag{4.8}$$

$$\frac{Y_r''}{Y_r}-\frac{X_r'}{X_r}\frac{Y_r'}{Y_r}+(\phi_r')^2=m_3X_r^2\ . \tag{4.9}$$

So, the problem has been reduced to solve two decoupled systems of ordinary differential equations for the spatial and time parts of the metric and field. The former is given by the solution to Eqs. (4.5), (4.6), and (4.8), while the latter is given by the solution to Eqs. (4.4), (4.7), and (4.9).

We will first proceed to solve the space part. Introducing a new variable $W\equiv(\ln Y_r)'$ and using Eq. (4.4) we can rewrite the system formed by Eqs. (4.7) and (4.9) as

$$W'+2W^2+3\frac{X_r'}{X_r}W=-m_2X_r^2\ , \tag{4.10}$$

$$W'+2W^2-\frac{X_r'}{X_r}W=m_3X_r^2\ . \tag{4.11}$$

A single equation for X_r can be obtained from these equations:

$$\frac{X_r''}{X_r}-\left[2+\frac{m_3}{a}\right]\left[\frac{X_r'}{X_r}\right]^2+2aX_r^2=0\ . \tag{4.12a}$$

Then W and subsequently Y_r can be found from the equation

$$X_r'W=-aX_r^3\ , \tag{4.12b}$$

where $a\equiv\frac{1}{4}(m_2+m_3)$. Equation (4.12a) can be integrated once to give

$$(X_r')^2=\left[C_1X_r^{2m_3/a}+\frac{2a^2}{m_3}\right]X_r^4\ . \tag{4.13}$$

Both Y_r and ϕ_r can be obtained from (4.4) and (4.12b) once this equation has been solved:

$$Y_r=Y_0\exp\left[-a\int\frac{X_r^3}{X_r'}dr\right]\ , \tag{4.14a}$$

$$\phi_r=\phi_0+a\int\frac{X_r^3}{X_r'}dr\ . \tag{4.14b}$$

In order to carry out the integration of Eq. (4.13) we have to specify the values of the arbitrary constant. The general solution can be obtained in terms of complicated

Gauss hypergeometrical functions; however, to construct the general solution would only obscure the features we want to highlight. Instead, we shall explore the consequences when special values are chosen for these constants.

(i) $a=0$. In this case we can see from Eq. (4.12b) that $X_r'=0$, then Eqs. (4.10) and (4.11) become the same equation for W, when solved for Y_r we get

$$Y_r=\begin{cases} Y_0\cos^{1/2}(\sqrt{2m_2}r), & m_2>0\ , \\ Y_0 r^{1/2}, & m_2=0\ , \\ Y_0\sinh^{1/2}(\sqrt{-2m_2}r), & m_2<0\ , \end{cases} \tag{4.15}$$

with $X_r(r_0)=1$, this in turn gives, for ϕ_r,

$$\phi_r=\begin{cases} \phi_0-\frac{1}{2}\ln[\cos(\sqrt{2m_2}r)], & m_2>0\ , \\ \phi_0-\frac{1}{2}\ln(r), & m_2=0\ , \\ \phi_0-\frac{1}{2}\ln[\sinh(\sqrt{-2m_2}r)], & m_2<0\ . \end{cases} \tag{4.16}$$

There is a special solution that appears as a particular subcase when both X_r' and W' are zero. The solution is then given for $m_2\le 0$ by

$$\begin{aligned} X_r&=X_0\ ,\\ Y_r&=Y_0e^{(-m_2/2)^{1/2}X_0r}\ ,\\ \phi_r&=\phi_0-\left[\frac{-m_2}{2}\right]^{1/2}X_0r\ . \end{aligned} \tag{4.17}$$

(ii) $a=\pm 2m_3/n$. For this case the solution to (4.13) is a finite series that depends on the value of n. For simplicity let $n=\pm 2$, then the solution is given by

$$X_r^2=\begin{cases} \dfrac{b}{1-C_1b^2r^2}, & n=2,\ b>0\ , \\ \dfrac{1}{b}\sec\left[\dfrac{\sqrt{C_1}}{b}r\right], & n=-2,\ b>0\ . \end{cases} \tag{4.18}$$

The other solutions have been ignored as they are either imaginary or not regular at the origin of coordinates. These solutions give from Eqs. (4.14a) and (4.14b) for Y_r and ϕ_r the following:

$$Y_r=\exp\left[\int\frac{-aX_rdr}{\sqrt{C_1}(X_r^{\pm 2}+b^2)^{1/2}}\right]\ , \tag{4.19}$$

$$\phi_r=\phi_0-a\int\frac{X_rdr}{\sqrt{C_1}(X_r^{\pm 2}+b^2)^{1/2}}\ , \tag{4.20}$$

the term $X_r^{\pm 2}$ corresponds to $n=\pm 2$.

Now we will solve for the time-dependent part of the functions, the system of equations for this case is given by Eqs. (4.5), (4.6), and (4.8). From Eq. (4.5) we get ϕ_t as a function of X_t and Y_t:

$$\phi_t=\phi_0+\ln\left[\frac{Y_t}{X_t}\right]\ . \tag{4.21}$$

With the help of this equation and introducing a new logarithmic variable $P=\partial_t(\ln Y_t)$ we rewrite (4.6) and (4.8) as

$$\dot P+2P^2-\frac{\ddot X_t}{X_t}-\frac{\dot X_t}{X_t}P=\frac{m_2}{X_t^2}\ , \tag{4.22}$$

$$\dot P+2P^2+\frac{1}{2}\frac{\ddot X_t}{X_t}+\left[\frac{\dot X_t}{X_t}\right]^2-\frac{\dot X_t}{X_t}P=\Lambda\ . \tag{4.23}$$

From these two equations we can get one for X_t:

$$\frac{\ddot X_t}{X_t}+\frac{2}{3}\left[\frac{\dot X_t}{X_t}\right]^2=\tfrac{2}{3}\Lambda-\frac{2}{3}\frac{m_2}{X_t^2}\ . \tag{4.24}$$

This equation can be integrated once to give

$$\left[\frac{\dot X_t}{X_t}\right]^2=\tfrac{2}{5}\Lambda-\frac{m_2}{X_t^2}\ . \tag{4.25}$$

This is identical to Friedmann's equation for a geometric mean scale factor in the presence of a cosmological constant and curvature $m_2\equiv k$ (which can be made $k=\pm 1,0$), and its solutions are well known:

$$X_t=\begin{cases} \left[\dfrac{5m_2}{2\Lambda}\right]^{1/2}\cosh\left[\left[\dfrac{2\Lambda}{5}\right]^{1/2}t\right], & m_2>0\ , \\ e^{(2\Lambda/5)^{1/2}t}, & m_2=0\ , \\ \left[\dfrac{-5m_2}{2\Lambda}\right]^{1/2}\sinh\left[\left[\dfrac{2\Lambda}{5}\right]^{1/2}t\right], & m_2<0\ . \end{cases} \tag{4.26}$$

With Eq. (4.16) or (4.17) we can calculate Y_t, but since all the solutions tend asymptotically toward the case $m_2=0$ we will only calculate Y_t and ϕ_t in this limit:

$$Y_t=Y_0e^{(9\Lambda/40)^{1/2}t}\sinh^{1/2}\left[\left[\frac{9\Lambda}{10}\right]^{1/2}t\right]\ , \tag{4.27}$$

$$\phi_t=\phi_0+\left[\frac{9\Lambda}{40}\right]^{1/2}t-\tfrac{1}{2}\ln\sinh\left[\left[\frac{9\Lambda}{10}\right]^{1/2}t\right]\ . \tag{4.28}$$

Clearly for large times we get $X_t\simeq Y_t\simeq e^{(2\Lambda/5)^{1/2}t}$ and $\phi_t=\phi_0$ as expected. Now we can construct the full solution

$$\begin{aligned} ds^2=&-dt^2+X_t^2X_r^2dr^2\\ &+Y_t^2Y_r^2(d\theta^2+\sin^2\theta d\varphi^2)\ . \end{aligned}$$

In the large-time limit we get

$$ds^2=-dt^2+e^{2(2\Lambda/5)^{1/2}t}[X_r^2dr^2+Y_r^2(d\theta^2+\sin^2\theta d\varphi^2)]\ . \tag{4.29}$$

In the case where $a=0$ we recover exactly the de Sitter solution in its open or flat version. In all the other cases we can choose the coordinates such that one of the functions, say, $Y_r=\rho$ then (4.29) becomes

$$ds^2=-dt^2+e^{2(2\Lambda/5)^{1/2}t}\times[Z_\rho{}^2 d\rho^2+\rho^2(d\theta^2+\sin^2\theta\, d\varphi^2)] \quad (4.30)$$

with

$$Z_\rho{}^2=\frac{Z_0{}^2}{\rho^2}\left[X_\rho{}^{(2m_3|a)}+\frac{2a^2}{m_3}\right] \quad (4.31)$$

and x_ρ is given by the solution to Eq. (4.13). The curvature scalar is given by Eq. (3.18) by substituting $f(\rho)$ for Z_ρ.

In order to understand the results obtained we must remember that the coordinates used are comoving coordinates, so even though inhomogeneities are not disappearing inside a comoving volume they certainly are for an observer measuring physical volume. Inhomogeneities get frozen out and then just pushed outside the observable horizon. This effect had been pointed out earlier in Ref. 16.

V. CONCLUSIONS

By explicitly solving in an exact form Einstein's equations for an inhomogeneous spherically symmetric metric of the Tolman-Bondi type coupled to a real scalar field with a potential that has a flat region, we studied in detailed the isotropization and homogenization of the Universe on scales of the order of the horizon today, in accord with the prediction of the "weak" no-hair theorem.[5] Two cases were studied: the first with a homogeneous scalar field, the second with a general one. In the first case the homogeneity severely restricts the functional dependence of the metric coefficients with time. In fact it forces the model to expand isotropically for all time, the rate at which it does it approach the de Sitter rate asymptotically. The second case is more interesting as truly inhomogeneous solutions can be found. The separability of the metric coefficients into space and time parts demanded the scalar field to be of the form $\phi(r,t)=\phi_r(r)+\phi_t(t)$. The time evolution was very similar to the homogeneous case, while the space evolution was calculated exactly.

Even though the equations were solved exactly by a well-defined procedure, this failed to capture all the features of the general solution. The solutions found have one less arbitrary function than the general solution should. For these to have been general we would have required six independent functions $f_i(r,t_0)$, which can be identified with $(X(r,t_0),\dot X(r,t_0),Y(r,t_0),\dot Y(r,t_0),\phi(r,t_0),\dot\phi(r,t_0))$. The problem arises due to the fact that both X and Y are separable functions and this, as was shown earlier on, determines the form of ϕ written in Eq. (4.3). Then $\dot\phi(r,t)=\dot\phi(t)$ on every hypersurface. Here is where the generality is lost. (I am in debt to Michael Turner for pointing out this to me.) We could interpret this as an indication that the general solution to these equations if far more complex than that given in Eqs. (4.1)—(4.3). It would nevertheless be extremely useful to know if these solutions are stable against small perturbations that do not decouple the functional dependence of the metric components into their space and time parts [e.g., $Y(r,t)=Y_r(r)+Y_t(t)+Y_1(r,t)$, $Y_1(r,t)$ very small].

We have argued that the approximation of the potential as a flat one, at least initially, is not too bad as it is possible to show that in general the damping term (friction) is larger for these models than for the homogeneous and isotropic model. The second stage of inflation, that of reheating, is assumed to proceed as in the homogeneous and isotropic model. It has been shown that in all cases the universe tends to a state of isotropic expansion which approaches the de Sitter rate for large times, so inflating the Universe. In this sense the Universe is "isotropizing." However, the homogeneity and flatness are a bit more subtle. It is always possible to choose the arbitrary functions in the solution in a judicious way and then recover the Robertson-Walker metric, in which case the models indeed become homogeneous globally. We could as well choose these functions to mimic, say, a Bianchi type V. Nevertheless, in general these functions can only be determined through boundary conditions. The important point to notice is that the time evolution decouples from the spatial variation, the former becomes identical to the de Sitter case. The inhomogeneities are frozen out and then are pushed out of the observers' horizon by the rapid expansion of the Universe. So we could conclude that globally the models do not become homogeneous [they do in the case $\phi=\phi(t)$]. However, inflation can still do the trick. The three-space is not maximally symmetric as we would expect if the no-hair theorem was absolutely correct. Nevertheless, since the three space is only expanding in a conformal way, than an observer living in this three-space would see the Universe around it becoming homogeneous and flat. For an "outside observer," this is not the case, the Universe is just growing in volume but is not changing its curvature or is becoming more homogeneous. The exact analogy is that of an observer living on the surface of an ellipsoid, if the ellipsoid expands very fast, for this observer the surface of the ellipsoid will appear more an more like a flat region, but for an observer much larger than the ellipsoid, it is always curved. This effect is precisely what is observed to happen in these inhomogeneous models as they evolve from an initial highly anisotropic and inhomogeneous phase. This can be seen by calculating the spatial curvature and showing that it goes exponentially to zero during the inflationary period in the frame of the observer.

ACKNOWLEDGMENTS

I would like to thank Lars Jensen for some very helpful conversations and Michael Turner for encouraging and illuminating suggestions. This work was supported in part by the DOE and NASA.

[1]A. H. Guth, Phys. Rev. D **23**, 347 (1981).
[2]A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
[3]A. D. Linde, Phys. Lett. **108B**, 389 (1982); **129B**, 177 (1983).
[4]M. S. Turner, in *The Inflationary Paradigm,* proceedings of the Cargèse School on Fundamental Physics and Cosmology, edited by J. Audouze and J. Tran Thanh Van (Editions Frontieres, Gif-Sur-Yvette, 1985).
[5]L. G. Jensen and J. Stein-Schabes, in proceedings of the First Course of the International School of Particle Astrophysics, Erice, Italy, 1986 (unpublished); Phys. Rev. D **35**, 1146 (1987).
[6]R. M. Wald, Phys. Rev. D **28**, 2118 (1983).
[7]L. G. Jensen and J. A. Stein-Schabes, Phys. Rev. D **34**, 931 (1986).
[8]M. S. Turner and L. Widrow, Phys. Rev. Lett. **57**, 2237 (1986); G. Steigman and M. S. Turner, Phys. Lett. **128B**, 295 (1983).
[9]A. Albrecht, R. H. Brandenberger, and R. A. Matzner, Phys. Rev. D **32**, 1280 (1985).
[10]R. C. Tolman, Proc. Nat. Acad. Sci. **16**, 511 (1930); H. Bondi, Mon. Not. R. Astron. Soc. **107**, 410 (1947).
[11]J. D. Barrow and J. Stein-Schabes, Phys. Lett. **103A**, 315 (1984).
[12]P. Szekeres, Commun. Math. Phys. **41**, 55 (1975).
[13]G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977); S. W. Hawking and I. G. Moss, Phys. Lett. **110B**, 35 (1982).
[14]M. A. H. Mac Callum, in *Cargèse Lectures in Physics,* edited by E. Shatzman (Gordon and Breach, New York, 1973), Vol. 6; G. F. R. Ellis, *ibid.*
[15]W. B. Bonnor, Mon. Not. R. Astron. Soc. **167**, 55 (1974); W. B. Bonnor and N. Tomimura, *ibid.* **175**, 85 (1976).
[16]J. A. Frieman and M. S. Turner, Phys. Rev. D **30**, 265 (1984); R. Brandenberger and R. Kahn, *ibid.* **29**, 2172 (1984); J. Bardeen, P. Steinhardt, and M. S. Turner, *ibid.* **28**, 679 (1983).